# The conjugacy problem and other algorithmically related questions

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Universitat Politècnica de Catalunya

EMS-SCM joint meeting

Barcelona

May 28th, 2015.

# Mathematics

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Algebra

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Group Theory

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Discrete groups

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focus on algorithmic questions

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- J. González-Meneses, E. Ventura, Twisted conjugacy in the braid group, **Israel Journal of Mathematics** 201 (2014), 455–476.
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# Outline

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# Presentations of groups

**Definition**

*A finite presentation of a (discrete) group G is*

$$G = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle.$$

- $a_1, \dots, a_n$ are the generators;
- $r_1, \dots, r_m$ are the relators;
- elements of $G$ are words (i.e., non-commutative! formal products) of the $a_i^{\pm 1}$'s, subject to the rules $r_j = 1$.

**Example**

- $\mathbb{Z} = \langle a \mid - \rangle$; $\quad a^5 \cdot a^{-3} = a^2$
- $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid ab = ba \rangle$; $\quad ba \cdot ba^{-2} = a^{-1}b^2$
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*Which group is* $G = \langle a, b \mid a^{-1}ba = b^2,\ b^{-1}ab = a^2 \rangle$ *?*

$$a^{-1}ba = b^2 \Rightarrow b^{-1}a^{-1}ba = b \Rightarrow (b^{-1}ab)^{-1}a = b$$
$$\Rightarrow a^{-2}a = b \Rightarrow a^{-1} = b.$$

*But then*

$$a^{-1}ba = b^2 \Rightarrow a^{-1} = b^2 = a^{-2} \Rightarrow a = 1$$
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*Hence,* $G = 1$ *is the trivial group.*

It is not easy, in general, to recognize $G$ from a given presentation.

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# Dehn's problems

## Word Problem, $WP(G)$

*For any given presentation $G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$, find an algorithm $\mathcal{W}$ with:*

- ***Input:*** *a word $w(a_1, \ldots, a_n)$ on the $a_i^{\pm 1}$'s;*
- ***Output:*** *"yes" or "no" depending on whether $w =_G 1$.*

## Conjugacy Problem, $CP(G)$

*For any given presentation $G = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$, find an algorithm $\mathcal{C}$ with:*

- ***Input:*** *two words $u(a_1, \ldots, a_n)$ and $v(a_1, \ldots, a_n)$ ;*
- ***Output:*** *"yes" or "no" depending on whether u and v are conjugate in G, $u \sim_G v$ (i.e., $v =_G g^{-1}ug$ for some $g \in G$).*

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# Dehn's problems

## Isomorphism Problem

*Find an algorithm $\mathcal{I}$ with:*

- ***Input:*** *two presentations* $G_i = \langle a_1, \ldots, a_{n_i} \mid r_1, \ldots, r_{m_i} \rangle$, $i = 1, 2$;
- ***Output:*** *"yes" or "no" depending on whether* $G_1 \simeq G_2$ *as groups.*

## Theorem (Novikov '55; Boone '58)

*There exist finitely presented groups with unsolvable word problem.*

## Theorem (Adyan '57; Rabin '58)

*The Isomorphism Problem is unsolvable.*

## Theorem (Miller '71)

*There exists a finitely presented group G with solvable word problem but unsolvable conjugacy problem.*

# Dehn's problems

**Isomorphism Problem**

*Find an algorithm $\mathcal{I}$ with:*

- ***Input:*** *two presentations* $G_i = \langle a_1, \ldots, a_{n_i} \mid r_1, \ldots, r_{m_i} \rangle$, $i = 1, 2$;
- ***Output:*** *"yes" or "no" depending on whether* $G_1 \simeq G_2$ *as groups.*

**Theorem (Novikov '55; Boone '58)**

*There exist finitely presented groups with unsolvable word problem.*

**Theorem (Adyan '57; Rabin '58)**

*The Isomorphism Problem is unsolvable.*

**Theorem (Miller '71)**

*There exists a finitely presented group G with solvable word problem but unsolvable conjugacy problem.*

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# Outline

Step 1:

Find a problem you like

(2004)

# Conjugacy problem for free-by-cyclic groups

**Definition**

*Let $F_n = \langle a_1, \ldots, a_n \mid - \rangle$ be a free group on $\{a_1, \ldots, a_n\}$ ($n \geq 2$), and let $\varphi \in Aut(F_n)$. The free-by-cyclic group $F_n \rtimes_\varphi \mathbb{Z}$ is defined as*

$$F_n \rtimes_\varphi \mathbb{Z} = \langle a_1, \ldots, a_n, t \mid t^{-1} a_i t = a_i \varphi \rangle.$$

**Observation**

*The word problem in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$ is solvable.*

**Open problem since 2004**

*Solve the conjugacy problem in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.*

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Let's consider an example: $M_\varphi = \langle a, b, t \mid t^{-1}at = a\varphi,\ t^{-1}bt = b\varphi \rangle$

$$\begin{array}{rcl} \varphi\colon F_2 & \to & F_2 \\ a & \mapsto & ab \\ b & \mapsto & aba \end{array} \qquad \begin{array}{rcl} \varphi^{-1}\colon F_2 & \to & F_2 \\ a & \mapsto & a^{-1}b \\ b & \mapsto & b^{-1}a^2 \end{array}$$

$$wt = t(w\varphi) \qquad wt^{-1} = t^{-1}(w\varphi^{-1})$$

$$\begin{aligned} tab^{-1}t^{-1}at^2a &= tab^{-1}t^{-1}t(ab)ta = tab^{-1}abta \\ &= tab^{-1}at(aba)a = tab^{-1}ataba^2 \\ &= tab^{-1}t(ab)aba^2 \\ &= tat(a^{-1}b^{-1}a^{-1})ababa^2 = tatba^2 \\ &= tt(ab)ba^2 = t^2ab^2a^2. \end{aligned}$$

**Lemma**

*Every element from $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$ has a unique normal form:*

$$t^r w \quad \text{for some } r \in \mathbb{Z},\ w \in F_n.$$

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Step 2:

Push the problem into your

favorite territory

(2005)

# Converting it into a free group problem

Let $t^r u$, $t^s v$, $t^k g$ be arbitrary elements in $M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$. Then,

$$\begin{aligned}(g^{-1}t^{-k})(t^r u)(t^k g) &= g^{-1}t^r(u\varphi^k)g \\ &= t^r(g\varphi^r)^{-1}(u\varphi^k)g.\end{aligned}$$

$$t^r u \sim_{M_\varphi} t^s v \iff r = s \quad \& \quad v \sim_{\varphi^r} (u\varphi^k) \text{ for some } k \in \mathbb{Z}.$$

**Definition**

*For $\phi \in Aut(G)$, two elements $u, v \in G$ are said to be $\phi$-twisted conjugated, denoted $u \sim_\phi v$, if $v = (g\phi)^{-1}ug$ for some $g \in G$.*

**Twisted Conjugacy Problem, $TCP(G)$**

*The twisted conjugacy problem for $G$, denoted $TCP(G)$: Given $\phi \in Aut(G)$ and $u, v \in G$ decide whether $u \sim_\phi v$.*

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Step 3:

Solve it

(2005)

# $CP(F_n \rtimes_\varphi \mathbb{Z})$ is solvable

**Theorem (Bogopolski–Martino–Maslakova–V., 2005)**

*$TCP(F_n)$ is solvable.*

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**Proof.** Given $t^r u, t^s v \in M_\varphi = F_n \rtimes_\varphi \mathbb{Z}$.

- $t^r u \sim_{M_\varphi} t^s v \iff r = s \;\&\; v \sim_{\varphi^r} (u\varphi^k)$ for some $k \in \mathbb{Z}$.
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Step 4:

Ups ... a technical problem!

(2005)

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- **Case 1:** $r \neq 0$
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- **Case 2:** $r = 0$
  - Still infinitely many $k$'s to check:

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  - Fortunately, this is precisely Brinkmann's result:

**Theorem (Brinkmann, 2006)**

*Given an automorphism $\phi\colon F_n \to F_n$ and $u, v \in F_n$, it is decidable whether $v \sim u\phi^k$ for some $k \in \mathbb{Z}$.*

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# Outline

Step 5:

Intuition always ahead

(2006)

# A crucial comment

Armando Martino: *"The whole argument essentially works the same way in presence of more stable letters, i.e., for free-by-free groups"*

**Definition**

*Let $F_n = \langle x_1, \ldots, x_n \mid \ \rangle$ be the free group on $\{x_1, \ldots, x_n\}$ ($n \geq 2$), and let $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$. The free-by-free group $F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$ is*

$$M_{\varphi_1,\ldots,\varphi_m} = F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m = \langle x_1, \ldots, x_n,\ t_1, \ldots, t_m \mid t_j^{-1} x_i t_j = x_i\varphi_j \rangle.$$

But this must be wrong ...

**Theorem (Miller '71)**

*There exist free-by-free groups with unsolvable conjugacy problem.*

Surprise was that ... ... Armando was "essentially" right !!

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Step 6:

Extend as much as possible

(2007)

# Outline

# Orbit decidability

**Definition**

*Let $X$ be a set. A collection of maps $A \subseteq \mathrm{Map}(X, X)$ is said to be orbit decidable (O.D.) if there is an algorithm $\mathcal{A}$ with:*

- ***Input:** two elements $x, y \in X$;*
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**Definition**

*For $X$, $A \subseteq \mathrm{Map}(X, X)$, the A-orbit of $x \in X$ is $\mathcal{O}(x) = \{x\alpha \mid \alpha \in A\}$.*

**Observation**

*O.D. is membership in A-orbits.*

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*The conjugacy problem for group G, CP(G), is just the O.D. for $A = \mathrm{Inn}(G) = \{\gamma_g \colon G \to G,\ x \mapsto g^{-1}xg \mid g \in G\} \trianglelefteq Aut(G)$.*

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# Short exact sequences

**Observation**

(i) *For $\varphi \in Aut(F_n)$, we have the natural short exact sequence:*

$$\begin{array}{ccccccccc} 1 & \to & F_n & \to & F_n \rtimes_\varphi \mathbb{Z} & \to & \mathbb{Z} & \to & 1 \\ & & & & x_i & \mapsto & 1 & & \\ & & & & t & \mapsto & t & & \end{array}$$

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(iii) *And their action subgroups are, respectively, $\langle \varphi \rangle \leqslant Out(F_n)$ and $\langle \varphi_1, \ldots, \varphi_m \rangle \leqslant Out(F_n)$.*

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(iii) *And their action subgroups are, respectively, $\langle \varphi \rangle \leqslant Out(F_n)$ and $\langle \varphi_1, \ldots, \varphi_m \rangle \leqslant Out(F_n)$.*

# Short exact sequences

## Observation

(i) *For $\varphi \in Aut(F_n)$, we have the natural short exact sequence:*

$$\begin{array}{ccccccccc} 1 & \to & F_n & \to & F_n \rtimes_\varphi \mathbb{Z} & \to & \mathbb{Z} & \to & 1 \\ & & & & x_i & \mapsto & 1 & & \\ & & & & t & \mapsto & t & & \end{array}$$

(ii) *For $\varphi_1, \ldots, \varphi_m \in Aut(F_n)$, we have the natural short exact sequence:*

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# Short exact sequences

**Definition**

*Consider an arbitrary short exact sequence of groups,*

$$1 \to F \to G \to H \to 1.$$

*Given $g \in G$, consider $\gamma_g : G \to G$, which restricts to an automorphism $\gamma_g|_F : F \to F$. Then, the action subgroup of the short exact sequence is:*

$$A = \{\gamma_g|_F \mid g \in G\} \leqslant Aut(F)$$

# Short exact sequences

Idea: ... our argument extends to arbitrary short exact sequences (... satisfying the conditions needed).

To solve $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$ we have needed:

- $TCP(F_n)$,
- orbit decidability of $\langle \varphi_1, \ldots, \varphi_m \rangle \in \mathrm{Aut}(F_n)$,
- computability up and down the short exact sequence.

These conditions (plus two more) will suffice ...

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These conditions (plus two more) will suffice ...

# The main result

**Theorem (Bogopolski-Martino-V., 2008)**

*Let*

$$1 \longrightarrow F \stackrel{\alpha}{\longrightarrow} G \stackrel{\beta}{\longrightarrow} H \longrightarrow 1$$

*be an algorithmic short exact sequence of groups such that*

(i) *$TCP(F)$ is solvable,*

(ii) *$CP(H)$ is solvable,*

(iii) *there is an algorithm which, given an input $1 \neq h \in H$, computes a finite set of elements $z_{h,1}, \ldots, z_{h,t_h} \in H$ such that*

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

*Then,*

$$CP(G) \text{ is solvable} \iff A_G = \left\{ \begin{array}{rccc} \gamma_g \colon & F & \to & F \\ & x & \mapsto & g^{-1}xg \end{array} \,\middle|\, g \in G \right\} \leqslant$$

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**Proposition (Bogopolski–Martino–V., 2008)**

*Torsion-free hyperbolic groups (in particular, free groups) satisfy hypothesis (ii) and (iii).*

So, they all fit well as $H$.

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# Outline

# Free-by-free groups

**Theorem (Bogopolski–Martino–Maslakova–V., 2005)**

*$TCP(F_n)$ is solvable.*

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**Theorem (Brinkmann, 2006)**

*Cyclic subgroups of $Aut(F_n)$ are O.D.*

**Corollary (Bogopolski–Martino–Maslakova–V., 2005)**

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**Theorem (Whitehead '36)**

*The full $Aut(F_n)$ is O.D.*

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*If $\langle \varphi_1, \ldots, \varphi_m \rangle = Aut(F_n)$ then $CP(F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$ is solvable.*

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$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m \longrightarrow F_m \longrightarrow 1$$

**Observation (linear algebra)**

*$TCP(\mathbb{Z}^n)$ is solvable.*

So,

$$CP(\mathbb{Z}^n \rtimes_{M_1,\ldots,M_m} F_m) \text{ is solvable } \Leftrightarrow \langle M_1, \ldots, M_m \rangle \leqslant GL_n(\mathbb{Z}) \text{ is O.D.}$$

Observe that now $M_i \in \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$ are just $n \times n$ invertible matrices.

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*Cyclic subgroups of $GL_n(\mathbb{Z})$ are O.D.*

**Corollary (Remeslennikov '69)**

*$\mathbb{Z}^n$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.*

**Observation (elementary)**

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# Braid-by-free groups

Consider the braid group on $n$ strands, given by the classical presentation:

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geqslant 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (1 \leqslant i \leqslant n-2) \end{array} \right\rangle .$$

- $CP(B_n)$ is solvable.
- And the automorphism group is easy:

**Theorem (Dyer–Grossman '81)**

*$|Out(B_n)| = 2$; more precisely, $Aut(B_n) = Inn(B_n) \sqcup Inn(B_n) \cdot \varepsilon$, where $\varepsilon \colon B_n \to B_n$ is the automorphism which inverts all generators, $\sigma_i \mapsto \sigma_i^{-1}$.*

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*$|Out(B_n)| = 2$; more precisely, $Aut(B_n) = Inn(B_n) \sqcup Inn(B_n) \cdot \varepsilon$, where $\varepsilon \colon B_n \to B_n$ is the automorphism which inverts all generators, $\sigma_i \mapsto \sigma_i^{-1}$.*

# Braid-by-free groups

Consider the braid group on $n$ strands, given by the classical presentation:

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geqslant 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (1 \leqslant i \leqslant n-2) \end{array} \right\rangle.$$

- $CP(B_n)$ is solvable.
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Consider Thompson's group $F$:

$$F = \left\{ f\colon [0,1] \to [0,1] \mid f \begin{array}{l} -\text{increasing and piecewise linear,} \\ -\text{with finitely many dyadic breakpoints,} \\ -\text{slopes being powers of 2.} \end{array} \right\}$$

- $CP(F)$ is solvable.
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**Theorem (Brin '97)**

*For every $\varphi \in Aut(F)$, there exists $\tau \in EP_2$ such that $\varphi(g) = \tau^{-1} g \tau$, for every $g \in F$.*

$$F \trianglelefteq EP_2 = \left\{ f\colon \mathbb{R} \to \mathbb{R} \mid \begin{array}{l} f \text{ is p.l., dyadic bkp., slopes } 2^n \\ \text{eventually periodic} \end{array} \right\}.$$

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**Conjecture**

*$k-CP(F)$ (i.e., conjugacy problem for k-tuples) is solvable.*

**Proposition (Burillo–Matucci–V. 2010)**

*If conjecture is true then $Aut(F)$ and $Aut^+(F)$ are orbit decidable.*

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*If conjecture is true and $\varphi_1, \ldots, \varphi_m \in Aut(F)$ generate either $Aut(F)$ or $Aut^+(F)$, then $CP(F \rtimes_{\varphi_1,\ldots,\varphi_m} F_m)$ is solvable.*

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# Outline

# Free-by-free negative results

**Theorem (Miller '71)**

*There exist free-by-free groups with unsolvable conjugacy problem.*

**Corollary**

*There exist 14 automorphisms $\varphi_1, \ldots, \varphi_{14} \in Aut(F_3)$ such that $\langle \varphi_1, \ldots, \varphi_{14} \rangle \leqslant Aut(F_3)$ is orbit undecidable.*

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# Finding orbit undecidable subgroups

**Observation (Bogopolski–Martino–V., 2008)**

*Let $F$ be a group, and let $A \leqslant B \leqslant Aut(F)$ and $u \in F$ be such that $B \cap Stab^*(u) = 1$. Then, $A$ is O.D. $\Rightarrow$ $MP(A, B)$ solvable.*

***Proof.*** *Given $\varphi \in B \leq Aut(F)$, let $w = u\varphi$ and*

$$\{\phi \in B \mid u\phi \sim w\} = (B \cap Stab^*(u)) \cdot \varphi = \{\varphi\}.$$

*So, $u$ can be mapped to a conjugate of $w$ by some automorphism in $A$* $\Leftrightarrow$ $\varphi \in A$. $\square$

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***Proof.*** *By Mihailova's construction:*

- *Take a group $U = \langle a_1, a_2 \mid r_1, \ldots, r_m \rangle$ with unsolvable word problem;*
- *Consider $A = \{(v, w) \mid v =_U w\} \leqslant F_2 \times F_2$;*
- *Easy to see that $A = \langle (a_1, a_1), (a_2, a_2), (r_1, 1), \ldots, (r_m, 1) \rangle$ so, $A$ is finitely generated;*
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**Corollary (Bogopolski–Martino–V., 2008)**

*$Aut(F_r)$ contains f.g. orbit undecidable subgroups, for $r \geqslant 3$.*

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- Aut $(B_n)$ does not contain $F_2 \times F_2$;
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# (Free abelian)-by-free negative results

## For free abelian groups

**Corollary (Bogopolski–Martino–V., 2008)**

$GL_d(\mathbb{Z})$ *contains f.g. orbit undecidable subgroups, for* $d \geqslant 4$.

*Proof.*

- $F_2 \simeq \left\langle P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \leq_{24} GL_2(\mathbb{Z})$.
- $Stab(1,0) = \{M \mid (1,0)M = (1,0)\} = \left\{ \begin{pmatrix} 1 & 0 \\ n & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$.
- $\langle P, Q\rangle \cap Stab(1,0) = \left\langle \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \right\rangle$.
- Choose a free subgroup $F_2 \simeq \langle P', Q'\rangle \leq \langle P, Q\rangle$ such that $\langle P', Q'\rangle \cap Stab(1,0) = \{I\}$ and consider

$$B = \left\langle \left(\begin{array}{c|c} P' & 0 \\ \hline 0 & I \end{array}\right), \left(\begin{array}{c|c} Q' & 0 \\ \hline 0 & I \end{array}\right), \left(\begin{array}{c|c} I & 0 \\ \hline 0 & P' \end{array}\right), \left(\begin{array}{c|c} I & 0 \\ \hline 0 & Q' \end{array}\right) \right\rangle \leq GL_4(\mathbb{Z}).$$

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# (Free abelian)-by-free negative results

## For free abelian groups

**Corollary (Bogopolski–Martino–V., 2008)**

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- *Note that $B \simeq F_2 \times F_2$.*
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**Proposition (Bogopolski–Martino–V., 2008)**

*Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.*

**Definition**

*A f.g. subgroup $A \leqslant GL_d(\mathbb{Z})$ is orbit decidable is there exists an algorithm $\mathcal{A}$ which, given two vectors $u, v \in \mathbb{Z}^n$ decides whether $v = uM$ by some matrix $M \in A$.*

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**Theorem (Bogopolski–Martino–V., 2008)**

*There exist 14 matrices $M_1, \ldots, M_{14} \in GL_d(\mathbb{Z})$, for $d \geqslant 4$, such that $\langle M_1, \ldots, M_{14} \rangle \leqslant GL_d(\mathbb{Z})$ is orbit undecidable.*

**Corollary (Bogopolski–Martino–V., 2008)**

*There exists a $\mathbb{Z}^4$-by-$F_{14}$ group with unsolvable conjugacy problem.*

**Question**

*Does $GL_3(\mathbb{Z})$ contain orbit undecidable subgroups ?*

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# Automata groups

**Proposition (Šunić–V., 2010)**

*For $d \geqslant 6$, the group $GL_d(\mathbb{Z})$ contains orbit undecidable, free subgroups.*

So, for $d \geqslant 6$, there exists a group of the form

$$\Gamma = \mathbb{Z}^d \rtimes F_m \leqslant \mathbb{Z}^d \rtimes GL_d(\mathbb{Z})$$

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**Theorem (Šunić–V., 2010)**

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Next step:

What about TCP in

your favorite group ?

# THANKS