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**Author: Sebastià Mijares i Verdú**

**Advisor: Enric Ventura Capell**

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BARCELONATECH
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# ALGEBRAIC EXTENSIONS IN FREE GROUPS AND STALLINGS GRAPHS

Sebastià Mijares i Verdú

Master in Advanced Mathematics and Mathematical Engineering (MAMME), FME.
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Advisor:

Enric Ventura Capell

# Autorització de difusió

Sebastià Mijares i Verdú

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# Resum en català


En aquest treball, estudiem extensions algebraiques en grups lliures fent un ús extensiu dels grafs de Stallings com una potent eina, tant per a resoldre problemes abstractes com a nivell algorísmic.

En el primer capítol construïm els grafs de Stallings detalladament, establint els termes i notació que emprem arreu del treball. Concluïm el capítol amb el resultat fonamental que ens garanteix l'existència d'una bijecció pràctica entre autòmates (grafs) i subgrups de grups lliures, que ens permet analitzar els grups lliures mitjançant dits autòmates.

En el segon capítol presentem diverses aplicacions clàssiques dels grafs de Stallings, com el problema de la pertinença, el càlcul del rang, bases, o l'índex de subgrups, i notablement les interseccions de subgrups. Aquests resultats no només il·lustren el potencial dels grafs de Stallings com a eina algebraica, sinó també ens són útils en el nostre posterior estudi de les extensions algebraiques.

En el tercer capítol explorem les extensions, i en particular les extensions algebraiques, de grups lliures. Comencem establint uns quants resultats clàssics com el teorema de Takahasi; després revisitem una conjectura de Miasnikov, Ventura, i Weil[4], la qual va ser desmentida posteriorment per Parzanchevski i Puder[8] i per Kolodner[3], i la qual reformulem i demostrem, passant a proposar dos nous tipus d'extensió, les extensions sobrejectives (en anglès *onto extensions*) i les injectives (en anglès, *into extensions*). Finalment, estudiem el conjunt d'extensions algebraiques d'un subgrup com a conjunt parcialment ordenat, i presentem alguns resultats sobre la seva estructura (estructura de reticle, distributivitat, i ordres totals).

En aquest tercer capítol, són aportacions noves els Teoremes 3.5, 3.7, 3.8, 3.9, 3.10, i tot allò referent a les extensions sobrejectives i injectives.


## Mots clau


Grafs de Stallings, autòmates, grups lliures, extensions algebraiques, extensions sobrejectives.

# Abstract

In this thesis, we study algebraic extensions in free groups making an extensive use of Stallings graphs as a powerful tool, both as a means to approach problems abstractly as well as computationally.

In the first chapter, we construct Stallings graphs in detail, establishing the terms and notation we'll use throughout the paper. We conclude that chapter with the fundamental result that states that there is a useful bijection between automata (graphs) and subgroups of a free group, enabling us to study free groups using said automata.

In the second chapter we describe many classical algebraic applications of Stallings graphs regarding subgroups of free groups, such as the membership problem, computing rank, basis, or index of subgroups, and most notably intersections of subgroups. These results not only serve as a means to illustrate the power of Stallings graphs as an algebraic tool, but are also important for our later applications in the study of algebraic extensions.

In the third chapter we explore extensions, and in particular algebraic extensions, of free groups. We open by establishing some classical results, such as Takahasi's theorem; we then revisit a conjecture by Miasnikov, Ventura, and Weil[4], later disproved by Parzanchevski and Puder[8] and by Kolodner[3], and which we reformulate and prove, going further to propose new kinds of extensions (onto and into extensions). Finally, we study the set of algebraic extensions of a subgroup as a partially ordered set, and provide some results regarding its structure (lattice structure, distributivity, and total orders).

In this last chapter, Theorems 3.5, 3.7, 3.8, 3.9, 3.10, and everything regarding into and onto extensions are new findings in this subject.

## Keywords

Stallings graphs, automata, free group, algebraic extensions, onto extensions.

# Contents

# Chapter 1

# Stallings graphs

Stallings graphs, named after John R. Stallings who introduced them in his 1983 paper[10], are a very useful tool for the study of finitely generated subgroups of free groups by the means of directed graphs. These not only provide a more intuitive approach to certain problems in group theory (such as the Marshall Hall problem, or Howson's theorem), but also algorithmic tools to solve these and many others (see 2). In this chapter, we will show the construction of these graphs from the subgroup, and some core properties we will exploit throughout, most notably the bijection between finitely generated subgroups of a free group and Stallings graphs (as a type of $A$-automata).

We will introduce the concepts following the notation from lecture notes by Enric Ventura[11], which are also described in more compact form in the paper by Miasnikov, Ventura, and Weil[4].

## 1.1 $A$-Automata

Let $A = \{a_1, ..., a_r\}$ be a set of symbols, and let $F(A)$ be the free group on that set of symbols, of rank $r$. An ***A*-automaton**, $\mathcal{A}$, is a directed, labelled graph, with labels in $A \cup A^{-1}$ and a basepoint. More formally, $\mathcal{A} = (V, E, q_0)$, where $V$ is a set whose elements we call *vertices*, $E \subset V \times A \times V$ is a set of labelled oriented *edges*, and $q_0 \in V$ is a distinguished vertex we call the *basepoint*. We also require the underlying graph to be connected. Observe that $\mathcal{A}$ admits loops (edges connecting the same vertex), but no parallel edges with the same label.

We say an automaton is **involutive** if $(p, a, q) \in E \Leftrightarrow (q, a^{-1}, p) \in E$ for all edges in $E$. We will in general assume all automata to be involutive, even if we do not depict inverse edges.

**Definition 1.1.** *Let $\mathcal{A}$ be an $A$-automaton. We define a* ***path*** *(of length $r$) as a sequence $\gamma = \{v_i e_i v_{i+1}\}_{i=1,\ldots,r} \subseteq V$ such that for every $i$ we have an edge, $e_i$, from $v_i$ to $v_{i+1}$. The choice of the edge is important, since two vertices may be, connected by different edges. Observe these paths respect the orientation of the edges.*

*The* ***label of a path*** *$\gamma = \{v_i\}_{i=1,\ldots,r+1}$ is the word formed by concatenating the labels of the edges $(v_i, v_{i+1})$ in $\gamma$. This is a word on $A$, and it corresponds to an element in the free group $F(A)$. We say $\gamma$ is* ***reduced*** *if the next edge to $e_i = (v_i, a_i, v_{i+1})$ is not the its inverse, $e_{i+1} \neq (v_{i+1}, a_i^{-1}, v_i)$.*

We will denote $p \xrightarrow{u} q$ a path from $p$ to $q$ with label $u$.

**Lemma 1.1.** *Let $p \xrightarrow{u} q$ be a path. If $u$ is reduced, then the path is reduced. The converse is not true in general.*

**Definition 1.2.** *The* ***language*** *of an automaton $\mathcal{A}$ is the set of labels of paths $q_0 \xrightarrow{u} q_0$, and we write it as $L(\mathcal{A})$.*

Note this fixes the words we can construct in our automata to a certain class of interest, and our goal will be to provide the conditions so that the language of an automaton is the set of words in a particular subgroup. For that, we must also discuss *trimness* and *determinism*.

**Definition 1.3.** *An $A$-automaton is said to be* ***trim*** *if it has no vertices of degree 1, except maybe the basepoint. To trim the automaton is to recursively remove all vertices of degree 1 from $\mathcal{A}$.*

**Definition 1.4.** *An A-automaton is said to be **deterministic** if:*

$$(p, a, q), (p, a, q') \in E \Rightarrow q = q'$$

**Lemma 1.2.** *Let $\mathcal{A}$ be a deterministic A-automaton. Then:*

1. *If $\exists p \xrightarrow{u} q$ and $p \xrightarrow{u} q'$, then $p \xrightarrow{u} q = p \xrightarrow{u} q'$.*

2. *If $\exists p \xrightarrow{u} q$ and $p' \xrightarrow{u} q$, then $p \xrightarrow{u} q = p' \xrightarrow{u} q$.*

3. *If $\exists p \xrightarrow{uvv^{-1}w} q$, then $\exists p \xrightarrow{uw} q$.*

4. *A path $p \xrightarrow{u} q$ is reduced if, and only if, $u$ is reduced.*

Observe trimness removes vertices that don't change anything in our language (after reducing the words, that is), and determinism provides strong conditions on word reduction, so that the language of our automaton is quite close (and indeed equal) to a subgroup of the free group.

We can also define automata homomorphisms, which will have strong restrictions when our automata are deterministic.

**Definition 1.5.** *Let $\mathcal{A} = (V, E, q_0)$ and $\mathcal{A}' = (V', E', q_0')$ be two A-automata. We define a **homomorphism** $\mathcal{A} \to \mathcal{A}'$ as a map $\phi : V \to V'$ such that $\phi(q_0) = q_0'$ and $(p, a, q) \in E \Rightarrow (\phi(p), a, \phi(q)) \in E'$.*

**Theorem 1.1.** *Let $\mathcal{A} = (V, E, q_0)$ and $\mathcal{A}' = (V', E', q_0')$ be two A-automata, with $\mathcal{A}'$ deterministic. Then:*

$$L(\mathcal{A}) \subseteq L(\mathcal{A}') \Leftrightarrow \exists \phi : \mathcal{A} \to \mathcal{A}' \textit{ homomorphism}$$

*And in that case, $\phi$ is unique.*

***Proof:*** *Suppose, firstly, that there exists such a homomorphism $\phi : \mathcal{A} \to \mathcal{A}'$. Then, given $u \in L(\mathcal{A})$, we have a path $q_0 \xrightarrow{u} q_0$ in $\mathcal{A}$, which by $\phi$ becomes a path $q_0' \xrightarrow{u} q_0'$ with the same label, so $u \in L(\mathcal{A}')$. Therefore, $L(\mathcal{A}) \subseteq L(\mathcal{A}')$.*

*Now suppose $L(\mathcal{A}) \subseteq L(\mathcal{A}')$. Let $q \in V$ be a vertex of $\mathcal{A}$. Let $u$ be the label of a path from $q_0$ to $q$, and let $v$ be the label of a path from $q$ to $q_0$. Then, $uv \in L(\mathcal{A})$, so $uv \in L(\mathcal{A}')$, and thus there exists a closed path in $\mathcal{A}'$ with said label. Since $\mathcal{A}'$ is deterministic, there exists a unique $q'$ in $V'$ with paths labeled $u$ from $q'_0$ to $q'$ and $v$ from $q'$ to $q'_0$. We define $\phi(q) = q'$, which by the uniqueness of $q'$, is well defined, and by 1.2 taking a different path in $\mathcal{A}$, we would obtain the same (reduced) path in $\mathcal{A}'$.*

**Corollary 1.1.1.** *If $\mathcal{A}$ is deterministic, the only homomorphism $\mathcal{A} \to \mathcal{A}$ is the identity.*

**Corollary 1.1.2.** *Let $\mathcal{A}, \mathcal{A}'$ be two deterministic and trim $A$-automata. Then:*

$$L(\mathcal{A}) = L(\mathcal{A}') \Leftrightarrow \mathcal{A} \cong \mathcal{A}'.$$

Therefore, we can already see a good identification between deterministic and trim $A$-automata and their language, from this corollary, hinting that not only these are a useful representation of subgroups of the free group, but also that this representation is unique for every subgroup, so there is no ambiguity and we can derive stronger results. This will be shown rigorously in Theorem 1.3.

## 1.2 Construction of the Stallings graph

As we've seen in the previous section, a deterministic and trim $A$-automaton can be used to model a subgroup of the free group, since the closed paths from the basepoint behave as words in the subgroup (the language of the automaton). In this section, given a *finitely generated subgroup* of $F(A)$, the free group, we will construct an $A$-automaton whose language is precisely the words in the subgroup. We will call this automaton the **Stallings graph of the subgroup**, which is the central object in this paper.

Let $W \subset F(A)$ be a finite set of reduced words. We construct the **flower automaton** in the natural way, by taking a basepoint and attaching to it a closed path derived from each word (see Figure 1.1). Observe that the flower automaton is in general not deterministic at the basepoint, since different words may start with the same symbol, although it is trim, involutive, and the language of the flower automaton is precisely $\langle W \rangle$.

Because of these observations, we may be tempted to represent $H = \langle W \rangle \leq F(A)$ with said flower automaton, or to at least modify slightly the automaton so that it is also deterministic, without losing any of the previous properties. Indeed, that is what the Stallings graph of $H$ will be. To make the flower automaton deterministic, we will apply a series of operations called *foldings* which will achieve this objective. Figure 1.1 illustrates these folding operations.

**Definition 1.6.** *Let $\mathcal{A}$ be an $A$-automaton, and suppose $e = (p, a, q)$ and $e' = (p, a, q')$ are two different edges. Let $\mathcal{L} = \{\{f, f'\} : f = (p', b, q), f' = (p', b, q'), p' \neq p\}$ be the set of edges with the same label and origin going to $q$ and $q'$ respectively.*

*Consider the automaton $\mathcal{A}'$ obtained by identifying $q = q'$, so $e = e'$ and $f = f'$ for all $\{f, f'\} \in \mathcal{L}$. We define this construction of $\mathcal{A}'$ as a **Stallings folding**.*

*The number $l = |\mathcal{L}|$ is called the **loss** of the folding. A folding is called **critical** when $l > 0$.*

**Remark 1.1.** *Let $\mathcal{A}'$ be a folding of $\mathcal{A}$ with loss $l$. Then, $|V'| = |V| - 1$ and $|E'| = |E| - 1 - l$.*

Applying these foldings, we eventually obtain a deterministic automaton, and since we only apply a finite amount of them (due to the previous remark), this process finishes.

**Theorem 1.2.** *Let $\mathcal{A}'$ be a folding of $\mathcal{A}$. Then, there is a homomorphism $\phi : \mathcal{A} \to \mathcal{A}'$, and in fact $L(\mathcal{A}) = L(\mathcal{A}')$.*

***Proof:*** *Let $\mathcal{A}'$ be a folding of $\mathcal{A}$ from identifying $q = q'$ as in our definition. We define $\phi : \mathcal{A} \to \mathcal{A}'$ simply by sending every vertex to itself, except for $q'$ which will go to $q$ (as in the folding). This map is indeed a homomorphism, since by the nature of the folding it satisfies the definition.*

*Suppose we apply all the foldings until we obtain a deterministic automaton $\mathcal{A}'$, and we consider the homomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ resulting from those foldings (composition of them).*

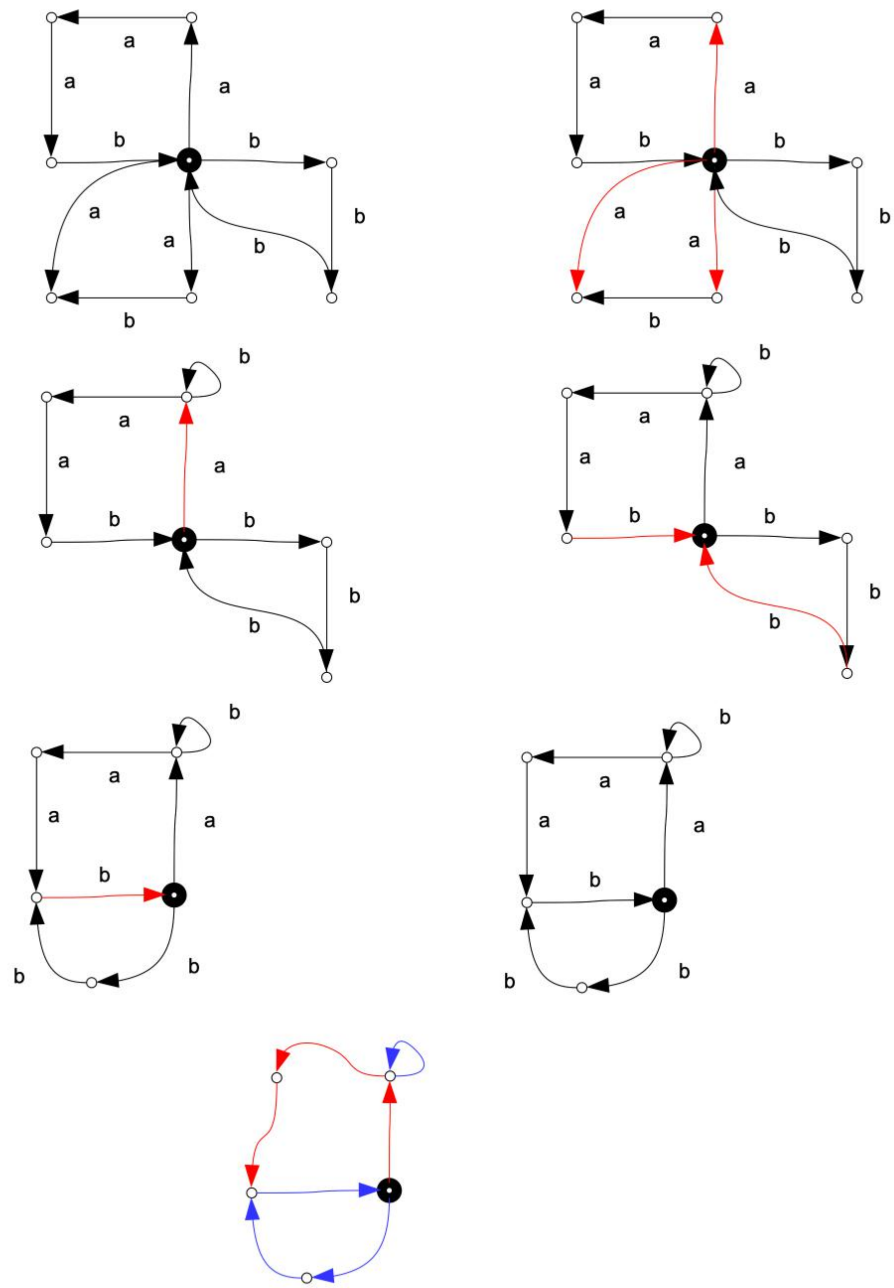


**Figure 1.1**: *Flower automaton of* $W = \{a^3b, b^3, aba^{-1}\}$ *and its foldings to obtain the Stallings graph of* $H = \langle W \rangle$*. Read the sequence of foldings top to bottom, right to left.*

*Then, by Theorem 1.1, $L(\mathcal{A}) \subseteq L(\mathcal{A}')$. We therefore must then show that $L(\mathcal{A}') \subseteq L(\mathcal{A})$. For that, returning to the case for a single identification, any labeled path in $\mathcal{A}'$ lifts to a labeled path in $\mathcal{A}$, trivially, so every closed path on the basepoint after the folding can be lifted to a closed path on the basepoint in the previous automaton and with the same label. Therefore, $L(\mathcal{A}') \subseteq L(\mathcal{A})$, and we conclude $L(\mathcal{A}') = L(\mathcal{A})$ after all foldings (and every folding individually).*

**Corollary 1.2.1.** *Let $W \subset F(A)$ be a finite set of reduced words, let $\mathcal{F}(W)$ be the flower automaton, and let $\mathcal{A}$ be the deterministic automaton obtained by iteratively folding the flower automaton. Then, $\mathcal{A}$ is deterministic and trim, and $L(\mathcal{A}) = \langle W \rangle$.*

This enables us to conclude that the automaton we obtained is a good representation of the finitely generated subgroup. This construction is useful because of several reasons: firstly, we have an effective manner to compute words in our subgroup, simply walking through paths starting and ending in our basepoint; secondly, determinism ensures those paths are well defined, so there is no ambiguity in the choice of the next step; finally, trimness ensures there are no redundant vertices in our graph. This suggests we may have some form of uniqueness, however, we may also think this construction is not necessarily unique, and the result may change depending on the steps we take in the foldings. Importantly, this is *not* the case, as we will see in the following section.

## 1.3 Bijection between Stallings graphs and finitely generated subgroups

So far, we've constructed an automaton from a finite set of words, $W$, deterministic and trim, whose language corresponds to the subgroup generated by those words in $F(A)$. In this section, we'll see not only that the result of the construction is independent from the steps taken in it, but also that it's independent of the particular set of generators we start with, so we have a one-to-one correspondence between finitely generated subgroups of $F(A)$ and deterministic and trim automata with labels in $A$. In other words, they're a well defined, and unique representation of them.

**Proposition 1.1.** *Let $\mathcal{A}$ be the result from the previous construction from a finite set $W \subset F(A)$, and let $H = \langle W \rangle \leq F(A)$. Then, $\mathcal{A}$ depends only on $H$, and will be called the* ***Stallings graph*** *of $H$, denoted $\Gamma(H)$.*

***Proof:*** *Suppose that, changing the order in the foldings from the flower automaton, we obtain different deterministic and trim automata, $\mathcal{A}_1$ and $\mathcal{A}_2$. Both of them satisfy $L(\mathcal{A}_1) = L(\mathcal{A}_2) = \langle W \rangle$, so by Corollary 1.1.2, $\mathcal{A}_1 \cong \mathcal{A}_2$. This also applies for different choices of the generators: in both cases, we'd obtain deterministic and trim automata whose languages are identical, therefore they must be isomorphic automata.*

**Theorem 1.3.** *We have a bijection:*

$$\begin{array}{lcl} \{H \leq_{f.g.} F(A)\} & \longleftrightarrow & \{\textit{deterministic and trim } A\textit{-automata}\} \\ H & \to & \Gamma(H) \\ L(\mathcal{A}) & \leftarrow & \mathcal{A} \end{array}$$

*which is also algorithmic and fast in both directions.*

***Proof:*** *From the previous result (1.1), we only must show that, given $\mathcal{A}$ a deterministic and trim $A$-automata, $L(\mathcal{A})$ is a finitely generated subgroup of $F(A)$.*

*Let $\mathcal{A} = (V, E, q_0)$ be a deterministic and trim $A$-automata, and let $T \subset \mathcal{A}$ be a spanning tree. Let $S \subseteq E$ be the set of edges of $\mathcal{A}$ outside $T$. We will show that $S$ determines a generating set of $L(\mathcal{A})$, so it is equal to a subgroup of $F(A)$.*

*For every edge $s \in S$, $T \cup \{s\}$ has exactly one cycle. Consider the path $h_s$ from and to $q_0$ formed by said cycle and the path from and to $q_0$ along $T$. More formally, if $s = (v, x_s, v')$, consider the path in $T \cup \{s\}$ $q_0 \xrightarrow{u} v \xrightarrow{x_s} v' \xrightarrow{u'} q_0$, and take $h_s = u x_s u' \in L(\mathcal{A})$. Consider an arbitrary $x \in L(\mathcal{A})$. Then, $x$ is the label of a closed path $q_0 \xrightarrow{x} q_0$. We can factor $x$ as a product of the $h_s$'s by following the labels of $x$ along $T$, until we find an edge $s \in S$. This will happen at least once, as $T$ is a tree and has no cycles , and we will take $h_s$ for that edge. Again, this $h_s$ (or more inclusively, $h_s^{-1}$ could be as well), will be well defined, as* *the path from the basepoint to each vertex of $s$ along $T$ is uniquely defined. Once again, we follow the label of $x$ along $T$ until we find another edge from $S$ and we repeat this process. This factors $x$ in terms of the elements $h_s^{\pm 1}$, and thus, these constitute a set of generators of $L(\mathcal{A})$.*

*We therefore conclude that $L(\mathcal{A})$ is a finitely generated subgroup of $F(A)$, since we found a finite set of generators. The uniqueness of this finitely generated subgroup (over trim and deterministic A-Automata) follows from the 1.1.2, so indeed, we have a bijection as we intended to show.*

**Corollary 1.3.1** (Nielsen's Theorem)**.** *Every finitely generated subgroup of a free group is free.*

***Proof:*** *Let $H \leq_{f.g.} F(A)$ be finitely generated, and let $\Gamma(H)$ be its Stallings graph, which by the previous theorem is uniquely defined. Then, we will see that the set of generators constructed earlier, $\{h_1, ...., h_s\}$, from a spanning tree $T$, is indeed a free basis, and there is no relations between them.*

*Let $w = h_{i_1}^{m_1} ... h_{i_d}^{m_d} \; : \; m_j \in \{-1, 1\}$ be a reduced word on the generators, where reduced means that, if $h_{i_j} = h_{i_{j+1}}$, then $m_j = m_{j+1}$.*

*Consider $h_{i_j}^{m_j} h_{i_{j+1}}^{m_{j+1}}$ a cancelling pair in $w$. We have two alternatives: either $h_{i_j} = h_{i_{j+1}}$ or not. In both cases, we will show that cancellations can only take place along the spanning tree. If they're different generators, observe they correspond to different edges, $e = (p, a, q)$ and $e' = (p', a', q')$. In particular, these edges are the only ones in each path outside the spanning tree, and since the Stallings graph is deterministic, they can't cancel out the labels $a, a'$ (two paths having the same tags must be equal, and having inverse tags implies they're the same path reversed). Similarly, if $h_{i_j} = h_{i_{j+1}}$, then $m_j = m_{j+1}$. We can write the path as*

$$q_0 \xrightarrow{u} p_1 \xrightarrow{v_1} p \xrightarrow{a} q \xrightarrow{v_2} p_1 \xrightarrow{u^{-1}} q_0$$

*Thus $h_{i_j} = uv_1av_2u^{-1}$, and the only cancellation possible is up to $u^{-1}u$. In other words, $h_{i_j}^k = u(v_1av_2)^k u^{-1}$, which is a general fact of any word in $H$. Therefore, the letter $a$ given* *by the edge outside the spanning tree cannot be cancelled.*

*In all, we conclude that, since $w$ is reduced, cancellations can only take place up to the letters given by the corresponding edges outside the spanning tree, thus the length of the word in terms of the ambient basis is at least the length of the word on the generators. Therefore, $w = 1$ implies that the $d = 0$, that is, the word on the generators is the empty word.*

**Remark 1.2.** *From the previous proof, we can observe that each spanning tree of the Stallings graph determines a basis of $H$. These basis may not be different for each spanning tree, but they will have the same number of elements (indeed, the rank of the group, as we will see in section 2.2).*

**Theorem 1.4** (Nielsen-Schreier)**.** *Every subgroup of a free group is free.*

The proof of this theorem for infinitely generated subgroups of $F(A)$ is identical to that of Nielsen's, although it's beyond our scope here, taking infinite automata. This realization mimics the historical finding of both theorems, hence the name Nielsen-Schreier.

# Chapter 2

# Algebraic applications of Stallings graphs

Having properly defined what the Stallings graph of a finitely generated subgroup is, the natural question would be *"What is it good for?"*. Indeed, quite a lot, as we're about to see. Many algebraic problems regarding subgroups of free groups can be algorithmically solved using Stallings graphs, and we present some of said applications in this chapter.

## 2.1 Membership problem

Let $H \leq_{f.g.} F(A)$ be a finitely generated subgroup. The **membership problem** on $H$ is the decisional problem of, given $g \in F(A)$, deciding whether $g \in H$ or not.

**Theorem 2.1.** *The membership problem for a finitely generated subgroup in $F(A)$ is solvable.*

***Proof (algorithm):*** *Let $H \leq_{f.g.} F(A)$ be a finitely generated subgroup, and let $g \in F(A)$. Let $\Gamma(H)$ be the Stallings graph of $H$. Expressing $g$ as a word $g = a_{i_1}^{m_1}...a_{i_k}^{m_{i_k}}$, we start at $q_0 = v_1$, the basepoint of $\Gamma(H)$: for every letter $a_{i_j}$ in the word $g$, if there is an edge outgoing from the current vertex, $v_j$, whose label is $a_{i_j}$, we set $v_{j+1}$ to be the endpoint of said edge, and continue the process. If not, the word is not in the subgroup. The word is in the subgroup if this algorithm runs through all the letters and the final vertex is the basepoint, $v_{k+1} = q_0$.*

*The algorithm is illustrated in 2.1.*

Moreover, if $g \in H$, we can give an algorithm that writes $g$ in terms of the original generators of $H$, which we illustrate firstly with an example:

**Example 2.1.** *Let $H = \langle a^3b, b^3, aba^{-1}\rangle \leq F(\{a,b\}) = F_2$ be our example subgroup (whose Stallings graph is in 1.1 and 2.2). Consider $g = ab^7a^2baba^{-1}b^{-3}$. We can check $g \in H$ as follows: We start at the basepoint, and we check if there is any edge with label $a$. After verifying that, we move to that vertex. We now search for an edge leaving this vertex with label $b$; it is indeed there, and it is in fact a loop, so we move to the same vertex in this step, as well as in the following 6 steps. After that, we follow the outgoing edge labelled $a$, moving to the next vertex, and we proceed in this fashion until we finally end up at the basepoint. Since our path ended precisely at the basepoint, $g \in H$ is verified.*

To write an element of the subgroup in terms of the original generators, we will *lift* the path up to the flower automaton, requiring the storage of the whole tower of foldings. Let $H = \langle W\rangle$ be our finitely generated group, and let $\mathcal{F}(W)$ be the flower automaton. Consider the chain of automata from the chain of foldings to obtain the Stallings graph:

$$\mathcal{F}(W) = \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots \to \mathcal{A}_n = \Gamma(H)$$

Let $g \in H$ be a word we know to be in $H$, and let $\gamma$ be the corresponding path $q_0 \xrightarrow{g} q_0$. To **lift** the path from $\Gamma(H)$ to $\mathcal{A}_{n-1}$, we undo the folding (of two edges) in $\mathcal{A}_{n-1}$. If the folded edge was not a part of $\gamma$, then the path remains unchanged and we move to the next folding. If the edge was in $\gamma$, we have two alternatives. Let $p \xrightarrow{a_i} q$ be the folded edge. Then, in $\mathcal{A}_{n-1}$, either we had $p \xrightarrow{a_i} q$ and $p \xrightarrow{a_i} q'$ (case 1), or we had $p \xrightarrow{a_i} q$ and $p' \xrightarrow{a_i} q$ (case 2).

In *case 2*, there are no changes in our path in $\mathcal{A}_{n-1}$. In *case 1*, we check the path in both alternatives as in the membership problem, and one of them will be valid. We change $\gamma$ into that new path and proceed with the next lift. This process continues until we reach $\mathcal{A}_0$, the flower automaton, in which case we have the product of the generators explicitly.

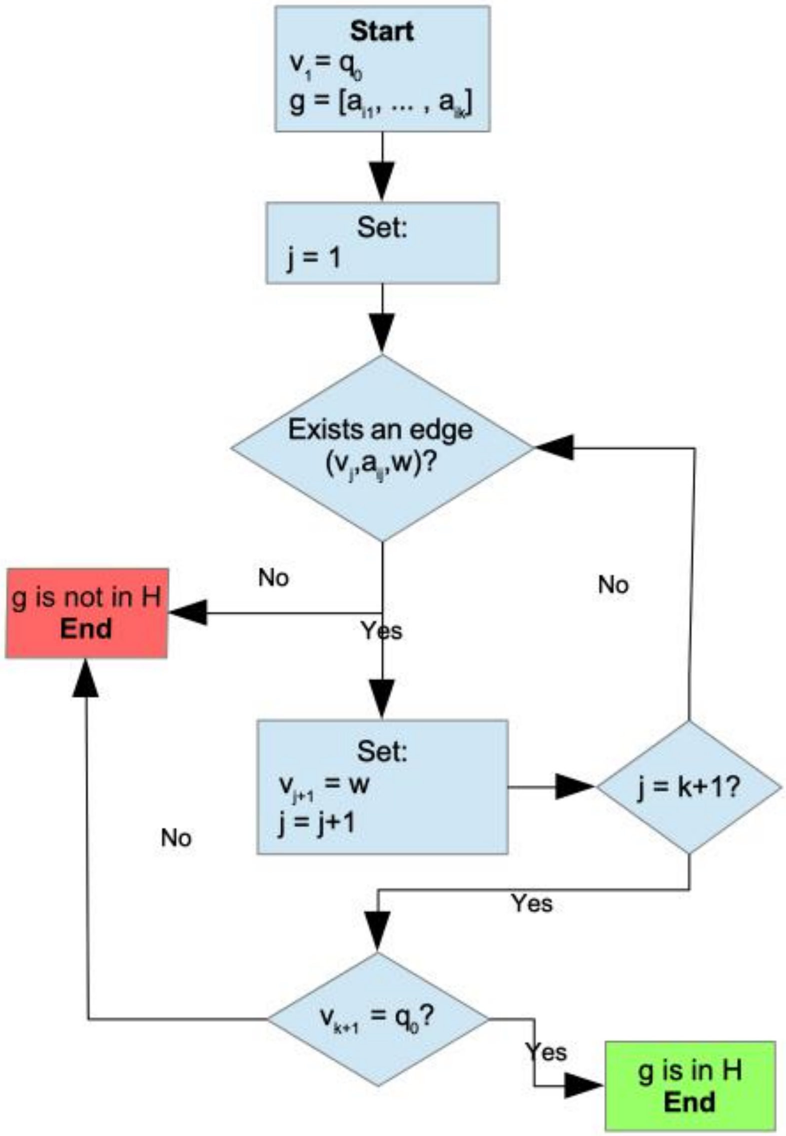


**Figure 2.1**: *Membership problem algorithm diagram 2.1*

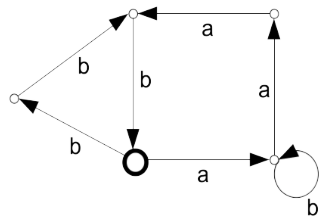


**Figure 2.2**: *Stallings graph of* $\langle a^3b, b^3, aba^{-1}\rangle$

## 2.2 Computation of rank and basis

Stallings graphs also provide an effective tool to compute the rank of a subgroup, and a basis for that subgroup (and in fact, a whole family of bases).

To calculate both the rank and a basis of $H \leq_{f.g.} F(A)$ we'll use *spanning trees* of the Stallings graph. Given $T \subset \Gamma(H)$ a spanning tree, and given $S \subset E$ the set of edges in the graph outside the spanning tree, we can construct a basis of $H$, with exactly one independent element for every edge in $S$. Every edge in $S$, $e \in S$, closes a cycle in the spanning tree, so we can consider *the* closed path in $T$ that goes through $e$, which will be a non trivial element of $H$. Doing the same for every element in $S$ gives a finite set of elements in $H$, $W \subset H$, which will be our basis.

**Proposition 2.1.** *Let* $H \leq_{f.g.} F(A)$*, let* $\Gamma(H)$ *be its Stallings graph, and let* $T \subset \Gamma(H)$ *be a spanning tree. Let* $W$ *be the set of elements of* $H$ *constructed from the previous algorithm. Then,* $W$ *is a basis for* $H$*.*

***Proof:*** *See proof of the Nielsen Theorem, 1.3.1.*

**Corollary 2.1.1.** *Let* $H \leq_{f.g.} F(A)$*, and let* $\Gamma(H)$ *be its Stallings graph, with* $n$ *vertices and* $m$ *edges. Then, the rank of* $H$ *is:*

$$r(H) = m - n + 1$$

## 2.3 Conjugation and normality of subgroups

Another algebraic problem is the **conjugacy problem**, that is, deciding when two given subgroups are conjugate. Once again, we can efficiently solve this problem using Stallings graphs, and by extension, the problem of normality.

**Definition 2.1.** *Recall that in a trim and deterministic automaton, the only vertex allowed to have degree 1 is the basepoint. This implies that, in such case, if the basepoint were an ordinary vertex, it would be trimmed out, and perhaps several vertices (and edges) would be as well after that. We call the subgraph that would be trimmed out if the basepoint were an ordinary vertex the* ***tail*** *of a Stallings graph (more generallly, the tail of a deterministic and trim automaton).*

**Remark 2.1.** *If we ignore inverse edges, the tail is a path, going from the basepoint to the first vertex with degree strctly greater than 1.*

*Observe as well that, if $q_0 \xrightarrow{u} q$ is such labeled path, then all words in the language of the automaton start with $u$ and end in $u^{-1}$. This trivially implies that, if we collapse the tail and make that vertex $q$ the basepoint of the automaton, the languages of the new and the previous automaton will be conjugate from each other by $u$, and therefore isomorphic.*

**Theorem 2.2.** *Let $H, K \leq_{f.g.} F(A)$ be two subgroups of the free group, and let $\Gamma(H), \Gamma(K)$ be the corresponding Stallings graphs. Then $H$ and $K$ are conjugate if, and only if, $\Gamma(H)$ and $\Gamma(K)$ coincide except for the basepoint and possible tail. Moreover, the conjugating word is given by the path from one basepoint to the other.*

***Proof:*** *Suppose first that the Stallings graphs coincide except for the basepoint (no tail). Let $q_0 \xrightarrow{u} q_0'$ be the path from the basepoint of $H$ to the basepoint of $K$. Clearly, we see every word $x \in K$ corresponds uniquely to a word $y \in H$ conjugated by $u$: $x = u^{-1}yu$. This enables us to conclude that $K = H^u$. If they coincide except for tails on one or both basepoints the proof still holds following the same procedure.*

*Suppose, now, that $K = H^u$. Considering $W = \{h_1, ..., h_n\}$ a basis of $H$, we have a basis of $K$ taking $W^u = \{u^{-1}h_1u, ..., u^{-1}h_nu\}$, so $H = \langle W \rangle$ and $K = \langle W^u \rangle$. Let $\mathcal{F}(W)$ and $\mathcal{F}(W^u)$ be the corresponding flower automata. Since all words in $\mathcal{F}(W^u)$ start and end in the same labeled edges, these will all fold into a tail, $q_0' \xrightarrow{u^{-1}} p$, and the rest of the automaton will be identical to $\mathcal{F}(W)$, where $p = q_0$, the basepoint of $\mathcal{F}(W)$. Every folding we apply to $\mathcal{F}(W)$ to obtain $\Gamma(H)$ also applies to $\mathcal{F}(W^u)$ in the same way, so at the end of that chain of foldings we obtain $\Gamma(H)$ with a tail from $\mathcal{F}(W^u)$, as we wanted to show.*

**Corollary 2.2.1.** *Let $H \leq_{f.g.} F(A) = F_r$. Then, $H \trianglelefteq F_r$ if, and only if, $\Gamma(H)$ is* ***complete*** *($2r$-regular), and* ***symmetric*** *(it is isomorphic to any change in the basepoint).*

This way, we have an efficient algorithm to determine normality and conjugacy of subgroups.

## 2.4 Finite index subgroups

A strong application of Stallings graphs is in determining when a finitely generated subgroup is of finite index, for which there is, again, an efficient algorithm. For that, we define a **complete** automaton with labels in $A$ ($|A| = r$) as one with an edge going in and an edge going out of any vertex and for every label, so all edges have degree $2r$. Given $\mathcal{A}$ a deterministic automaton, we define the **full Schreier graph** $\tilde{\mathcal{A}}$ as the previous automaton appending infinite trees to each edge to make it complete (trees isomoprhic to half of the Cayley graph of $F_r$). Note that the full Schreier graph of a deterministic automaton is also deterministic.

**Remark 2.2.** *Let $H \leq F(A)$ be finitely generated, and let $\Gamma(H)$ be its Stallings graph. Then, $L(\tilde{\Gamma}(H)) = L(\Gamma(H))$.*

A notation we'll use frequently is, given $H \leq F(A)$, both $H \backslash F(A)$ and $F(A)/H$. These are both quotient sets for the following equivalence relation:

$$g \sim g' \Leftrightarrow g' = hg \;:\; h \in H$$

The quotient to the right or to the left denotes if multiplication by an element in $H$ is from the right or from the left:

$$F(A)/H \text{ corresponds to } g \sim gh \ : \ h \in H$$

$$H\backslash F(A) \text{ corresponds to } g \sim hg \ : \ h \in H$$

**Theorem 2.3.** *Let $H \leq_{f.g.} F(A)$ be finitely generated. Let $T$ be a spanning tree of $\tilde{\Gamma}(H)$. For every vertex $p \in \tilde{V}$, we define $I_p = label_T(q_0, p)$. Then, we have a bijection:*

$$\phi : \tilde{V} \leftrightarrow H\backslash F(A)$$

$$p \mapsto H \cdot I_p$$

***Proof:*** *Since $\tilde{\Gamma}(H)$ is deterministic, it's clear that $I_p$ is unique for each $p \in \tilde{V}$, so we'll show that for any $g \in F(A)$ there exists $p \in \tilde{V}$ such that $g = I_p$. As the full Schreier graph is complete, it has an edge for every label (including inverses) at each vertex, so for any word $g \in F(A)$, we can find a vertex whose part $I_p = g$, simply following the path described by such word (for each letter, we follow the edge labelled with said letter). Moreover, the vertex we reach is unique, again, by determinism, thus we have a bijection, as we intended to prove.*

**Corollary 2.3.1.** *Let $H \leq_{f.g.} F(A)$. Then, $H$ is of finite index if, and only if, $\Gamma(H)$ is complete. In such case, $[F(A) : H] = |V|$.*

***Proof:*** *Suppose, firstly, that $\Gamma(H)$ is complete. Then, clearly, $\tilde{\Gamma}(H) = \Gamma(H)$, and by the previous theorem, we have a bijection between the vertices in $\Gamma(H)$ and the equivalence classes in $H\backslash F(A)$, that is, $H$ is of finite index.*

*Supposing, now, that $H$ is of finite index, we have a finite amount of equivalence classes (cosets) $H, Hg_1, ..., Hg_n$, which by the bijection in the previous theorem, corresponds to finitely many vertices in the full Schreier graph of $\Gamma(H)$. This implies that $\Gamma(H)$ is complete, as otherwise we'd be adding infinitely many vertices in incorporating at least one half of the Cayley graph of $F(A)$.*

*Observe that the number of cosets, the index of the subgroup, is the number of vertices in $\Gamma(H)$, again, by the bijection in the previous theorem.*

**Corollary 2.3.2** (Schreier index formula)**.** *Let $H \leq F(A)$ be of finite index. Then, $H$ is finitely generated and $r(H) - 1 = [F(A) : H](|A| - 1)$.*

**Corollary 2.3.3.** *If $1 \neq H \leq F(A)$ is normal and finitely generated, then it's of finite index.*

This way, not only we have an algorithm to determine whether $H$ is of finite index or not, but also to find a set of representatives of the cosets. The algorithm to determine whether $H$ is of finite index is simply applying the criterion in 2.3.1, checking every vertex in $\Gamma(H)$ and its degree, which requires $O(|V|)$ time. To find representatives of the cosets, we proceed, again, following the criterion laid out in 2.3: fixing some spanning tree, $T$, for every vertex in $\Gamma(H)$, we simply compute the label of the path $q_0 \to p$ along $T$, what we called $I_p$.

## 2.5 Intersection problem

Stallings graphs have had a very successful application in the computation of intersections of subgroups. Not only that, but we can use them to show that free groups satisfy the *Howson property*.

**Definition 2.2.** *Let $G$ be a group. We say $G$ satisfies the **Howson property** if the intersection of any two finitely generated subgroups of $G$ is also finitely generated.*

**Definition 2.3.** *Let $\mathcal{A}$ and $\mathcal{A}'$ be two $A$-automata. We define the **pullback** as the automaton:*

$$\mathcal{A} \times_A \mathcal{A}' = (V \times V', \tilde{E}, (q_0, q_0'))$$

*Where:*

$$\tilde{E} = \{((p, p'), a, (q, q')) \mid (p, a, q) \in E, (p', a, q') \in E'\} \subseteq (V \times V') \times (V \times V') \times A$$

**Example 2.2.** *Consider the subgroups of* $F_2$ *given in Figure 2.3,* $H = \langle bab, b^{-2}a^2b, b^{-1}aba^{-1}b\rangle$ *and* $K = \langle bab, ba^{-1}b^2, ba^{-2}b\rangle$*. Their pullback is the larger automaton in the picture, with 25 vertices. We can observe that the pullback is deterministic, which is a general fact, and it's not trim, which also holds in general. Moreover, the pullback is connected, which in general isn't true either.*

*As we will see in Theorem 2.4, the pullback's language coincides precisely with the intersection of the subgroups. In this case,* $H \cap K = \langle bab, ba^{-1}b^2\rangle$*.*

**Remark 2.3.** *The pullback of deterministic automata is also deterministic, but it is not necessarily connected nor trim. Figure 2.3 is a good illustration of this.*

**Theorem 2.4.** *Let* $H, K \leq_{f.g.} F(A)$*. Then,* $\Gamma(H \cap K)$ *is the trimming of the connected component of* $\Gamma(H) \times_A \Gamma(K)$ *containing the basepoint.*

***Proof:*** *Let* $\mathcal{A}$ *be the connected component of* $\Gamma(H) \times_A \Gamma(K)$ *containing the basepoint* $(q_0, q_0')$*. Let* $u \in H \cap K$*. Then, there exist paths* $q_0 \xrightarrow{u} q_0$ *and* $q_0' \xrightarrow{u} q_0'$ *in each Stallings graph, which we can write as:*

$$q_0 a_0 p_1 a_1 p_2 \ldots p_n a_n q_0$$

$$q_0' a_0 p_1' a_1 p_2' \ldots p_n' a_n q_0'$$

*Therefore, as* $(p_i, a_i, p_{i+1}) \in E$ *and* $(p_i', a_i, p_{i+1}') \in E'$*, we have* $((p_i, p_i'), a_i, (p_{i+1}, p_{i+1}')) \in \tilde{E}$*, so we have a path:*

$$(q_0, q_0')a_0(p_1, p_1')\ldots(p_n, p_n')a_n(q_0, q_0')$$

*That is,* $(q_0, q_0') \xrightarrow{u} (q_0, q_0')$*, thus* $H \cap K \subseteq L(\mathcal{A})$*.*

*Let us now take a closed path* $(q_0, q_0') \xrightarrow{u} (q_0, q_0')$*. Just like before, we can write this path as:*

$$(q_0, q_0')a_0(p_1, p_1')\ldots(p_n, p_n')a_n(q_0, q_0')$$

*Which implies we have edges* $((p_i, p_i'), a_i, (p_{i+1}, p_{i+1}'))$ *in* $\tilde{E}$*, thus we have edges* $(p_i, a_i, p_{i+1})$ *and* $(p_i', a_i, p_{i+1}')$ *in each Stallings graph, concluding we have paths*

$$q_0 a_0 p_1 a_1 p_2 \ldots p_n a_n q_0$$

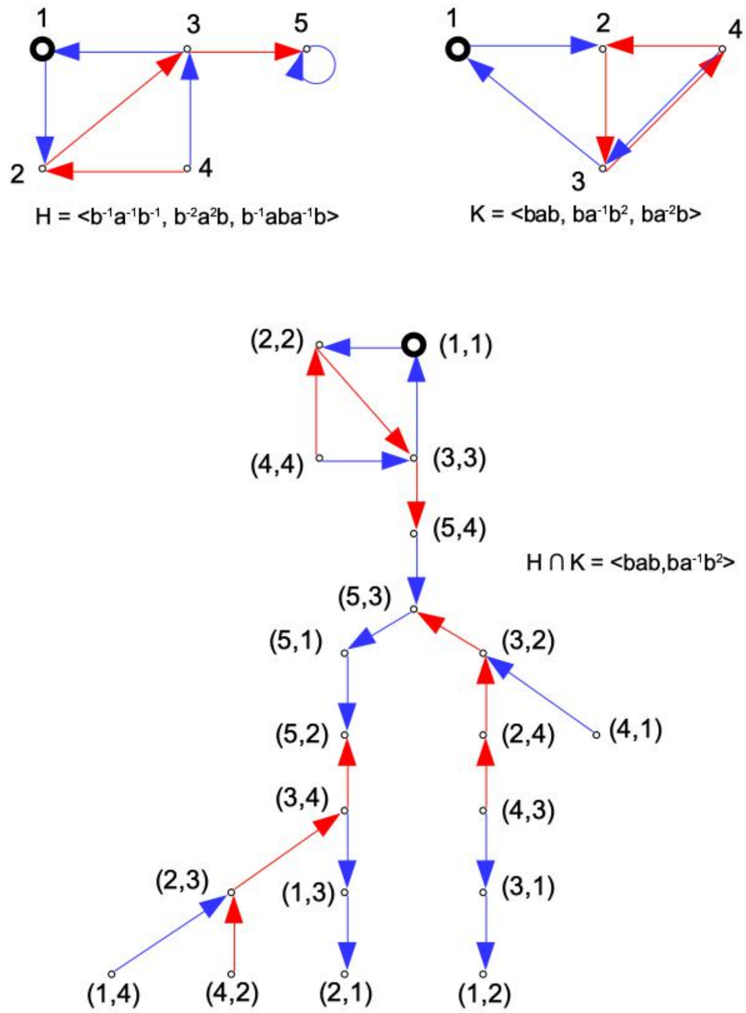


**Figure 2.3**: *Stallings graphs of* $H = \langle bab, b^{-2}a^2b, b^{-1}aba^{-1}b\rangle$, $K = \langle bab, ba^{-1}b^2, ba^{-2}b\rangle$, *and their pullback, which by Theorem 2.4 is the Stallings graph of the intersection (before trimming).*

$$q_0' a_0 p_1' a_1 p_2' ... p_n' a_n q_0'$$

*That is,* $q_0 \xrightarrow{u} q_0$ *and* $q_0' \xrightarrow{u} q_0'$*, thus* $L(\mathcal{A}) \subseteq H \cap K$*, finally finding that* $L(\mathcal{A}) = H \cap K$*, as we intended to show.*

**Corollary 2.4.1.** *Free groups satisfy the Howson property.*

With the previous construction, a natural question to ask is regarding the rank of the intersection of the subgroups. Indeed, an upper bound can be provided, as the following theorem shows.

**Definition 2.4.** *Let* $H \leq_{f.g.} F(A)$*. We define the* ***reduced rank*** *as* $\tilde{r}(H) = max\{0, r(H) - 1\}$*. This is precisely* $r(H) - 1$ *for any non-trivial subgroup.*

**Lemma 2.1** (Handshaking lemma)**.** *Let* $G = (V, E)$ *be a graph, and let* $d(p)$ *be the degree of the vertex* $p$*. Let* $m = |E|$ *be the number of edges in* $G$*. Then:*

$$\sum_{p \in V} d(p) = 2m$$

**Theorem 2.5** (Hanna and Walter Neumann[6 7])**.** *Let* $H, K \leq_{f.g.} F(A)$*. Then,* $\tilde{r}(H \cap K) \leq 2\tilde{r}(H)\tilde{r}(K)$*. Moreover:*

$$\sum_{HgK \in H \backslash F(A)/K} \tilde{r}(H^g \cap K) \leq 2\tilde{r}(H)\tilde{r}(K)$$

***Proof:*** *Suppose* $H \cap K$ *is non-trivial, so neither* $H$ *or* $K$ *is trivial. We can see that, by the Handshaking Lemma 2.1 and 2.1.1, for every vertex in* $\Gamma(H)$*:*

$$\sum_{p \in V(\Gamma(H))} (d(p) - 2) = 2\tilde{r}(H)$$

*Consider the projection maps* $\pi_H : V(\Gamma(H \cap K)) \to V(\Gamma(H))$ *and* $\pi_K : V(\Gamma(H \cap K)) \to V(\Gamma(K))$ *defined as:*

$$\pi_H((p, p')) = p \ ; \ \pi_K((p, p')) = p'$$

*Trivially,* $\pi = (\pi_H, \pi_K) : V(\Gamma(H \cap K)) \to V(\Gamma(H)) \times V(\Gamma(K))$ *is injective (it is in fact the identity), and for every* $(p,p') \in V(\Gamma(H \cap K))$*, it's clear that* $2 \leq d((p,p')) \leq d(\pi_H((p,p'))), d(\pi_K((p,p')))$*, thus:*

$$0 \leq d((p,p')) - 2 \leq min\{d(\pi_H((p,p'))), d(\pi_K((p,p')))\} - 2 \leq$$

$$\leq (d(\pi_H((p,p'))) - 2)(d(\pi_K((p,p'))) - 2)$$

*Therefore, we can conclude that:*

$$\begin{array}{rcl} 2\tilde{r}(H \cap K) & = & \sum_{p \in V(\Gamma(H \cap K))} d(p) - 2 \\ & \leq & \sum_{(p,p') \in V(\Gamma(H \cap K))} (d(\pi_H((p,p'))) - 2)(d(\pi_K((p,p'))) - 2) \\ & \leq & \sum_{(p,p') \in V(\Gamma(H)) \times V(\Gamma(K))} (d(p) - 2)(d(p') - 2) \\ & = & (\sum_{p \in V(\Gamma(H))} (d(p) - 2))(\sum_{p' \in V(\Gamma(K))} (d(p') - 2)) \\ & = & 4\tilde{r}(H)\tilde{r}(K) \end{array}$$

*Thus,* $\tilde{r}(H \cap K) \leq 2\tilde{r}(H)\tilde{r}(K)$.

Hanna Neumann conjectured that the factor 2 in the previous theorem could be removed, though she never succeeded in showing it. However, the conjecture was finally proven in 2011 by Joel Friedman[2] and Igor Mineyev[5] working independently.

**Theorem 2.6** (Hanna Neumann Conjecture)**.** *Let* $H, k \leq_{f.g.} F(A)$*. Then,* $\tilde{r}(H \cap K) \leq \tilde{r}(H)\tilde{r}(K)$*. Moreover:*

$$\sum_{HgK \in H \backslash F(A)/K} \tilde{r}(H^g \cap K) \leq \tilde{r}(H)\tilde{r}(K)$$

***Proof:*** *The complete proof can be found in the paper by Dicks[1].*

# Chapter 3

# Extensions in a free group

Given subgroups $H \leq K \leq F(A)$, we say $K$ is an **extension** of $H$. In this chapter we will study extensions in free groups, with the goal of showing Takahasi's Theorem. A **free factor** of $H \leq F(A)$ is a subgroup $H' \leq H$ such that $H = H' * \tilde{H}$ (free product) for some $\tilde{H}$, that is, there is a basis of $H'$ that extends to a basis of $H$, and we will write it as $H' \leq_{f.f.} H$.

**Lemma 3.1.** *Let $H, K \leq_{f.g.} F(A)$. Then:*

$$H \leq K \Leftrightarrow \exists \phi_{H,K} : \Gamma(H) \longrightarrow \Gamma(K) \text{ homomorphism}$$

*If it exists, then $\phi_{H,K}$ is unique.*

***Proof:*** *Direct consequence of Proposition 1.1.*

**Lemma 3.2.** *If $H <_{f.f.} K$, then $rk(H) < rk(K)$.*

***Proof:*** *Let $H <_{f.f.} K$. Then, a basis of $H$ extends to a basis of $K$, so it has strictly fewer elements, so smaller rank.*

**Proposition 3.1.** *Let $H \leq K \leq F(A)$ be an extension of finitely generated subgroups, and let $\phi_{H,K}$ be the unique homomorphism from Lemma 3.1. Then, if $\phi_{H,K}$ is injective, then $H$ is a free factor of $K$. The converse is not true.*

***Proof:*** *Since $\phi_{H,K}$ is injective, $\Gamma(H)$ is a subgraph of $\Gamma(K)$ with the same basepoint, so we can consider vertices and edges in $\Gamma(H)$ to be also in $\Gamma(K)$. Let $T$ be a spanning tree of $\Gamma(H)$, and let $\tilde{T}$ be a spanning tree of $\Gamma(K)$ containing $T$. Let $S$ be the set of edges in*

*$\Gamma(H)$ that aren't in $T$. As we have seen before, these determine a basis of $H$, $H_S$. Let $\tilde{S}$ be the set of edges of $\Gamma(K)$ that aren't in $\tilde{T}$. Then, $S \subset \tilde{S}$, as if an edge in $S$ were in $\tilde{T}$, it would close a cycle in $T$, and thus a cycle in $\tilde{T}$, as $T \subset \tilde{T}$. Therefore, if $K_{\tilde{S}}$ is the basis of $K$ given by $\tilde{S}$, we can clearly see that $H_S \subset K_{\tilde{S}}$, so a basis of $H$ is entirely contained in a basis of $K$, so we conclude $H$ is a free factor of $K$.*

*As a counterexample to the reverse implication, note that $\langle ab \rangle$ is clearly a free factor of $\langle a, b \rangle$, but their Stallings graphs have two vertices and one vertex respectively, so the homomorphism can't be injective.*

**Definition 3.1.** *Let $H \leq K \leq F(A)$ be an extension of finitely generated subgroups, and let $\phi_{H,K}$ be the unique homomorphism from Lemma 3.1. If $\phi_{H,K}$ is surjective, then we say $K$ is an **A-principal overgroup** of $H$. The set of A-principal overgroups of $H \leq F(A)$ is called the **A-fringe** of $H$, and we write it as $\mathcal{O}_A(H)$.*

**Remark 3.1.** *We define surjectivity of the homomorphisms on both vertices and edges, not just on the vertices, as this has a stronger topological interpretation. As a corollary, if $\mathcal{A}$ is a strict subgraph of $\mathcal{A}'$, then the injective homomorphism between them is not surjective (even if they have the same number of vertices).*

**Remark 3.2.** *It must be also observed that the bijection between subgroups and Stallings graphs is not monotonous with subgraphs: $H \leq K$ does not imply that $\Gamma(H)$ is a subgraph of $\Gamma(K)$ or vice versa. In general, it could be either or neither of the two.*

The $A$-fringe of $H \leq_{f.g.} F(A)$ is finite, and it can be computed. Indeed, if we consider all identifications of vertices in $\Gamma(H)$ (all equivalence relations in $V$), of which there are finitely many, we obtain a set of automata. If we trim and fold said automata so they're deterministic and trim, we get all possible automata with surjective homomorphisms $\Gamma(H) \to \mathcal{A}$. The fact that this covers all possible surjective homomorphisms is trivial from the construction: every surjective homomorphism must be the result of grouping the vertices in some partition, which gives an equivalence relation in our set of vertices.

**Lemma 3.3** (Intersection of free factors)**.** [4] *Let* $H, K, L, \{H_i\}_{i\in I}, \{K_i\}_{i\in I}$ *be subgroups of a free group. Then:*

- $H \leq_{f.f.} L \leq_{f.f.} K \Rightarrow H \leq_{f.f.} K$
- *If* $H_i \leq_{f.f.} K_i : \forall i \in I$*, then* $\bigcap_{i\in I} H_i \leq_{f.f.} \bigcap_{i\in I} K_i$

**Example 3.1.** *Let* $H = \langle a^3, abab^{-1}a^{-1}\rangle$ *be a subgroup of the free group, whose Stallings graph is in Figure 3.1. Then, we have 15 different partitions of* $\{1, 2, 3, 4\}$*, each giving an identification of the vertices in* $\Gamma(H)$*. However, if we perform foldings on said identifications to ensure our automata are deterministic, we find exactly 3 different deterministic and trim automata, which correspond to the principal overgroups of* $H$*:* $H, H_1 = \langle a, bab^{-1}\rangle$*, and* $H_2 = F_2$*.*

**Theorem 3.1** (Takahasi[4])**.** *Let* $H \leq_{f.g.} F(A)$*. Then, there exists a finite number of extensions of* $H$*,* $H \leq H_0, ..., H_n$ *(where* $H_0 = H$*), such that any other extension* $H \leq K$ *is a free multiple of one of the* $H_i$*'s.*

***Proof:*** *We'll show that the A-fringe satisfies the conditions of the theorem. Let* $H \leq K$ *be an extension of* $H$*. Let* $\phi_{H,K} : \Gamma(H) \to \Gamma(K)$ *be the homomorphism from 3.1. Observe that the image of this homomorphism is another deterministic and trim automaton, which is a quotient of* $\Gamma(H)$*. Let* $L_{H,K} \leq_{f.g.} F(A)$ *be such that* $\Gamma(L_{H,K}) = \phi_{H,K}(\Gamma(H))$*. Then, it's clear that* $L_{H,K} \in \mathcal{O}_A(H)$ *is in the A-fringe, as its Stallings graph is constructed as a quotient of* $\Gamma(H)$*, and we have an injective homomorphism* $\Gamma(L_{H,K}) \to \Gamma(K)$*, so by 3.1,* $L_{H,K}$ *is a free factor of* $K$*.*

*Therefore, for every extension of* $H$*,* $K$*, there is an element in the A-fringe that is a free factor of* $K$*. Since, as we saw earlier (3), the A-fringe is finite, we have a finite set of extensions of* $H$ *satisfying the conditions of the theorem.*

**Definition 3.2.** *Let* $H \leq_{f.g.} F(A)$*, and let* $\mathcal{T}$ *be a set of extensions of* $H$ *satisfying the conditions of Takahasi's theorem 3.1. We say* $\mathcal{T}$ *is a* ***Takahasi family****.*

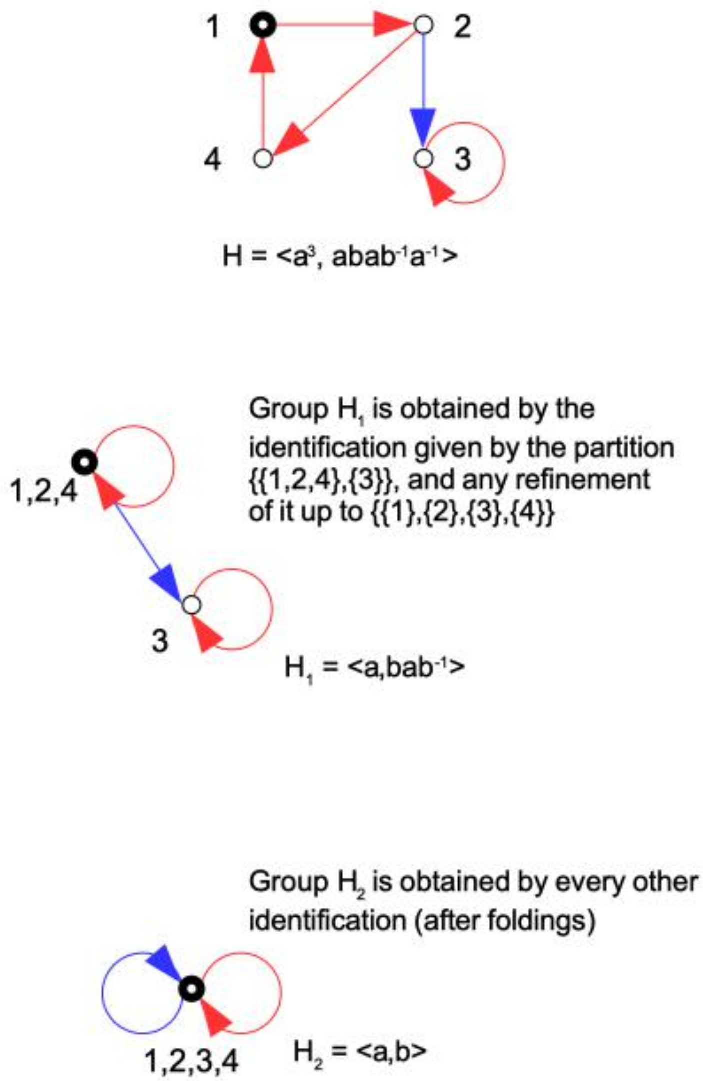


**Figure 3.1**: *Stallings graph of* $H = \langle a^3, abab^{-1}a^{-1} \rangle$, *and its principal overgroups*

Takahasi's Theorem gives powerful insight on the structure of subgroups of free groups, and on their extensions. However, the set we obtained, the $A$-fringe, depends on the ambient basis of the free group we choose, $A$. A natural question follows: does such a finite set of extensions necessarily depend on the ambient basis? Or can it be constructed so that it only depends on the subgroup in question? Is there a minimal Takahasi family?

As we'll see in the following subsection, we can construct such a finite family of subgroups, independent from $A$, which will be the set of *algebraic extensions* of $H$, and which will, moreover be an inclusion minimal Takahasi family.

## 3.1 Algebraic extensions

A special kind of group extensions are those called *algebraic extensions*. Let $H \leq K \leq F(A)$ be a group extension. We say $x \in K$ is **algebraic** over $H$ if for every free factor of $K$ containing $H$, $H \leq L \leq_{f.f.} K$, then $x \in L$. Otherwise, we say $x$ is **transcendent**.

**Lemma 3.4.** *Let $H \leq K \leq F(A)$ be a group extension.*

- *If $x, y \in K$ are algebraic over $H$, then $xy, x^{-1}, y^{-1}$ are algebraic over $H$ as well.*
- *If $x \in K$ is transcendent over $H$, then $x^{-1}$ is transcendent over $H$ as well.*
- *If $x, y \in K$ are transcendent over $H$, then $xy$ is not transcendent over $H$ in general.*

***Proof:*** *Let $x, y \in K$ be algebraic, so for every free factor $H \leq L \leq_{f.f.} K$, we have $x, y \in L$. Since $L$ is a group, we deduce: $x^{-1}, y^{-1}, xy \in L$. This implies the second statement: if $x$ is transcendent, there exists $H \leq L \leq_{f.f.} K$ such that $x \notin L$, but if $x^{-1}$ were algebraic, $x^{-1} \in L$, which would imply $x \in L$, so we'd have contradiction.*

**Definition 3.3.** *We say an **extension** $H \leq K$ is **algebraic** if all elements in $K$ are algebraic over $H$. We will write it as $H \leq_{alg} K$.*

Although it would be tempting to deduce that these algebraic extensions are related to algebraic extensions of fields, given some "similarity" in their construction and their naming, we must note they are fundamentally **unrelated**. Some similarities do hold, such as that the composition of algebraic extensions is again an algebraic extension (3.5), but many key others don't, such as intermediate extensions of an algebraic extension also being algebraic, which holds for fields, but not here.

We have equivalent definitions for algebraic extensions, as can be seen in the following proposition.

**Proposition 3.2** (Characterisation of algebraic extensions)**.** *Let $H \leq K \leq F(A)$ be a group extension. The following are equivalent:*

1. *$H \leq K$ is algebraic.*

2. *$H$ is not contained in any proper free factor of $K$.*

3. *$\exists X \subseteq K$ such that $\langle X \cup H \rangle = K$ and $x$ is algebraic over $H \ : \ \forall x \in X$.*

***Proof:*** *Suppose $H \leq K$ is algebraic, and suppose $H$ is contained in a proper free factor of $K$, $H \leq L <_{f.f.} K$. Then, every $x \in K \setminus L$ is transcendent, and thus $K$ can't be algebraic, thus we have contradiction and $1 \Rightarrow 2$.*

*Suppose $H$ is not contained in any proper free factor of $K$. Then, $H \leq L \leq_{f.f.} K \Rightarrow L = K$, so every $x \in K$ is contained in every extension of $H$ that is a free factor of $K$, so $H \leq K$ is an algebraic extension. Therefore, $2 \Rightarrow 1$*

*Let $H \leq K$ be algebraic. Then, given $X$ a set of generators of $K$, all elements in $X$ are algebraic, and $K = \langle X \cup H \rangle$, so $1 \Rightarrow 3$.*

*Finally, suppose $\exists X \subseteq K$ such that $\langle X \cup H \rangle = K$ and $x$ is algebraic over $H \ : \ \forall x \in X$. Since all elements in $X \cup H$ are algebraic, by Lemma 3.4, we deduce all elements in $K$ are algebraic, and thus $H \leq K$ is an algebraic extension.*

**Proposition 3.3** (Characterisation of free factors)**.** *Let $H \leq K \leq F(A)$ be a group extension. Then, $H$ is a free factor of $K$ if, and only if, $H \leq K$ is* ***purely transcendental****, that is, all elements in $K \setminus H$ are transcendental over $H$.*

***Proof:*** *Suppose the extension is not purely transcendental, and let $x \in K \setminus H$ be algebraic over $H$. Then, for every free factor of $K$ containing $H$, $H \leq L \leq_{f.f.} K$, $x \in L$. In particular, $H$ is not a free factor, as that would imply we have a free factor containing $H$, itself, but not containing $x$.*

*Suppose now that the extension is purely transcendental. Then, since all elements in $K \setminus H$ are transcendental, there exists $H \leq L_x \leq_{f.f.} K$ such that $x \notin L_x$ for every $x \in K \setminus H$. Therefore, $\bigcap_{x \in K \setminus H} L_x = H$, and by Lemma 3.3, $H$ is a free factor of $K$.*

Algebraic extensions are of interest in this context of Stallings graphs because they are a computable set with good properties. More specifically, the set of algebraic extensions is a Takahasi family, which is minimal under inclusion and, unlike the Takahasi families we constructed previously, does not depend on the alphabet of the ambient free group (its basis). This result (3.2) provides further insight on the structure of subgroups of the free group and its extensions.

**Theorem 3.2.** *The set of algebraic extensions of $H \leq F(A)$, $AE(H)$, is finite, does not depend on $A$, and is the inclusion-minimal Takahasi family for $H$.*

***Proof*[4]*:*** *By the proof of Takahasi's Theorem (3.1), the $A$-fringe of $H$ is a Takahasi family. Let $H \leq_{alg} K$ be an algebraic extension. Then, $\exists H' \in \mathcal{O}_A(H) | H \leq H' \leq_{f.f.} K$. By the characterisation of algebraic extensions, 3.2, this implies $H' = K$, so $K \in AE(H)$, and $AE(H) \subseteq \mathcal{O}_A(H)$ for any $A$. Finiteness of $AE(H)$ thus follows immediately.*

*To see that $AE(H)$ is the inclusion-minimal Takahasi family, let $\mathcal{T}$ be a Takahasi family of $H$. Then, for any $K \in AE(H)$, there exists $H' \in \mathcal{T} | H \leq H' \leq_{f.f.} K$ as before, which again implies $K = H'$. Therefore, $AE(H) \subseteq \mathcal{T}$ for any Takahasi family.*

**Corollary 3.2.1.** *Let $H \leq F(A)$. The set $AE(H)$ is computable.*[4]

For the computation of $AE(H)$, we must compute the $A$-fringe of $H$ for some ambient basis $A$, and remove free products in it, that is, if $K, K' \in \mathcal{O}_A(H)$ and $K \leq_{f.f.} K'$, then $K' \notin AE(H)$. Some algorithms for that purpose have been proposed, such as the one by Silva and Weil[9], which uses only graphical tools, and is based on the following Theorem 3.3.

**Definition 3.4.** *Let $\Gamma$ be an $A$-automaton, $w$ a word on $A$, and let $p, q$ be vertices in $\Gamma$. An* ***expansion*** *of $\Gamma$ given by $p, q$ and $w$ is a new automaton obtained by adding to $\Gamma$ a new path labelled $w$ from $p$ to $q$:*

$$p \overset{w_1}{\to} p_1 \overset{w_2}{\to} \cdots \overset{w_l}{\to} q.$$

*If $w$ is reduced, we say it's a* ***reduced expansion****.*

**Theorem 3.3** (Silva-Weil[9])**.** *Let $H, K \leq_{f.g.} F(A)$ be finitely generated subgroups such that $d = rk(K) - rk(H) > 0$. Then, $H \leq_{f.f.} K$ if, and only if, $\Gamma_A(H)$ can be transformed into a subgraph of $\Gamma_A(K)$ by a sequence of $d' \leq d$ vertex identifications (pairwise), and the subsequent necessary foldings to make the automaton deterministic. Moreover, the transformation of $\Gamma_A(H)$ previously described is a subgraph of $\Gamma_A(K)$ by exactly $d - d'$ reduced expansions.*

**Proposition 3.4** (Composition of free factors)**.** *Let $H \leq K_i \leq K \leq F(A)$ be a group extension for $i \in \{1, 2\}$. Then:*

1. *If $H \leq_{f.f.} K_1$ and $K_1 \leq_{f.f.} K$, then $H \leq_{f.f.} K$.*
2. *If $H \leq_{f.f.} K$, then $H \leq_{f.f.} K_1$. However, $K_1 \leq K$ need not be purely transcendental.*
3. *If $H \leq_{f.f.} K_1$ and $H \leq_{f.f.} K_2$, then $H \leq_{f.f.} (K_1 \cap K_2)$. However, $H \leq \langle K_1 \cup K_2 \rangle$ need not be purely transcendental.*

***Proof[4]:*** *Let $H \leq_{f.f.} K_1 \leq_{f.f.} K$. Then, $K_1 = H * H'$ and $K = K_1 * K_1' = H * H' * K_1' = H * K'$ thus $H \leq_{f.f.} K$.*

*For statement number 2, we observe that, by the characterisation of free factors, all elements in $K \setminus H$ are transcendental, and in particular, all elements in $K_1 \setminus H$ are as well. The same reasoning applies to the third statement.*

*For a counterexample of the second part of the third statement, consider $H = \langle aba^{-1}b^{-1} \rangle$, $K_1 = \langle aba^{-1}, b \rangle$, $K_2 = \langle bab^{-1}, a \rangle$. Then, $H$ is a free factor of both $K_i$'s, but $\langle K_1 \cup K_2 \rangle = \langle a, b \rangle$, and $H \leq_{alg} \langle a, b \rangle$.*

**Proposition 3.5** (Composition of algebraic extensions)**.** *Let $H \leq K_i \leq K \leq F(A)$ be a group extension for $i \in \{1, 2\}$. Then:*

1. *If $H \leq_{alg} K_1$ and $K_1 \leq_{alg} K$, then $H \leq_{alg} K$.*
2. *If $H \leq_{alg} K$, then $K_1 \leq_{alg} K$. However, $H \leq K_1$ need not be an algebraic extension.*
3. *If $H \leq_{alg} K_1$ and $H \leq_{alg} K_2$, then $H \leq_{alg} \langle K_1 \cup K_2 \rangle$. However, $H \leq (K_1 \cap K_2)$ need not be an algebraic extension.*

***Proof*[4]*:*** *For the first statement, suppose $H \leq_{alg} K_1 \leq_{alg} K$, and suppose $H$ is contained in a proper free factor of $K$, $L \leq_{f.f.} K$. Then, by Proposition 3.4, $H \leq (L \cap K_1) \leq_{f.f.} K_1$, so, as $H \leq_{alg} K_1$, we have $K_1 \leq L$. This, in turn, implies $L = K$, and thus $H \leq_{alg} K$.*

*For the second statement, let $H \leq_{alg} K$ and let $K_1$ be an intermediate extension. Then, $K_1$ cannot be contained in a proper free factor of $K$, as that would also include $H$, thus $K_1 \leq_{alg} K$.*

*Finally, suppose $H \leq \langle K_1 \cup K_2 \rangle$ is not algebraic, and let $H \leq L \leq_{f.f.} \langle K_1 \cup K_2 \rangle$ be a proper free factor. By 3.4, $(L \cap K_i) \leq_{f.f.} K_i$, so $H$ would be contained in proper free factors of $K_1, K_2$, contradicting that they're algebraic extensions.*

*As a counterexample for the second part of the third statement, consider the subgroup $H = \langle aba^{-1}b^{-1}, a^{-1}b^{-1}ab \rangle$. Then, $H \leq_{alg} K_1, K_2$, where $K_1 = \langle a, bab^{-1}, b^2 \rangle$ and $K_2 = \langle a^2, ab, ab^{-1} \rangle$, but:*

$$H \leq_{f.f.} K_1 \cap K_2 = \langle aba^{-1}b^{-1}, a^{-1}b^{-1}ab, a^2, b^2, ba^2b \rangle$$

**Example 3.2.** *Let $H = \langle a^3, a^2b^2a \rangle$ be as in 3.2. We denote with "tuples" the partitions that underlie the vertex identifications we perform in $\Gamma(H)$. Firstly, we perform only pairwise identifications, that is, we identify only two vertices in $\Gamma(H)$, and then fold and trim the resulting automata. These yield the Stallings graphs of $K_1, K_2, K_3$, and $K_4$ (the specific identifications are listed in the Figure 3.2).*

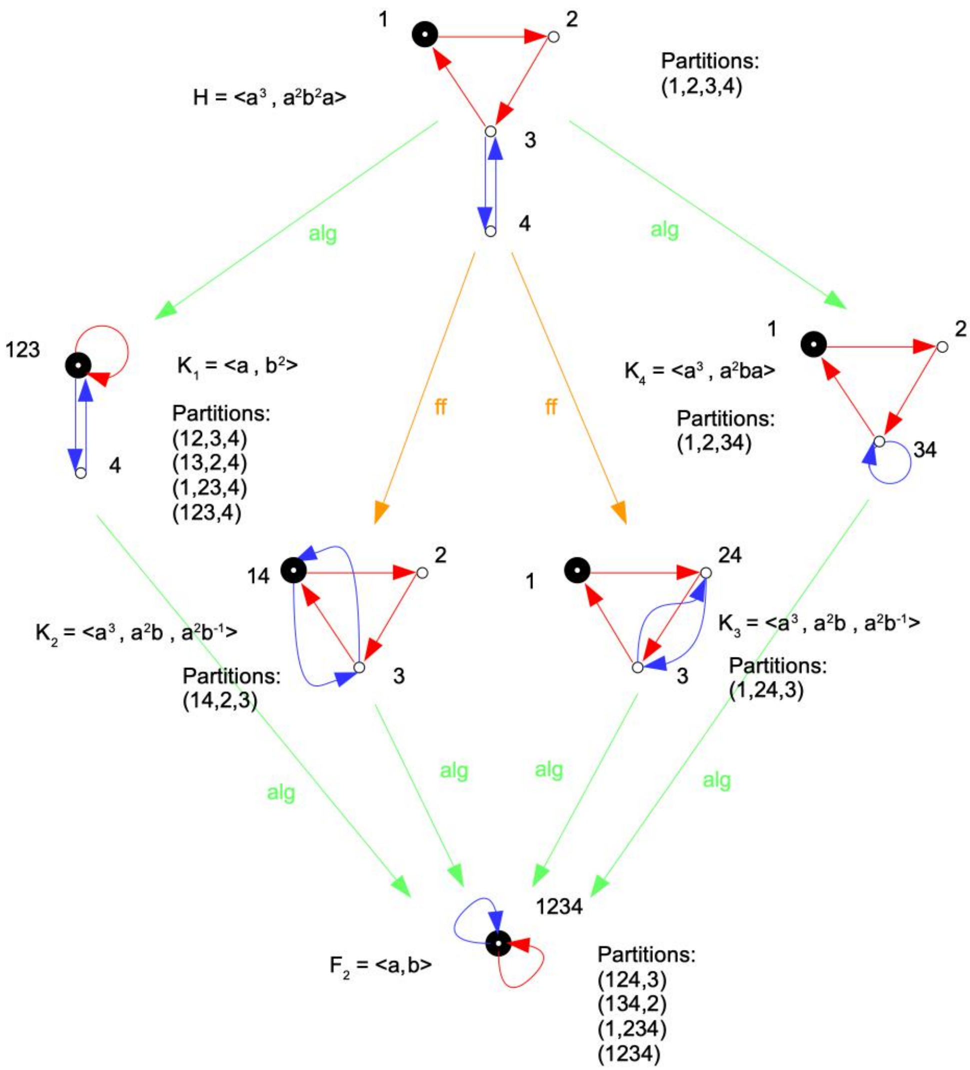


**Figure 3.2**: *A-fringe of* $H = \langle a^3, a^2b^2a \rangle$ *and its algebraic extensions.*

*We can immediately compute their ranks, finding that $rk(K_1) = rk(K_4) = rk(H) = 2$, so the cannot be free multiples of $H$, and thus are algebraic extensions of $H$, $K_1, K_4 \in AE(H)$. On the other hand, we notice that $rk(K_2) = rk(K_3) = 1 + rk(H)$, and that they are the result of one step (in terms of quotients, one pairwise identification in the vertices). Therefore, $H$ is a free factor of both $K_2$ and $K_3$.*

*We perform the same procedure on $\Gamma(K_1)$ and $\Gamma(K_4)$, finding exactly one more extension, common to both, $F_2$, which again has the same rank, so it must be an algebraic extension of $K_1, K_4$, and by transitivity, of $H$. This concludes our computation of $AE(H)$.*

*If we continue this process on $K_2$ and $K_3$, we find that $F_2$ is also an algebraic extension of them.*

## 3.2 A conjecture by Miasnikov-Ventura-Weil

In their paper[4], Miasnikov, Ventura, and Weil conjectured that the set of algebraic extensions of $H \leq_{f.g.} F(A)$ is the intersection of the $A$-fringe of $H$ over all bases $A$. They wrote: *"[...] we conjecture that $AE(H) = \bigcap_A \mathcal{O}_A(H)$, where $A$ runs over all the bases of $F(A)$. As noticed [earlier], this is the case when $H \leq_{f.i.} F(A)$ or $H \leq_{f.f.} F(A)$, but nothing is known in general."*

**Conjecture 3.1** (Miasnikov-Ventura-Weil[4]). *Let $H \leq_{f.g.} F(A)$. Then:*

$$AE(H) = \bigcap_{f \in Aut(F(A))} \mathcal{O}_A(f(H))$$

However, this conjecture was disproven by Parzanchevski and Puder[8], and later extended by Kolodner[3].

**Proposition 3.6** (Parzanchevski-Puder[8]). *Let $H = \langle a^2b^2 \rangle \leq K = \langle a^2b^2, ab \rangle$ be a group extension. Then, $\phi_{H,K} : \Gamma_A(H) \to \Gamma_A(K)$ is surjective for every basis $A$ of the free group, but $K$ is not an algebraic extension of $H$ (it's a proper free multiple).*

*Therefore, $K$ is a principal overgroup of $H$ for any basis (and thus it's in every $A$-fringe of $H$), but is not an algebraic extension. This disproves Conjecture 3.1.*

In their paper, Parzanchevski and Puder asked whether the conjecture would still hold true if we expanded the changes of basis to every extension of $F(A)$ (in particular, $F_\infty$). Kolodner's paper disproved this conjecture as well, actually going much further, and we provide an intuitive overview of Kolodner's method.

Consider $X$ a countably infinite set, and let $A \subset X$ be finite. Let $H, K \leq F(A) \leq F(X)$ be subgroups such that $\phi : \Gamma_A(H) \to \Gamma_A(K)$ is a surjective homomorphism, so $K \in \mathcal{O}_A(H)$. Our goal is to see if, for any $f \in Aut(X)$, $K \in \mathcal{O}_{f(A)}(H)$ still holds.

**Definition 3.5.** *Let $\varphi : F(X) \to F(Y)$ be a group homomorphism. We say $\varphi$ is* ***non-degenerate*** *if $\varphi(x) \neq 1$ for every $x \in X$ (letters are not mapped to the trivial element).*

For every $\varphi \in Aut(F(X))$, we define the natural **functor** $\mathcal{F}_\varphi$ in the automata category, in which we replace every labeled edge $p \xrightarrow{a_i} q$ by the path:

$$p \xrightarrow{x_1} p_1 \xrightarrow{x_2} \cdots \xrightarrow{x_n} p_n \xrightarrow{x_{n+1}} q$$

Where $\varphi(a_i) = x_1 \cdots x_{n+1}$ is the image of the label by the automorphism. Note this functor is a very natural construction to obtain the Stallings graph of $\varphi(H)$. We illustrate this process in Figure 3.3.

We obtain a trim automaton, whose language is trivially $\varphi(H)$ and is not only trim but topologically equivalent to $\Gamma(H)$. Moreover, the homomorphism $\mathcal{F}_\varphi(\phi) : \mathcal{F}_\varphi(\Gamma(H)) \to \mathcal{F}_\varphi(\Gamma(K))$ is also obviously surjective for any automorphism $\varphi$ (and in fact, even if $\varphi$ is a non-degenerate homomorphism). However, this new automaton is, in general, not deterministic; different consecutive labels may have images that partially cancel each other out. Therefore, we have to perform foldings on $\mathcal{F}_\varphi(\Gamma(H))$, and $\mathcal{F}_\varphi(\Gamma(K))$.

We can perform foldings carefully and in a particular order that *mostly* preserves surjectivity. First, we make **foldings on the left hand side**: for every time we have $p \xrightarrow{x} q$ and $p' \xrightarrow{x} q$ in $\mathcal{F}_\varphi(\Gamma(H))$, we can identify $(p = p') \xrightarrow{x} q$ and do the same in $\mathcal{F}_\varphi(\Gamma(K))$, as these will map to $\phi(p) \xrightarrow{x} \phi(q)$ and $\phi(p') \xrightarrow{x} \phi(q)$. It could happen that $\phi(p) = \phi(p')$, in which case there would be no further identification on the right hand side. These simultaneous foldings

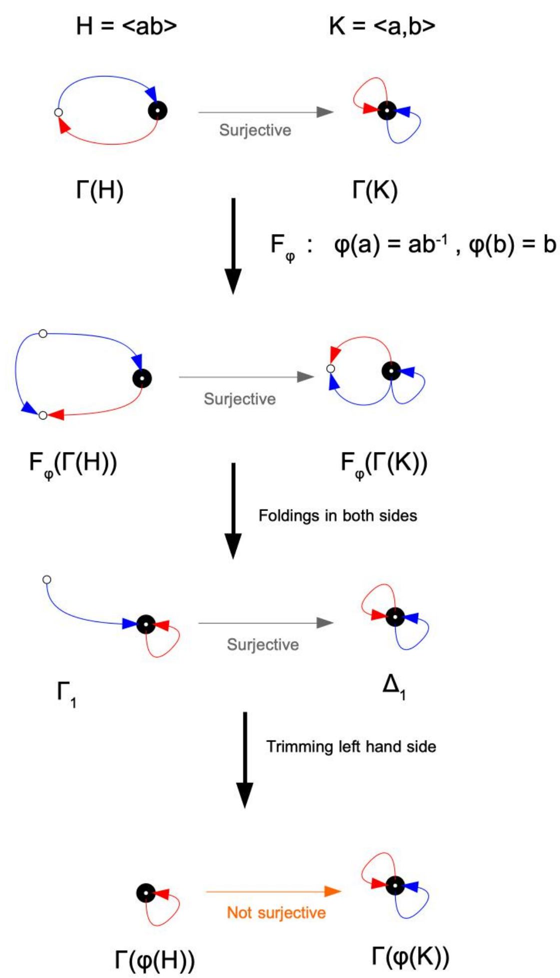


**Figure 3.3**: *Process of foldings of $\mathcal{F}_{\varphi}$ and the critical moment*

clearly preserve surjectivity, and we can make $\mathcal{F}_\varphi(\Gamma(H))$ into a deterministic automaton in this manner, $\Gamma_1$.

These foldings in $\mathcal{F}_\varphi(\Gamma(K))$ may not be sufficient to make it deterministic, but after we finish foldings on the left hand side, we can proceed to perform **foldings on the right hand side**, with no changes on the left hand automaton. These foldings trivially preserve surjectivity in our homomorphism, finally obtaining $\Delta_1$, deterministic, from $\mathcal{F}_\varphi(\Gamma(K))$, and we have the surjective homomorphism:

$$\Gamma_1 \to \Delta_1$$

After these foldings, our automata are deterministic, but they may not be trim. Before we continue, we observe that every untrimmed tail in $\Delta_1$ is the image of some untrimmed tail in $\Gamma_1$: if we have a vertex of degree 1 in $\Delta_1$, its preimages must have degree $\leq 1$ as well (as they're folded), thus it's in an tail in $\Gamma_1$. Removing these, we apply the same reasoning to the next vertex in the tail in $\Delta_1$, inductively concluding that every tail in $\Delta_1$ is the image of tails in $\Gamma_1$. In all, we will now **trim on the right hand side**: keeping in mind that for every tail we trim in $\Delta_1$ we also trim every tail in $\Gamma_1$ preimage of the first one, obtaining:

$$\Gamma_2 \to \Delta_2.$$

In this situation, $\Delta_2$ is deterministic and trim (it is in fact the Stallings graph of $\varphi(K)$), but $\Gamma_2$ may not be trim yet. The tails in $\Gamma_2$ cannot simply be removed as we may lose surjectivity in the process, and this is the **critical moment**.

After this finding, the question becomes apparent: *how do we control the appearance of the formation of tails in* $\Gamma_1$ *and* $\Delta_1$*?* Our problem now shifts into understanding both how these tails form and when they map into vertices that have already been covered (that is, when removing them preserves surjectivity).

In performing foldings, a tail forms when we have a vertex, $p$, such that all the edges coming out of it have the same label. These will all be identified into one, so after the folding process, $p$ has degree 1, and is at the tip of a tail. Returning to our construction, as we assume the images of each label to be reduced, this implies that such vertex $p$ will correspond to a vertex in the original Stallings graph, $\Gamma(H)$. If the edges outgoing $p$ have labels $a_1, \ldots, a_r$, $p$ will be at the tip of a tail after folding $\mathcal{F}_\varphi(\Gamma(H))$ if $\varphi(a_1), \ldots, \varphi(a_r)$ all **start with the same letter**, so we have cancellations.

With this, we only have one part of the story. For any set of labels $a_1, \ldots, a_r$, we can find such an automorphism $\varphi$, where $\varphi(a_1), \ldots, \varphi(a_r)$ start with the same letter, giving us a tail around $p$. However, this tail may map into another tail in $\Delta_1$, so it's trimmed by $\Gamma_2 \to \Delta_2$, or maybe it remains, but we do not lose surjectivity when removing it from $\Gamma_2$ after all. This question is very delicate indeed, and is at the heart of the tools developed in Kolodner's paper[3].

The paper introduces a category called *Free Groups with Restrictions* (FGR), formed by a set of symbols (an alphabet) and a set of "restrictions" that are unordered pairs of elements of said alphabet, which apply to the homomorphisms that we can have between objects of said category. Using these FGR objects, we can construct a recursive method that essentially checks all non-degenerate homomorphisms $\varphi : F(A) \to F(X)$, and finds whether $\mathcal{F}_\varphi(\phi)$ is surjective or not for each $\varphi$.

### 3.2.1 Kolodner's method

To construct the FGR category and Kolodner's method, we first have to introduce some notation and additional objects. Let $\Gamma, \Delta$ be $A$-automata. We define the **Whitehead graph** of an $A$-automaton $\Gamma$, $W(\Gamma)$, as a graph whose vertices are the set of labels, $A$, and whose edges are pairs $\{a_i, a_j^{-1}\}$ such that there exists:

$$q \overset{a_i}{\to} p \overset{a_j}{\leftarrow} q'$$

In $\Gamma$, and the two are different edges. Such pairs are called *2-paths* in $\Gamma$. If $\Gamma = \Gamma_A(F(A))$, we will generally denote $W(\Gamma) = W_A$ the set of all possible 2-paths in a fixed alphabet.

Another notation we'll use is $\tau : F(A) \setminus \{1\} \to A$, which will be the function that returns the last letter in each reduced word, so for example $\tau(b^2aba^{-1}b^2a^7) = a$.

In all, we construct the category of **Free Groups with Restrictions (FGR)**, whose objects will be pairs $(X, N)$, where $X$ is a set of symbols ("generators"), and $N \subseteq W_X$ ("restrictions"), and whose homomorphisms $\varphi \in Hom((X, N), (Y, M))$ are non-degenerate group homomorphisms $\varphi : F(X) \to F(Y)$ satisfying:

1. $\varphi(x) = y_1...y_r \Rightarrow \{y_j, y_{j+1}^{-1}\} \in M$ for every $x \in X$ and every $j = 1, ..., r-1$.

2. $\{x_1, x_2\} \in N \Rightarrow \{\tau(\varphi(x_1)), \tau(\varphi(x_2))\} \in M$ for every 2-path in $N$.

Importantly, observe that $Hom((X, \emptyset), (Y, W_Y))$ is the set of all non-degenerate homomorphisms $\varphi : F(X) \to F(Y)$. This is going to be the starting point for the surjectivity algorithm.

Now, fixing some FGR object $(X, N_X)$ and some $\{x, y\} \in W_X \setminus N_X$, we can give a **partition of the set of FGR homomorphisms** $Hom((X, N_X), (Y, W_Y))$. Let $\varphi \in Hom((X, N_X), (Y, W_Y))$ be an FGR homomorphism. Since $\{x, y\}$ is not in $N_X$, there may be cancellations in $\varphi(x)\varphi(y^{-1})$. These can come in exactly one of 5 types, which will define the 5 classes in our partition. Writing $\varphi(x) = ut$ and $\varphi(y) = vt$, so that $t$ is the maximal cancellation, we have the 5 following possibilities (3.4):

1. No cancellation: $t = 1$.

2. No absortion: $u, v, t \neq 1$.

3. $\varphi(x)$ absorbs $\varphi(y)$: $u \neq 1$ and $v = 1$.

4. $\varphi(y)$ absorbs $\varphi(x)$: $v \neq 1$ and $u = 1$.

5. Mutual anihilation: $u = v$.

Not only a homomorphism falls into exactly one of these options, but (in general) we can also find a homomorphism falling into each one of these options for a fixed $\{x, y\} \in W_X \setminus N_X$. Therefore, fixing such a 2-path, we can partition the set of FGR homomorphisms

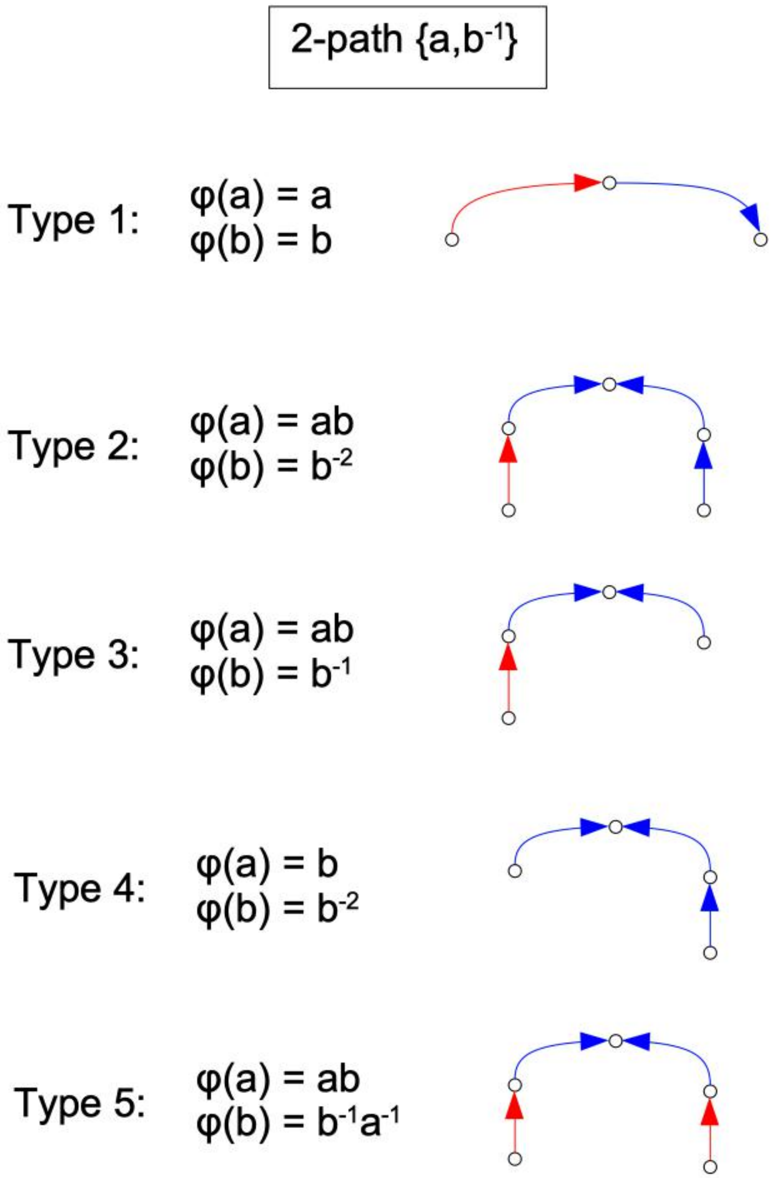


**Figure 3.4**: *Types of cancellations of FGR homomorphisms*

into 5 disjoint blocks accordingly. Going further, given $\varphi \in Hom((X, N_X), (Y, W_Y))$ and $\{x, y\} \in W_X \setminus N_X$, depending on the type of cancellation of $\varphi(x)\varphi(y^{-1})$ from before, we can define an FGR object $(U, N_U)$ and a *folding homomorphism* $\psi_{\{x,y\}} \in Hom((X, N_X), (U, N_U))$ such that there exists $\varphi' \in Hom((U, N_U), (Y, W_Y))$ satisfying $\varphi = \varphi' \circ \psi_{\{x,y\}}$.

For each type of cancellation, we have the following folding homomorphisms:

1. We have $U_1 = X$, $\psi^1_{\{x,y\}} = Id$, $N_{U_1} = N_X \cup \{x, y\}$, and $\varphi' = \varphi$.

2. We will have two cases, depending on whether $x = y^{-1}$ or not. **Firstly, if** $x \neq y^{-1}$, then $U_2 = X \cup \{s\}$ and $N_{U_2} = \{\{\tau(\psi^2_{\{x,y\}}(a))\tau(\psi^2_{\{x,y\}}(b))\} : \{a, b\} \in N_X\} \cup \{\{x, s^{-1}\}\{y, s^{-1}\}\{x, y\}\}$, and the homomorphisms will be:

$$\psi^2_{\{x,y\}}(z) = \begin{cases} xs & z = x \\ ys & z = y \\ z & else \end{cases} \quad \varphi'(z) = \begin{cases} u & z = x \\ v & z = y \\ t & z = s \\ \varphi(z) & else \end{cases}$$

**Secondly, if** $x = y$, then $U_2 = X \cup \{s\}$ and $N_{U_2} = \{\{\tau(\psi^2_{\{x,y\}}(a))\tau(\psi^2_{\{x,y\}}(b))\} : \{a, b\} \in N_X\} \cup \{\{x, s^{-1}\}\{x^{-1}, s^{-1}\}\{x, x^{-1}\}\}$, and the homomorphisms will be:

$$\psi^2_{\{x,y\}}(z) = \begin{cases} s^{-1}xs & z = x \\ z & else \end{cases} \quad \varphi'(z) = \begin{cases} u & z = x \\ t & z = s \\ \varphi(z) & else \end{cases}$$

3. We'll have $U_3 = X$ and $N_{U_3} = \{\{\tau(\psi^2_{\{x,y\}}(a))\tau(\psi^2_{\{x,y\}}(b))\} : \{a, b\} \in N_X\} \cup \{\{x, y^{-1}\}\}$, and the homomorphisms will be:

$$\psi^3_{\{x,y\}}(z) = \begin{cases} xy & z = x \\ z & else \end{cases} \quad \varphi'(z) = \begin{cases} u & z = x \\ \varphi(z) & else \end{cases}$$

4. We'll have $U_4 = X$ and $N_{U_4} = \{\{\tau(\psi^2_{\{x,y\}}(a))\tau(\psi^2_{\{x,y\}}(b))\} : \{a, b\} \in N_X\} \cup \{\{x^{-1}, y\}\}$, and the homomorphisms will be:

$$\psi^4_{\{x,y\}}(z) = \begin{cases} yx & z = y \\ z & else \end{cases} \quad \varphi'(z) = \begin{cases} v & z = y \\ \varphi(z) & else \end{cases}$$

5. We'll have $U_5 = X \setminus \{y\}$ and $N_{U_5} = \{\{\tau(\psi^2_{\{x,y\}}(a))\tau(\psi^2_{\{x,y\}}(b))\} : \{a,b\} \in N_X\}$, and the homomorphisms will be:

$$\psi^5_{\{x,y\}}(z) = \begin{cases} x & z = y \\ z & else \end{cases} \quad \varphi'(z) = \varphi(z)$$

Of course, this decomposition can be iterated recursively, applying it to $\varphi'$, and a natural question is whether this process would come to an end in a finite number of steps. Crucially, this is the case even if the alphabet grows in some of the steps, as we see in the follwing theorem.

**Theorem 3.4** (Kolodner[3]). *Let $X$ be finite, and let $\varphi \in Hom((X, N_X), (Y, W_Y))$. Then, $\varphi$ can be factored into $\varphi = \varphi' \circ \psi_k \circ \cdots \circ \psi_1$, where every $\psi_i : (U_{i-1}, N_{U_{i-1}}) \to (U_i, N_{U_i})$ is a folding homomorphism built as before, and $N_{U_k} = W_{U_k}$, so $\varphi'(x)\varphi'(y^{-1})$ does not cancel out for every distinct $x, y \in U_k$.*

***Proof:*** *Consider $h : Hom((X, N_X), (Y, W_Y)) \to \mathbb{N} \times \mathbb{N}$ a height function defined as $h(\varphi) = (\sum_{x \in X} len(\varphi(x)), |W_X \setminus N_X|)$, and consider $\mathbb{N} \times \mathbb{N}$ ordered with the lexicographic order. Then, $h(\varphi') < h(\varphi)$ for each decomposition $\varphi = \psi^i_{\{x,y\}} \circ \varphi'$ as we described before, for each case $i \in \{1, 2, 3, 4, 5\}$, therefore the factorization ends in a finite number of steps.*

Recalling the functor $\mathcal{F}_\varphi$ for some non-degenerate $\varphi : F(X) \to F(Y)$, we may loose surjectivity in a (deterministic and trim) automata homomorphism $\Gamma \to \Delta$ when we have foldings to make in $\mathcal{F}_\varphi(\Gamma)$. However, if $\mathcal{F}_\varphi(\Gamma)$ is deterministic, then $\mathcal{F}_\varphi(\Gamma \to \Delta)$ is surjective (assuming, of course, that $\Gamma \to \Delta$ was surjective from the start). If $\varphi : X \to Y$ and $\Gamma$ (deterministic and trim $X$-automaton) are such that $\mathcal{F}_\varphi(\Gamma)$ is deterministic (trimness is guaranteed), then we call the pair $(\Gamma, \varphi)$ a **stencil**.

For a fixed $\Gamma$, $(\Gamma, \varphi)$ won't be a stencil in general, but we can carefully restrict the set of homomorphisms so that all our "restricted" choices give us stencils for $\Gamma$. This is how the notion of a *stencil space* comes into play.

**Definition 3.6.** *Let $\Gamma$ be a deterministic and trim $X$-automaton. We say an FGR object $(X, N_X)$ is a **stencil space** of $\Gamma$ if $W(\Gamma) \subseteq N_X$.*

If $(X, N_X)$ is a stencil space for $\Gamma$, then $(\Gamma, \varphi)$ is a stencil for any FGR homomorphism $\varphi \in Hom((X, N_X), (Y, N_Y))$, and for any FGR object $(Y, N_Y)$.

Returning to our original problem, let $X$ be a finite alphabet contained in a countably infinite one, $Y$, and let $\Gamma \to \Delta$ be a surjective $X$-automaton homomorphism, where both are deterministic and trim. We will call the **surjectivity problem** the decisional problem of determining whether $\Gamma_Y(\mathcal{F}_\varphi(\Gamma \to \Delta))$ is surjective for any $\varphi \in Hom((X, N_X), (Y, W_Y))$ for some set of restrictions $N_X$ (keep in mind that $\Gamma_A(\cdot)$ is the folding and trimming of an $A$-automaton). Starting with $(X, \emptyset)$ as our FGR object, we'd be considering all non-degenerate homomorphisms $\varphi : F(X) \to F(Y)$.

Given a surjectivity problem, defined by an *ambient space* $(X, N_X)$ and an $X$-automata homomorphism $\Gamma \to \Delta$, we have three alternatives:

1. $\Gamma \to \Delta$ is not surjective, and the surjectivity problem resolves negatively.
2. $(X, N_X)$ is a stencil space for $\Gamma$, in which case the problem resolves positively.
3. Neither of the above, in which case it is unclear whether the problem resolves positively or negatively. We call this the **ambiguous case**.

This case analysis provides an apparent algorithm if we can further analyse the third situation. This is indeed the case, applying Kolodner's partition we saw earlier (3.2.1). In all, if our problem is in the third ambiguous case, we can break it into 5 subproblems from the partition, and we analyse these new ones recursively. The overall surjectivity problem will resolve positively if, and only if, all 5 subproblems resolve positively. If one of them doesn't, we will have found a non-degenerate homomorphism $\varphi : F(X) \to F(Y)$ such that $\Gamma_Y(\mathcal{F}_\varphi(\Gamma \to \Delta))$ is not surjective (see 3.5).

Due to the nature of our construction and what we've seen so far, this method is correct, and thanks to Theorem 3.4, we know there won't be infinite loops, since each homomorphism will appear after a finite number of steps. However, it can happen that, for every $k \in \mathbb{N}$, there is a homomorphism whose decomposition is of length $k$, so the overall process never ends, as we keep opening further and further cases. Therefore, this procedure may not be practical when the problem resolves positively, although in our recursion we may show that some cases we open are equivalent to others we've encountered before. The algorithm is, however, effective in problems that resolve negatively, as we'll show shortly.

The method is also not particularly efficient, as the number of cases grows exponentially.

**Example 3.3** (Kolodner's method)**.** *Let $H = \langle ab \rangle \leq K = \langle a, b \rangle$, take $X = \{a, b\}$, our ambient FGR object be $(X, \emptyset)$. Then, $\phi_{H,K}$ is surjective, and $W(\Gamma_X(H)) = \{\{a, b^{-1}\}, \{a^{-1}, b\}\}$, so clearly $\emptyset$ is not a stencil space for $\Gamma_X(H)$. Take $\Gamma = \Gamma_X(H)$ and $\Delta = \Gamma_X(K)$.*

*Let $\{a, b^{-1}\} \in W(\Gamma) \setminus \emptyset$. We have 5 subproblems:*

1. *Ambient FGR $(U_1, N_1) = (X, \{\{a, b^{-1}\}\})$ and $\psi_1 = id_X$.*

2. *Ambient FGR $(U_2, N_2) = (\{a, b, c\}, \{\{a, c^{-1}\}, \{b^{-1}, c^{-1}\}, \{a, b^{-1}\}\})$ and folding homomorphism:*
$$\psi_2(x) = \begin{cases} ac & x = a \\ c^{-1}b & x = b \end{cases}$$

3. *Ambient FGR $(U_3, N_3) = (\{a, b\}, \{\{a, b\}\})$ and folding homomorphism:*
$$\psi_3(x) = \begin{cases} ab^{-1} & x = a \\ b & x = b \end{cases}$$

4. *Ambient FGR $(U_4, N_4) = (\{a, b\}, \{\{a^{-1}, b^{-1}\}\})$ and folding homomorphism:*
$$\psi_4(x) = \begin{cases} a & x = a \\ a^{-1}b & x = b \end{cases}$$

5. *Ambient FGR $(U_3, N_3) = (\{a\}, \emptyset)$ and folding homomorphism $\psi_5(x) = a : \forall x \in X$*

*We consider case 3. We get $H_3 = \psi(H) = \langle a \rangle$ and $K_3 = \psi(K) = K$. Now, clearly, $\phi_{H_3,K_3}$ is **not** surjective, so the overall problem resolves negatively. In other words, there is a homomorphism $\varphi : F(X) \to F(X)$ (which is in fact an automorphism) such that $\phi_{\varphi(H),\varphi(K)}$ is not surjective.*

**Remark 3.3.** *As we noted earlier, this method may not end in a finite number of steps in cases when the problem resolves positively, thus it can't be referred to as a proper algorithm.*

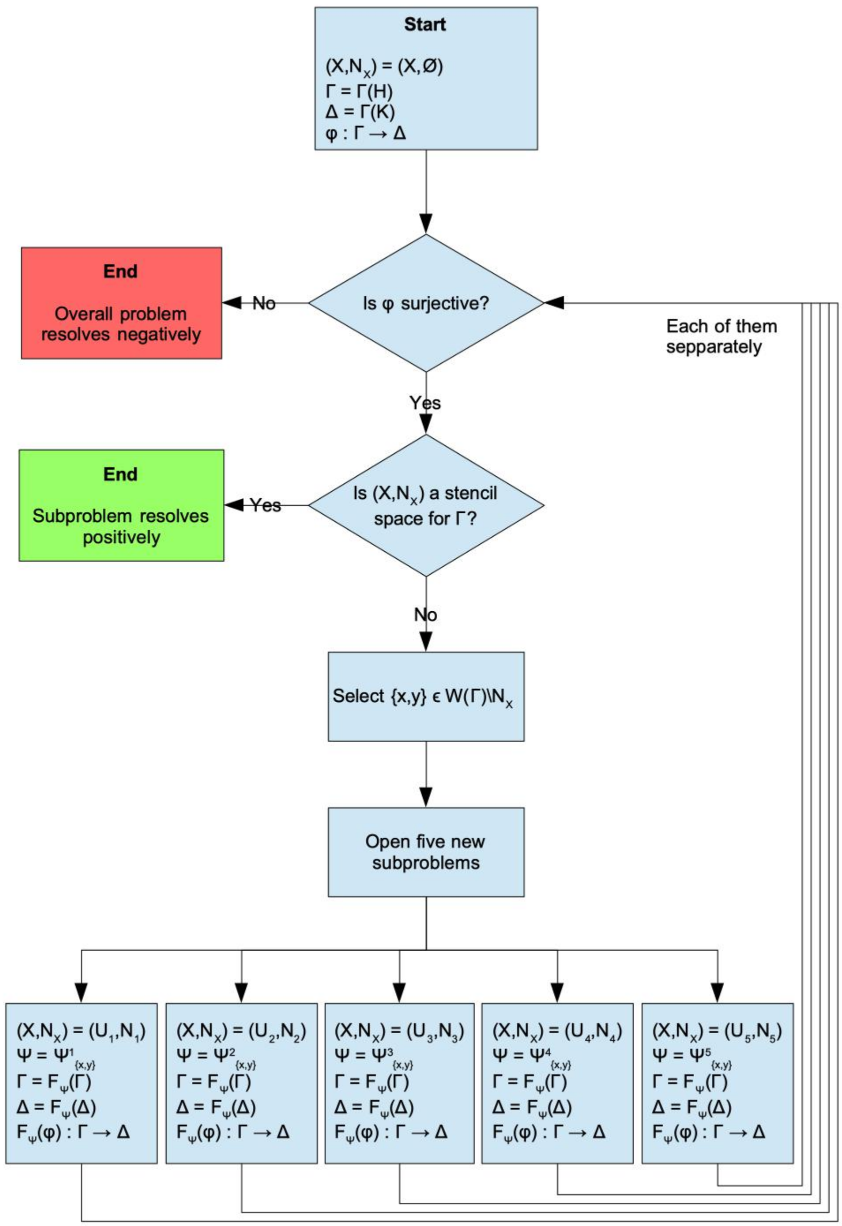


**Figure 3.5**: *Kolodner's method diagram*

### 3.2.2 Reformulation of the MVW conjecture

So far, the disproven reformulations of the MVW conjecture have been "upwards": considering automorphisms of larger ambient groups, starting at $F(A)$, and moving up to $F_\infty$. We can, however, consider a "downwards" reformulation of the original conjecture, where we take the automorphisms of any intermediate extension.

**Conjecture 3.2.** *Let $H \leq K \leq F(A)$ be an extension of finitely generated groups. Then:*

$$H \leq_{alg} K \Leftrightarrow \varphi(K) \in \mathcal{O}_A(\varphi(H)) \;:\; \forall \varphi \in Aut(L), K \leq L \leq F(A)$$

The key observation here is that, in general, given $L \leq F(A)$, $\varphi \in Aut(L) \not\Rightarrow \varphi \in Aut(F(A))$. It does hold true, however, if $L \leq_{f.f.} F(A)$, so here we're adding new automorphisms to the later conjectures, and in particular to the first conjecture.

We can give an example showing that Parzanchevski and Puder's Counterexample 3.6 no longer holds in this new form of the conjecture.

**Example 3.4.** *Let $H = \langle a^2b^2 \rangle \leq_{f.f.} \langle a^2b^2, ab \rangle = K \leq_{f.f.} L = \langle a^2b^2, ab, ba \rangle$. Consider $\varphi \in Aut(L)$ defined as:*

$$\varphi(a^2b^2) = aba^{-1}b^{-1} \;;\; \varphi(ab) = a^2b^2 \;;\; \varphi(ba) = ba$$

*Then, $\Gamma_A(\varphi(H))$ has 4 vertices and $\Gamma_A(\varphi(K))$ has 6, so $\varphi(K) \notin \mathcal{O}_A(\varphi(H))$ (see Figure 3.6).*

*Further, we can show this with an automorphism of $K$. Let $\psi \in Aut(K)$ be defined as:*

$$\psi(a^2b^2) = ab \;;\; \psi(ab) = a^2b^2$$

*Then, as in the previous case, $\Gamma_A(\psi(H))$ has fewer vertices than $\Gamma_A(\psi(K)) = \Gamma_A(K)$, so there can't be a surjective homomorphism (see Figure 3.6).*

This technique also applies to Kolodner's counterexample. In fact, it makes up the key idea underlying our proof of the conjecture (or rather, theorem). To show it (Theorem 3.5), we will first provide a couple of preliminary results that will ease our proof.

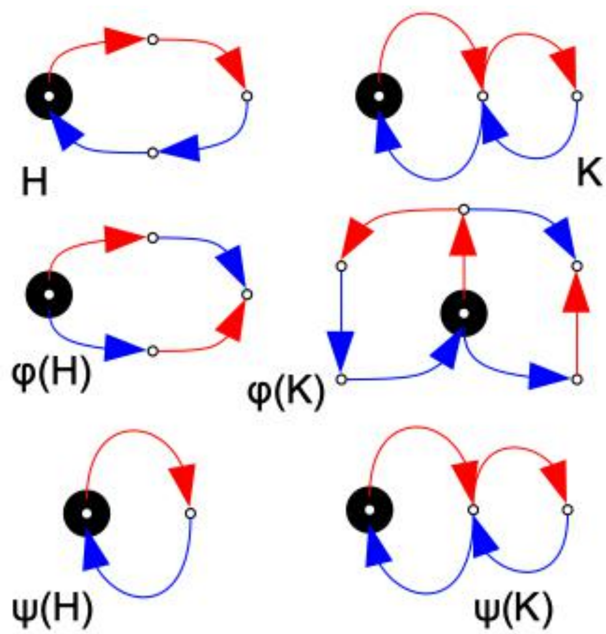


**Figure 3.6**: *Stallings graphs of subgroups in example 3.4*

**Lemma 3.5.** *For any $H \leq K$, either $H \leq_{alg} K$ or $H \leq_{alg} L <_{f.f.} K$.*

***Proof:*** *Direct consequence of the characterisation of algebraic extensions (3.2) and them being a Takahasi family (Theorem 3.2).*

**Lemma 3.6.** *Let $K = \langle k_1, ..., k_r \rangle$ be the basis obtained from the Stallings graph as in 1.3.1 by some spanning tree. Let $H = \langle k_1, ..., k_s \rangle$, with $s < r$. Then, $\Gamma_A(H)$ is a strict subgraph of $\Gamma_A(K)$.*

***Proof:*** *Let $T$ be the spanning tree of $\Gamma(K)$ from which we construct the basis and let $S$ be the set of edges outside $T$. Let $S' \subset S$ be the set of edges corresponding to $k_1, ..., k_s$. Then, clearly, $\Gamma(H)$ is the trimming of $T \cup S'$ (that is, adding the edges of $S'$ to the set of edges of $T$), and is therefore a strict subgraph of $\Gamma(K)$.*

**Theorem 3.5.** *Let $H \leq K \leq_{f.g.} F(A)$. Then:*

$$H \leq_{alg} K \Leftrightarrow \varphi(K) \in \mathcal{O}_A(\varphi(H)) \ : \ \forall \varphi \in Aut(L), K \leq L$$

***Proof:*** *First, suppose $H \leq_{alg} K$, and that there exist an extension $K \leq L$ and an automorphism $\varphi \in Aut(L)$ such that $H' = \varphi(H) \not\leq_{alg} \varphi(K) = K'$. In such case, there exists* *a proper free factor of $K'$ containing $H'$, $H' \leq J' <_{f.f.} K'$. Since $\varphi$ is an automorphism, we have $H \leq J = \varphi^{-1}(J') <_{f.f.} K$, which contradicts that $H \leq_{alg} K$, thus $\varphi(H) \leq_{alg} \varphi(K)$ for any automorphism $\varphi \in Aut(L)$ of any such extension $K \leq L$.*

*For the reverse implication, suppose $H \leq_{f.f.} K$, so we have a basis extension $H = \langle h_1, ..., h_r\rangle$ and $K = \langle h_1, ..., h_s\rangle$, with $r < s$. Consider the basis $\langle k_1, ..., k_s\rangle$ of $K$ obtained from a spanning tree of $\Gamma(K)$. Take the automorphism $\varphi \in Aut(K)$ defined as $\varphi(h_i) = k_i$, and call $H' = \varphi(H)$. Then, $H' = \langle k_1, ..., k_r\rangle$, and by Lemma 3.6, $\Gamma(H')$ is a strict subgraph of $\Gamma(K)$.*

*By Lemma 3.5, if $K$ is in the A-fringe of $H$ (so $\phi_{H,K}$ is surjective), and $H \not\leq_{alg} K$, either $H \leq_{f.f.} K$ or $H \leq_{alg} H' \leq_{f.f.} K$. We've covered the first possibility, so suppose $H <_{alg} H' <_{f.f.} K$. Then, by Lemma 3.1, we have unique homomorphisms $\phi_{H,H'}$, $\phi_{H,K}$, and $\phi_{H',K}$ between their corresponding Stallings graphs. Since they are unique, $\phi_{H,K} = \phi_{H',K} \circ \phi_{H,H'}$. By the first part of the proof, there is $\varphi \in Aut(K)$ such that $\phi_{\varphi(H'),K}$ is not surjective (and in fact injective). Therefore, by the composition of homomorphisms, $\phi_{\varphi(H),K}$ is not surjective, despite $\phi_{\varphi(H),\varphi(H')}$ being surjective.*

## 3.3 Onto extensions

The counterexamples to the original MVW conjecture by Parzanchevski, Puder[8], and Kolodner[3] open up the possibility of defining another class of free group extensions: *onto extensions*. These will be an expansion of algebraic extensions, and in this section we will see some results for onto extensions analogous to those for algebraic extensions.

We say an extension $H \leq K$ of finitely generated subgroups of $F(A)$ is an **onto extension** if the homomorphism between their Stallings graphs is surjective (onto) for any basis of the ambient group, that is:

$$\varphi(K) \in \mathcal{O}_A(\varphi(H)) \ : \ \forall \varphi \in Aut(F(A))$$

As we've seen before, algebraic extensions are also onto extensions, but onto extensions may not be algebraic in general.

If $H \leq K$ is onto, we will write it as $H \leq_{onto} K$, and $\Omega(H)$ will denote the set of all onto extensions of $H$. We therefore have:

$$AE(H) \subseteq \Omega(H) = \bigcap_{f \in Aut(F(A))} \mathcal{O}_A(f(H))$$

Proposition 3.5 regarding the composition and union of algebraic extensions is a particularly powerful result, which extends to onto extensions as well, as we show in the following results.

**Lemma 3.7.** *Let $W \subset F(A)$ be a finite set of reduced words, and let $\mathcal{F}(W)$ be the flower automaton. Then, no foldings produce vertices of degree 1, except for maybe the basepoint.*

***Proof:*** *For a folding to produce a vertex, p, of degree 1, all edges leaving p, listed as $(p, a, q_1), ..., (p, a, q_d)$, must have the same label. Since all words are reduced, the only place this may happen at the start is at the basepoint. This, however, does not change the labels of edges coming in or out of vertices; if anything, the number of labels outgoing any vertex after foldings may increase (in which case, such vertex will be the identification of two or more). Therefore, the only vertex of degree 1 we may obtain is the basepoint, and in the foldings process to obtain the Stallings graph of $\langle W \rangle$ we do not have to trim.*

Observe that the Stallings graph of $\langle H_1 \cup H_2 \rangle$ can be obtained by folding the union of their respective Stallings graphs identifying the basepoint. With this in mind, the previous lemma, and the process of foldings we followed in Kolodner's method, we obtain the following proposition.

**Proposition 3.7** (Union of onto extensions)**.** *Let $H_1 \leq_{onto} K_1$ and $H_2 \leq_{onto} K_2$ be onto extensions. Then, $\langle H_1 \cup H_2 \rangle \leq_{onto} \langle K_1 \cup K_2 \rangle$.*

***Proof:*** *Since they are onto, the homomorphism between their Stallings graphs is onto for every change of basis in the ambient free group. Consider their Stallings graphs in a particular basis, $\Gamma_A(H_i) \to \Gamma_A(K_i)$. As we've observed earlier, the Stallings graph of the union is obtained by folding the union of the Stallings graphs of the two subgroups,* *identifying their basepoint. Let $\Gamma$ and $\Delta$ be such unions of automata, which have the natural homomorphism $\Gamma \to \Delta$ inherited from the previous two, and it is trivially surjective.*

*Now, in an analogous fashion as in Kolodner's method, we perform foldings on $\Gamma$ and $\Delta$ to obtain the Stallings graphs of $H_1 \cup H_2$ and $K_1 \cup K_2$. Just as then, the homomorphism may no longer be surjective if at some point we have to trim a tail in the left hand side. However, by Lemma 3.7, we can assure this will not be the case: as noted in the first chapter, the process of foldings does not depend on the order. If $H_i = \langle W_i \rangle$, we can obtain $\Gamma$ by foldings on the flower automaton $\mathcal{F}(W_1 \cup W_2)$, and then continuing with said foldings obtain $\Gamma_A(\langle H_1 \cup H_2 \rangle)$. Therefore, by the lemma no tails appear in the entire process, and the same applies to the right hand side, so we do not lose surjectivity in the process.*

*In all, the homomorphism $\Gamma_A(\langle H_1 \cup H_2 \rangle) \to \Gamma_A(\langle K_1 \cup K_2 \rangle)$ is surjective, and since we can do the same for any change of basis of $F(A)$, we conclude $\langle H_1 \cup H_2 \rangle \leq_{onto} \langle K_1 \cup K_2 \rangle$.*

**Proposition 3.8** (Composition of onto extensions)**.** *If $H \leq_{onto} K \leq_{onto} L$, then $H \leq_{onto} L$. Moreover, if $H \leq_{onto} L$, and $H \leq K \leq L$, then $K \leq_{onto} L$.*

***Proof:*** *By 3.1, we have that $\phi_{H,K}$ and $\phi_{K,L}$ are surjective for every basis of $F(A)$, and $\phi_{H,L} = \phi_{K,L} \circ \phi_{H,K}$ for every basis of $F(A)$, thus $\phi_{H,L}$ is surjective for every basis of the ambient group as well.*

*Similarly, for the second part of the statement, we have that $\phi_{H,L}$ is onto, and $\phi_{H,L} = \phi_{K,L} \circ \phi_{H,K}$, which implies $\phi_{K,L}$ is onto.*

In all, it seems reasonable to consider also an analogous result to the Takahasi theorem for onto extensions. However, other than $\Omega(H)$ being finite, trivially by construction, we'd also need a dual notion to onto extensions, just like algebraic extensions and free factors. One may intuitively think such a dual notion could be that every homomorphism were injective, which we could call *into extensions*. Unfortunately, into extensions would be a subclass of free factors (Lemma 3.1), whereas onto extensions are weaker than algebraic ones, so their dual must also be a weaker notion extending free factors. We do not know what such a dual notion should be, nor whether a Takahasi analogous result is possible.

Nevertheless, into extensions seem easily definable, and in fact, they are the subject of the following section.

## 3.4 Into extensions

We say an extension $H \leq K$ is an **into extension**, $H \leq_{into} K$, if $\phi_{\varphi(H),\varphi(K)}$ is injective (into) for every $\varphi \in Aut(F(A))$. As it has been observed in 3.1, this implies that $H \leq_{f.f.} K$, as $\phi_{H,K}$ being injective in some basis implies it's a free factor.

**Example 3.5.** *Let $\langle a \rangle \leq \langle a, b \rangle$ be an extension. If we take $\varphi \in Aut(F_2)$ defined as $\varphi(a) = ab^2$, $\varphi(b) = b$, we see the homomorphism between the Stallings graphs of $\langle ab \rangle$ and $\langle a, b^2 \rangle$ is **not** injective (both over vertices and edges).*

This very simple example seems to indicate that into extensions may not exist. However, as we will show later on, we can not only provide examples of into extensions, but also characterise them in general. For that, we may follow an analysis similar to the one we did for Kolodner's method: let $H \leq_{f.f.} K \leq F(A)$ be an extension such that $\phi_{H,K}$ is injective (we may call $H$ a **graphical free factor** of $K$). We want to see whether $\phi_{\varphi(H),\varphi(K)}$ is also injective for some arbitrary automorphism $\varphi \in Aut(F(A))$.

Recall the functor $\mathcal{F}_\varphi$ we defined earlier, and let $\Gamma = \Gamma_A(H)$ and $\Delta = \Gamma_A(K)$ be the corresponding Stallings graphs. Then, we can take $\mathcal{F}_\varphi(\phi) : \mathcal{F}_\varphi(\Gamma) \to \mathcal{F}_\varphi(\Delta)$ as then. Observe we preserve injectivity so far. These new automata are trim and, in general, not deterministic, so we have to perform foldings on them. First, we carry out **foldings on the left hand side** (on $\mathcal{F}_\varphi(\Gamma)$), obtaining $\Gamma_1$. Since $\mathcal{F}_\varphi(\Gamma)$ is a subgraph of $\mathcal{F}_\varphi(\Delta)$, we can perform the exact same foldings on $\mathcal{F}_\varphi(\Delta)$, obtaining $\Delta_1$, thus we have the homomorphism:

$$\phi_1 : \Gamma_1 \to \Delta_1$$

This homomorphism is, clearly, still injective and well defined, as we have done the same foldings on both sides. However, we may still have to perform **foldings on the right hand**

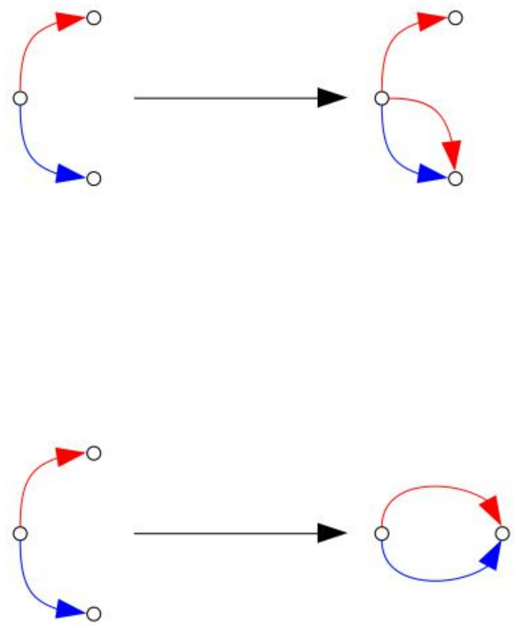

**Figure 3.7**: *Critical step for injective extensions*

**side**, and in doing that, **we may lose injectivity** (see Figure 3.7). In other words, this is our **critical step**, using previous terminology.

Just as in the surjectivity case, we have to determine when this critical step happens. As the picture illustrates (3.7), we only have to worry about edges that are outside $\Gamma_1$ but in $\Delta_1$ (seeing $\Gamma_1$ as a subgraph of $\Delta_1$). These can come in one only form: edges *outside* $\Gamma_1$ that connect two vertices *inside* $\Gamma_1$.

The other theoretical case we exclude is that of two vertices in $\Gamma_1$ connected by parallel edges (with the same label and direction) to a vertex out of $\Gamma_1$. We can exclude this situation by induction: in our ground case, suppose there exists some vertex $p$ in $\mathcal{F}_\varphi(\Delta)$, out of $\mathcal{F}_\varphi(\Gamma)$, and two vertices in $\mathcal{F}_\varphi(\Gamma)$ $q_1$ and $q_2$ with edges $(p, a, q_1)$ and $(p, a, q_2)$. Then, such vertices also exist in $\Gamma$ and $\Delta$, against them being deterministic; they cannot be mapped to the same label because $\varphi$ is an automorphism. For the inductive step (and informally), we'd have to obtain $p$ by identifying two vertices $p_1$ and $p_2$ connected to another "common" link behind, going back to the same issue as in the ground case.

**Definition 3.7.** *Let $\mathcal{A}$ be an automaton, with set of vertices $V$ and edges $E$, and let $V' \subset V$ be a subset of vertices containing the basepoint, $q_0$. We define the* ***induced automaton*** *(or* ***induced subgraph****) of $\mathcal{A}$ by $V'$, $\mathcal{A}[V']$, as the automaton with vertices $V'$, basepoint $q_0$, and edges $E' = \{(p, a, q) \in E : p, q \in V'\}$.*

We can express the previous finding in terms of induced subgraphs: if $\Gamma \subset \Delta$ is an induced subgraph, then foldings outside $\Gamma$ do not identify two distinct vertices in $\Gamma$. This key observation enables us to state the following theorem.

**Theorem 3.6.** *Let $H \leq_{f.f.} K \leq F(A)$ be a graphical free factor (i.e. $\phi_{H,K}$ is injective) such that $\Gamma_A(H)$ is an induced subgraph of $\Gamma_A(K)$. Then, $H \leq K$ is an into extension, $H \leq_{into} K$.*

***Proof:*** *For any automorphism $\varphi \in Aut(F(A))$, consider the previous procedure of foldings after the application of $\mathcal{F}_\varphi$. As we've seen, since $\Gamma_A(H)$ is an induced subgraph of $\Gamma_A(K)$, the critical step of foldings will not identify two vertices of $\Gamma_1$ in $\Delta_1$. Indeed: since it's an induced subgraph, there are no "additional" edges connecting two vertices of $\Gamma_A(H)$ in $\Gamma_A(K)$, so the only possibility is that two vertices in $\Gamma_A(H)$ are identified because they both have edges $(p, a, q_1)$ and $(p, a, q_2)$ with the same label with some $p$ outside $\Gamma_A(H)$. As we said earlier informally, this can be discarded inductively.*

*Suppose, as our ground case, there exists such vertex $p$ and such edges $(p, a, q_1)$ and $(p, a, q_2)$. Then, those edges are the image of two edges in $\Gamma_A(K)$ by $\mathcal{F}_\varphi$, $(p', x_1, q_1')$ and $(p', x_2, q_2')$, with $p$ outside $\Gamma_A(H)$ and the other two in it. Therefore, $\varphi(x_1) = \varphi(x_2) = a$, and since $\varphi$ is injective, $x_1 = x_2$. However, these two labels cannot coincide, as $\Gamma_A(K)$ is deterministic.*

*Suppose now that we obtain the previous situation (edges $(p, a, q_1)$ and $(p, a, q_2)$) after a sequence of foldings. In particular, $p$ is the result of an identification of two vertices, $p_1, p_1'$ which had, in the previous step $p_2$ such that $(p_2, a_1, p_1)$ and $(p_2, a_1, p_1')$ were edges in the automaton. Similarly, $p_2$ may be split in two similarly, obtaining two paths $p_{k+1} \overset{a_k}{\to} p_k \overset{a_{k-1}}{\to} \cdots \overset{a_1}{\to} p_1$ and $p'_{k+1} \overset{a_k}{\to} p'_k \overset{a_{k-1}}{\to} \cdots \overset{a_1}{\to} p'_1$. We have two possibilities: either these two paths are the image of the same edge by $\mathcal{F}_\varphi$, or they are not. In the first case, it's clear that we are in the same situation as in the ground case, contradicting determinism in $\Gamma_A(K)$. In the second case, we'd have two disjoint paths with the same labels (in the image), which lift to paths with different labels in $\Gamma_A(K)$. In other words, $\varphi(x_1 \cdots x_r) = \varphi(y_1 \cdots y_s) = a_{k+1} \cdots a_1$. This contradicts that $\varphi$ is an automorphism.*

*Therefore, after folding and trimming (which does not affect injectivity either, trivially), $\Gamma_A(\varphi(H))$ is, again, an induced subgraph of $\Gamma_A(\varphi(K))$. Therefore, $H \leq_{into} K$.*

**Corollary 3.6.1.** *If $H \leq_{into} F(A)$, then $H = F(A)$.*

**Example 3.6.** *The extension $H = \langle aba^{-1} \rangle \leq \langle aba^{-1}, bab^{-1} \rangle = K$ is into.*

These into extensions are, as we said, a subclass of free products, and some results for free factors also apply locally to into extensions.

**Proposition 3.9.** *Into extensions satisfy the following properties:*

1. *If $H \leq_{into} K \leq_{into} L$, then $H \leq_{into} L$.*
2. *If $H \leq_{into} L$ and $H \leq K \leq L$, then $H \leq_{into} K$, but not necessarily $K \leq_{into} L$.*
3. *If $H \leq_{into} K_1, K_2$, then $H \leq_{into} K_1 \cap K_2$.*

***Proof:*** *For 1, by the uniqueness of the homomorphism $\phi_{H,L} = \phi_{K,L} \circ \phi_{H,K}$, since $\phi_{K,L}$ and $\phi_{H,K}$ are injective, $\phi_{H,L}$ is also injective. The same argument holds for 2.*

*For statement 3, we simply observe the pullback. We have the homomorphisms $\phi_{H,K_1}$ and $\phi_{H,K_2}$, and by the definition of the pullback, it is easy to deduce that:*

$$\phi_{H,K_1 \cap K_2}(v) = (\phi_{H,K_1}(v), \phi_{H,K_2}(v))$$

*Thus, since both original homomorphisms are injective for any basis, by hypothesis, the one for the intersection is also injective directly for any basis, thus, the extension is into.*

In the case the extensions not only are into, but also $H$ is an induced subgraph of both, then we find good properties with the union of subgroups as well.

**Proposition 3.10.** *Let $H \leq K_1, K_2 \leq F(A)$ be such that $\Gamma_A(H)$ is an induced subgraph of both $\Gamma_A(K_1)$ and $\Gamma_A(K_2)$. Then, it is also an induced subgraph of $\Gamma_A(\langle K_1 \cup K_2 \rangle)$.*

***Proof:*** *It's clear that $\Gamma_A(\langle K_1 \cup K_2 \rangle)$ is constructed by joining $\Gamma_A(K_1)$ and $\Gamma_A(K_2)$ at their basepoints, and folding and trimming them. Since $\Gamma_A(H)$ is a subgraph of both, it will fold completely until we have one copy of $\Gamma_A(H)$ in the overall Stallings graph; we perform those foldings first.*

*Consider the resulting automaton after those two actions. Since $\Gamma_A(H)$ is an induced subgraph of both $\Gamma_A(K_1)$ and $\Gamma_A(K_2)$, no edge outside of $\Gamma_A(H)$ connects two vertices inside $\Gamma_A(H)$, and this still holds true now: otherwise, we'd have identified a vertex in $\Gamma_A(H)$ with a vertex outside $\Gamma_A(H)$, which in particular belongs in one or another of the $\Gamma_A(K_i)$s, contradicting the fact that $\Gamma_A(K_1)$ and $\Gamma_A(K_2)$ are deterministic.*

*Every folding missing now is a result of identifying vertices in $\Gamma_A(K_1)$ and $\Gamma_A(K_2)$ but outside $\Gamma_A(H)$, therefore $\Gamma_A(H)$ remains an induced subgraph of $\Gamma_A(\langle K_1 \cup K_2 \rangle)$.*

This last result tempts us to say that into extensions in general have such good behaviour under the "union" of subgroups, but that is not immediately the case; as it only applies to those into extensions in which the subgraph is also an induced subgraph. Nevertheless, we haven't found any into extensions in which this rule does not apply; in other words, it seems likely that Theorem 3.6 is indeed an equivalence, which we conjecture to be the case.

**Conjecture 3.3.** *Let $H \leq K \leq F(A)$ be an extension of finitely generated free groups. Then $H \leq_{into} K$ if, and only if, $\Gamma_A(H)$ is an induced subgraph of $\Gamma_A(K)$.*

*If this is true for some basis, it is also directly true for every basis.*

It appears that a proof could be straightforward, at least informally, following a similar procedure as we have, imitating Kolodner's idea. However, this requires having some control over what the images of an automorphism (not just a homomorphism) of $F(A)$ may be in order to discard conclusively that a non-induced subgraph may be an into extension, as well as some control over "how much" not of an induced subgraph we're dealing with.

## 3.5 The lattice of algebraic extensions

By Proposition 3.5 on the composition of algebraic extensions, we deduce that $AE(H)$ is a partially ordered set by the relation $\leq_{alg}$. In this section, we'll explore this poset and some of its properties (cardinality, structure, and others). Throughout the section, we'll assume the subgroups are finitely generated unless explicitly stated otherwise.

We will say an algebraic extension $H \leq_{alg} K$ is **covering** (or that $K$ *covers* $H$) if $H \leq_{alg} L \leq K \Rightarrow L = K$ or $L = H$. This terminology is directly taken from the notation in posets.

Observe that the set of algebraic extensions is never empty, as we always have that $H \in AE(H)$. This, combined with the following lemma, allows us to say that the set of algebraic extensions is unique for every subgroup.

**Lemma 3.8.** *Let $H <_{alg} K$. Then, $AE(K) \subset AE(H)$. Moreover, any sequence of finitely generated subgroups:*

$$H_0 \leq_{alg} H_1 \leq_{alg} ... \leq_{alg} H_n \leq_{alg} ...$$

*Is stationary.*

A basic question we may ask about our poset $AE(H)$ is whether it has a unique maximal element or not. As the following proposition shows (3.11), that is indeed the case.

**Proposition 3.11.** *Let $H \leq_{f.g.} F(A)$ be a subgroup. Then, the poset $(AE(H), \leq_{alg})$ has a unique maximal element.*

***Proof:*** *By the previous Lemma (3.5), either $H \leq_{alg} F(A)$, or $H \leq_{alg} K <_{f.f.} F(A)$. If $H \leq_{alg} F(A)$, then we're done, and $F(A)$ is such maximal element. If not, suppose we have two different maximal elements, $K_1, K_2 \in AE(H)$, which satisfy:*

$$K_i \leq_{alg} L : L \in AE(H) \Rightarrow L = K_i$$

*By 3.5, we know that $H \leq_{alg} \langle K_1 \cup K_2 \rangle$ is also an algebraic extension, and $K_i \leq_{alg} \langle K_1 \cup K_2 \rangle$ is also algebraic, so $K_1 = K_2 = \langle K_1 \cup K_2 \rangle$, and we have a unique maximal element.*

An important class of posets are lattices. A poset $P$ is a **lattice** if, for any $x, y \in P$ there exists unique and well defined **supremum** (join) and **infimum** (meet) elements in $P$:

$$x \vee y \in P \; : \; x \leq z, y \leq z \Rightarrow x \vee y \leq z$$

$$x \wedge y \in P \; : \; x \geq z, y \geq z \Rightarrow x \wedge y \geq z$$

We can show that, for any non-trivial $H \leq_{alg} F(A)$, the set of algebraic extensions, $AE(H)$, is a lattice, as in Theorem 3.7.

**Theorem 3.7.** *Let $H \leq_{alg} F(A)$ be a non-trivial subgroup. Then, $AE(H)$ is a lattice with meet and join operations defined, for every $K, L \in AE(H)$, as:*

$$K \vee L = \langle K \cup L \rangle$$

$$H \leq_{alg} K \wedge L \leq_{f.f.} K \cap L$$

*That is, $K \wedge L$ is the largest algebraic extension inside their intersection.*

***Proof:*** *By 3.5 and the fact that $AE(H)$ is a minimal Takahasi family, those two elements are indeed algebraic extensions of $H$ and they are uniquely defined. It's also trivial to check that these satisfy the definition of the join and meet operations: firstly, if $K, L \leq_{alg} R$, then $\langle K \cup L \rangle \leq_{alg} R$, trivially. Secondly, if $H \leq_{alg} R \leq_{alg} K, L$, then $H \leq_{alg} R \leq K \cap L$. By Proposition 3.4, we find that $K \wedge L \leq_{f.f.} R$, which implies $R = K \wedge L$.*

A strong property in lattices is *distribution*. A lattice is **distributive** if is satisfies:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) : \forall x, y, z \in P$$

Although the subgroups of $F(A)$ (or any group, for that matter) are a lattice, they are not a distributive one in general: in the lattice of subgroups of $F(A)$, join and meet operations are the generated by the union and intersection of subgroups. If we consider $X = \langle a \rangle$, $Y = \langle b \rangle$, and $Z = \langle ab \rangle$, we find:

$$X \wedge (Y \vee Z) = X \cap \langle b, ab \rangle = X \cap F_2 = X$$

$$(X \wedge Y) \vee (X \wedge Z) = 1 \vee 1 = 1$$

The lattice of algebraic extensions is **not** distributive either, as the following counterexample shows.

**Example 3.7** (Algebraic extensions aren't distributive)**.** *Consider $H = \langle a^2, b^2, ab^2a \rangle$, whose $A$-fringe is as in Figure 3.8. We deduce all the elements in the $A$-fringe are algebraic extensions due to the rank: no identification increases rank, so by the Silva-Weil Theorem 3.3 they are not free factors, and thus are algebraic.*

*Note that, in this lattice, we have, on the one hand:*

$$K_1 \wedge (K_2 \vee K_3) = K_1 \wedge F_2 = K_1$$

*But on the other hand:*

$$(K_1 \wedge K_2) \vee (K_1 \wedge K_3) = H \vee H = H$$

*As $K_1 \neq H$, we see the lattice does not satisfy the condition of distribution for all the algebraic extensions, thus $AE(H)$ is not a distributive lattice.*

Since the set of algebraic extensions is a lattice, although not necessarily distributive, we may raise the question: is any (finite) lattice isomorphic to the set of algebraic extensions of some subgroup of $F(A)$, for some $A$? In the following theorems we explore the structure

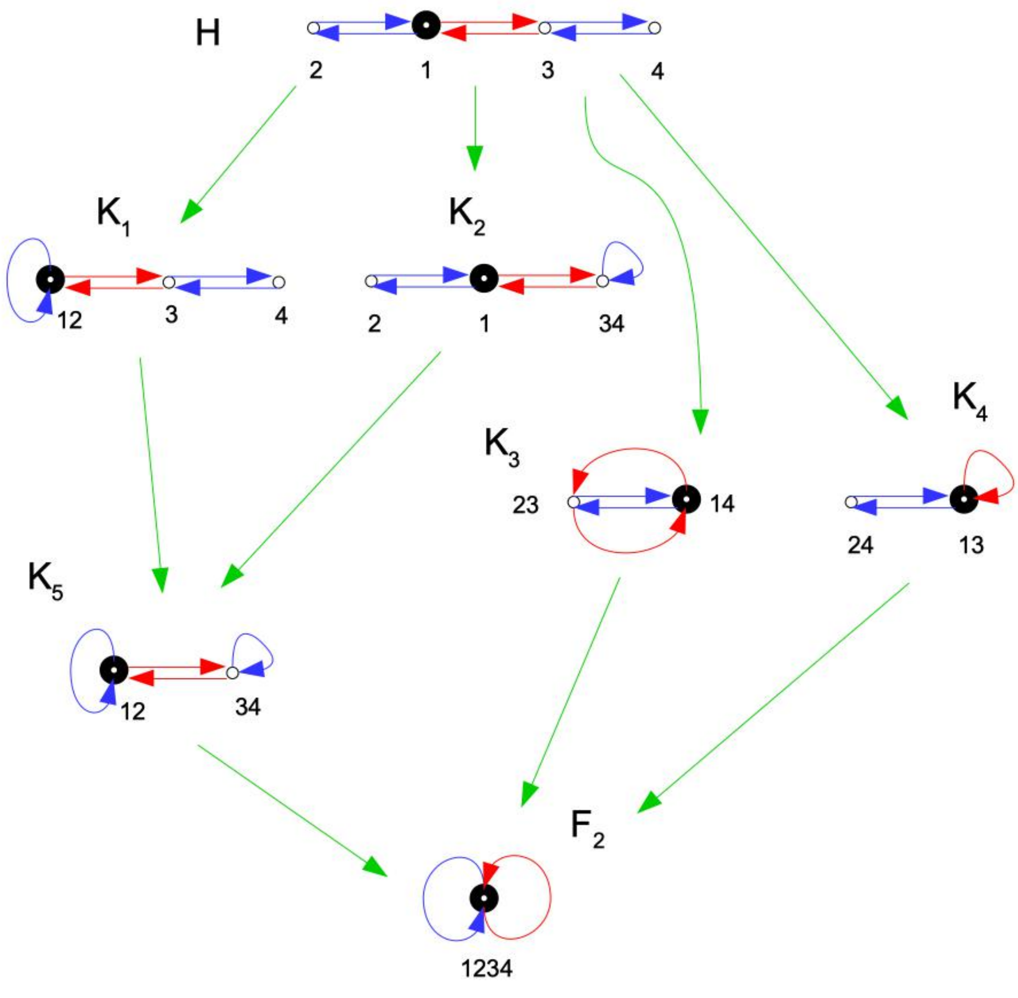


**Figure 3.8**: *Algebraic extensions lattice of $H = \langle a^2, b^2, ab^2a \rangle$, which is also the whole A-fringe.*

of the sets of algebraic extensions to answer this question, and perhaps characterise some particular structures, like singletons, pairs, chains, and more.

**Lemma 3.9.** *Let $H, H' \leq_{f.g.} F(A)$ be conjugate, $H' = g^{-1}Hg = H^g$ for some $g \in F(A)$. Then:*

$$AE(H') = AE(H)^g = \{g^{-1}Kg \ : \ K \in AE(H)\}$$

***Proof:*** *Let $K \in AE(H)$, so $H \leq L \leq_{f.f.} K \Rightarrow K = L$, and let $K' = K^g$. Trivially, $H' \leq K'$, and if $H' \leq L' \leq_{f.f.} K'$, then $H'^{g^{-1}} = H \leq L'^{g^{-1}} \leq_{f.f.} K'^{g^{-1}} = K$, which implies $K = L'^{g^{-1}}$, thus $L' = K'$ since conjugation is an automorphism. Therefore, $AE(H)^g \subseteq AE(H')$.*

*By a symmetric argument, we have the other inclusion, hence we conclude the two sets are equal.*

This lemma implies that, if some group's algebraic extensions has a certain structure, we can replicate this structure infinitely many times simply by conjugating. This can in fact be generalised, by an identical proof, to any automorphism (Lemma 3.10).

**Lemma 3.10.** *Let $H \leq_{f.g.} F(A)$ and let $\varphi \in Aut(F(A))$ be an automorphism. Then:*

$$AE(\varphi(H)) = \varphi(AE(H)) = \{\varphi(K) \ : \ K \in AE(H)\}$$

***Proof:*** *Identical to the proof for 3.9.*

Some of the structures we may want to characterise are those that are small (such as of one or two elements) or very simple (such as chains). Theorem 3.8 characterises singletons, and Theorem 3.9 characterises when there are only two algebraic extensions.

**Theorem 3.8.** *Let $H \leq_{f.g.} F(A)$. Then:*

$$H \leq_{f.f.} F(A) \Leftrightarrow AE(H) = \{H\}$$

***Proof:*** *Suppose $AE(H) = \{H\}$. Then, as the algebraic extensions of $H$ form a Takahasi family, we have that:*

$$H \leq K \Rightarrow H \leq_{f.f.} K$$

*In particular,* $H \leq_{f.f.} F(A)$. *Observe we showed a stronger result: if* $AE(H) = \{H\}$, *then it is a free factor of any extension of itself.*

*On the other hand, suppose that* $H \leq_{f.f.} F(A)$. *By Lemma 3.2,* $s = rk(H) \leq rk(F(A)) = r$. *If* $r = s$, *then* $H = F(A)$ *and the result is trivial, so suppose* $r > s$. *Since* $H$ *is a free factor, there exists a basis of* $H$, $\{h_1, ..., h_s\}$, *that extends to a basis of* $F(A)$, $B = \{h_1, ..., h_s, t_1, ..., t_{r-s}\}$, *thus* $F(A) = \langle A \rangle = \langle B \rangle$.

*With this change in the ambient basis, we now have that* $\Gamma_B(H)$ *has exactly one vertex and* $s$ *edges, all of which loops with labels* $h_1, ..., h_s$. *In this setup, it is clear that* $\mathcal{O}_B(H) = \{H\}$. *Since* $AE(H) \subseteq \mathcal{O}_B(H)$ *for any basis* $B$, *we conclude that* $AE(H) = \{H\}$.

In the case that $F(A) = F_2$ we can provide stronger conditions on when a subgroup is a free factor of the free group. We can say that, in general, the previous theorem tells us that $F(A)$ is an algebraic extension, but in $F_2$ we can strengthen that statement, and say that any non commutative subgroup of $F_2$ extends algebraically to $F_2$.

**Proposition 3.12.** *Let* $F_2 = F(\{a, b\})$, *and let* $H \leq F_2$. *Then,* $rk(H) \geq 2 \Rightarrow H \leq_{alg} F_2$.

***Proof:*** *Let* $H \leq F_2$ *be of rank at least 2. By 3.2, if* $K <_{f.f.} F_2$, *then* $rk(K) = 1$, *in which case* $L \leq K \Rightarrow rk(L) = 1$ *for any such* $L$. *Therefore,* $H$ *is not a subgroup of any proper free factor of* $F_2$, *so the extension is algebraic.*

This result does not generalise, however: consider $K = \langle ab, ac \rangle \leq_{f.g.} F_3$, and $H = \langle abacb^{-1}a^{-1}, ababacb^{-1}a^{-1}b^{-1}a^{-1}, ..., (ab)^s ac(ab)^{-s} \rangle$ for some $s \geq 3$. Then, clearly $H \leq K$ and it's finitely generated in $F_3$, with rank $s$. However, we can note that $K \leq_{f.f.} F_3$, as $\langle ab, ac, a \rangle = F_3$, so $H \leq F_3$ is not algebraic.

Just as we've characterised finitely generated subgroups whose set of algebraic extensions has exactly one element, we can characterise subgroups with exactly two algebraic extensions.

**Theorem 3.9.** *Let* $H \leq_{f.g.} F(A)$. *Then,* $AE(H) = \{H, F(A)\}$ *if, and only if, for any extension* $H \leq K < F(A)$, $H \leq_{f.f.} K$ *and* $H$ *is not a free factor of* $F(A)$.

***Proof:*** *Suppose first that $AE(H) = \{H, F(A)\}$. Then, since $AE(H)$ is a Takahasi family, for any extension $H \leq K < F(A)$ there is a subgroup in $AE(H)$ that is a free factor of $K$, thus $H \leq_{f.f.} K$. On the other hand, if $H$ is a free factor of any extension $H \leq K < F(A)$, and $AE(H)$ is a minimal Takahasi family, $H$ must be the only strict subgroup of $F(A)$ in $AE(H)$, therefore either $H \leq_{alg} F(A)$, or $H \leq_{f.f.} F(A)$. Since $H$ is not a free factor of the free group, then $AE(H) = \{H, F(A)\}$.*

**Corollary 3.9.1.** *Let $H \leq K \leq F(A)$ be finitely generated. Then, $AE(H) = \{H, K\}$ if, and only if, for any extension $H \leq L < K$, $H \leq_{f.f.} L$ and $H$ is not a free factor of $K$. Moreover, in such a case, $K \leq_{f.f.} F(A)$, and in fact $K \leq L \Rightarrow K \leq_{f.f.} L$.*

We may call **quasi primitive** subgroups those that satisfy the conditions of Theorem 3.9, that is, $|AE(H)| = 2$. Further, we can give some conditions in which $AE(H)$ is a chain (totally ordered as a poset).

**Theorem 3.10.** *Let $H \leq_{f.g.} F(A)$ be such that $AE(H)$ is a chain, so:*

$$AE(H) = \{H = H_0 \leq_{alg} H_1 \leq_{alg} \cdots \leq_{alg} H_n\}$$

*Then, for every extension $H \leq K$, $H_i \leq_{f.f.} K$ if, and only if, $i = max\{j \in [n] : H_j \leq K\}$.*

***Proof:*** *Let $i = max\{j \in [n] : H_j \leq K\}$ be such a maximum, and suppose $H_j \leq_{f.f.} K$ for some $j \leq i$, as by Takahasi there must be at least one such $H_j$ free factor of $K$. Since $H_i \leq_{f.f.} H_i$ and $H_j \leq_{f.f.} K$, by 3.3:*

$$H_i \cap H_j = H_j \leq_{f.f.} H_i = H_i \cap K$$

*But, by hypothesis, $H_j \leq_{alg} H_i$, which implies $i = j$.*

More generally, beyond characterising specific structures, we may inquire, as we said, whether for every lattice we may find a subgroup $H \leq_{f.g.} F(A)$ such that $(AE(H), \leq_{alg})$ is isomorphic to said lattice. This is what we will refer to as the **lattice realisation problem** of algebraic extensions.

For that, a preliminary matter we may tackle is, given some subgroup $K \leq_{f.g.} F(A)$, can we always find a subgroup $H \leq K$ such that $K$ covers $H$ in $AE(H)$? By Lemma 3.9, this implies finding infinitely many such subgroups.

**Proposition 3.13.** *Let $K \leq_{f.g.} F(A)$ be a finitely generated subgroup. Then, there exists a $H \leq K$ such that $K$ covers $H$ in $AE(H)$.*

***Proof:*** *Let $H \leq K$. If $H \leq_{alg} K$, then there exists at least one subgroup in $AE(H)$ with $L \leq_{alg} K$ such that $K$ covers $L$. Let $\{k_1, ..., k_r\}$ be a basis of $K$. Then $H = \langle k_1^2, k_2, ..., k_r \rangle \leq_{alg} H$ by Theorem 3.3, as it is a proper subgroup of the same rank.*

This may seem sufficient to show that we can construct any lattice as $AE(H)$ for some $H$: take one such subgroup of $F(A)$ (or any free group) as in 3.13, and conjugate it until we find as many as we want below our maximal element, then iterate this process on the ones we've constructed, and so on. However, this process breaks down, as the new subgroups can easily have trivial intersection, so there may be no common subgroup so that both are an algebraic extension of it. In general, we cannot control what their intersections will be.

Nevertheless, there seem to be no restrictions on what structures we may demand from our lattices. We may have arbitrarily long chains, and an arbitrary number of descendants in either direction (subgroups $H_i \leq_{alg} H$ or $H \leq_{alg} K_i$ for some $H$). In all, we conjecture that every lattice is realisable.

**Conjecture 3.4.** *Let $(L, \leq)$ be a finite lattice. Then, there exists $H \leq_{f.g.} F(A)$ for some finite alphabet $A$ such that $(AE(H), \leq_{alg}) \cong (L, \leq)$ as lattices.*

## 3.6 Open problems

In this chapter we have explored several questions which, despite our best efforts, we have not been able to fully resolve, and which we believe may be of interest in further research. We list them below, reasoning some of our findings and intuitions about them.

1. A "dual" notion to onto extensions and a version of Takahasi's theorem for onto extensions. Just as we extended results about algebraic extensions to onto extensions, we believe that the same can be done for Takahasi's theorem. However, this requires setting a "dual" to onto extensions, just like algebraic extensions and free factors are "dual" of each other. Into extensions seem poised as such a dual, as they satisfy properties analogous to free factors. However, this is not sufficient for an analog of Takahasi's Theorem, at least directly: the extension $\langle aba^{-1}b^{-1}\rangle \leq_{alg} \langle a, b\rangle$ is the only onto extension, but $\langle a, b\rangle \leq_{f.f.}$ is not into, so more research is needed on this topic to clear up the relationship (if any) between those notions.

2. We believe into extensions can be characterised in terms of induced subgraphs, as we noted in conjecture 3.3.

3. We believe an effective characterisation of subgroups with totally ordered sets of algebraic extensions is possible. A notable attempt was conjecturing that, if $H$ is covered in $AE(H)$ by exactly one subgroup $K \in AE(H)$, then $AE(H)$ is a chain (and vice versa). However, this turned out not to be true, taking for example $H = \langle a^2b^2, c, a^2cb^2\rangle$.

4. In line with 3, we believe characterisations for when the set of algebraic extensions is distributive (perhaps using forbidden structures as in Birkhoff's Theorem) can be obtained.

5. As we conjectured at the end of the previous section, we believe the lattice realisation problem resolves positively.

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