Orbit decidability and the conjugacy problem in groups

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Outline

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Orbit decidability

Definition

Let X be a set. A collection of maps $A \subseteq \mathrm{Map}(X, X)$ is said to be orbit decidable (O.D.) if there is an algorithm s.t., given $x, y \in X$, it decides whether $x\alpha = y$ for some $\alpha \in A$ (and, if so, finds such an α).

Definition

For X, $A \subseteq \mathrm{Map}(X, X)$, the A-orbit of $x \in X$ is $\mathcal{O}(x) = \{x\alpha \mid \alpha \in A\}$.

Observation

O.D. is membership in a given A-orbit.

(Zoom into the problem)

- *Geometry: take X = space, A = action;*
- *Algebra: take X = algebraic structure, $A \subseteq \mathrm{End}(X)$;*
  - *Our case: $X = G$ group, $A \subseteq \mathrm{End}(G)$, $A \subseteq \mathrm{Aut}(G)$.*

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Classical examples

Theorem (Whitehead 1936)

There is an algorithm to decide, given $u, v \in F_r$, whether there exists $\alpha \in \mathrm{Aut}(F_r)$ s.t. $u\alpha = v$.

In other words: $\mathrm{Aut}(F_r)$ is O.D.

Variations with tuples of words, subgroups, tuples of subgroups, modulo conjugation, etc.

All these are instances of the Orbit Decidability problem.

Observation

The conjugacy problem for G is just the O.D. for $A = \mathrm{Inn}(G) \leqslant \mathrm{Aut}(G)$.

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First examples: $G = \mathbb{Z}^d$

Observation (folklore)

The full group $\mathrm{Aut}(\mathbb{Z}^d) = \mathrm{GL}_d(\mathbb{Z})$ *is orbit decidable.*

Proof. *For $u, v \in \mathbb{Z}^d$, there exists $A \in \mathrm{GL}_d(\mathbb{Z})$ such that $v = uA$ if and only if $gcd(u_1, \ldots, u_d) = gcd(v_1, \ldots, v_d)$.*

Proposition (Bogopolski–Martino–V., 2008)

Finite index subgroups of $GL_d(\mathbb{Z})$ are O.D.

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Proposition (linear algebra)

For $A \in \mathrm{GL}_d(\mathbb{Z})$, the subgroup $\langle A \rangle \leqslant GL_d(\mathbb{Z})$ is O.D.

Proof. *(sketch)*

- *Given $A \in \mathrm{GL}_d(\mathbb{Z})$, $u, v \in \mathbb{Z}^d$, want to decide wether $uA^n = v$ for some $n \in \mathbb{N}$.*
- *Keep computing $u, uA, uA^2, uA^3, \ldots$ and compare with v.*
- *Denote λ the eigenvalue of A with maximum modulus. The projection of uA^n to E_λ grows faster than all other projections.*
- *So we can compute n_0 such that either $u, uA, uA^2, uA^3, \ldots, uA^{n_0}$ hits v, or either $uA^n \neq v$ for all n.*

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Brinkmann's result

Theorem (Brinkmann, 2006)

Cyclic groups of $\mathrm{Aut}(F_r)$ are orbit decidable. That is, given $\varphi \in \mathrm{Aut}(F_r)$ and $u, v \in F_r$, one can decide whether $v = u\varphi^n$ for some $n \in \mathbb{Z}$.

Proof.

- *Same idea as before: there is a computable n_0 such that either $u, u\varphi, u\varphi^2, u\varphi^3, \ldots, u\varphi^{n_0}$ hits v, or either $u\varphi^n \neq v$ for all n.*
- *The computation of n_0 is quite complicated, making strong use of train-tracks.*

Theorem (Brinkmann, 2006)

Cyclic groups of $\mathrm{Aut}(F_r)$ are orbit decidable up to conjugacy. That is, given $\varphi \in \mathrm{Aut}(F_r)$ and $u, v \in F_r$, one can decide whether $v \sim u\varphi^n$ for some $n \in \mathbb{Z}$ (i.e., $\langle\varphi\rangle \cdot \mathrm{Inn}(F_r)$ is O.D.).

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Whitehead problem and variations

Theorem (Whitehead'30)

The full group $\mathrm{Aut}(F_r)$ is orbit decidable. That is, given $u, v \in F_r$ one can decide whether $v = u\alpha$ for some $\alpha \in \mathrm{Aut}(F_r)$ (also for tuples).

This is a classical and very influential result.

Proposition (Bogopolski–Martino–V., 2008)

Finite index subgroups of $\mathrm{Aut}(F_r)$ are O.D.

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Theorem (Makanin, 1982)

The full $\operatorname{End}(F_r)$ *is orbit decidable. That is, given* $u, v \in F_r$ *one can decide whether* $v = u\alpha$ *for some* $\alpha \in \operatorname{End}(F_r)$ *(also for tuples).*

Proof. *It reduces to solving (a system of) equations over* F_r.

Theorem (Ciobanu–Houcine, 2010)

$\operatorname{Mon}(F_r)$ *is orbit decidable. That is, given* $u, v \in F_r$ *one can decide whether* $v = u\alpha$ *for some injective endomorphism* $\alpha \in \operatorname{Mon}(F_r)$ *(also for tuples).*

Corollary

For every f.g. $H \leqslant F_r$, $\operatorname{Stab}(H)$ *is O.D (also for tuples, and similarly for monos and endos).*

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Definition

A virtual endomorphism of G is a homomorphism $\varphi\colon H \to K$ between finite index subgroups $H, K \leqslant_{\text{fi}} G$.

Theorem (Rubió–V., w.p.)

The collection of virtual endos (resp. virtual monos, virtual autos) of F_r is O.D. (also for tuples).

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Other groups

Theorem (Collins, Zieschang, 1984)

Let $G_1, \ldots, G_n$ be freely indecomposable groups with $\mathrm{Aut}(G_i)$ being O.D. Then, its free product $G = G_1 * G_2 * \cdots * G_n$ has $\mathrm{Aut}(G)$ O.D.

Theorem (Levitt–Vogtman, 2000)

For a surface group G, $\mathrm{Aut}(G)$ is O.D. (also for tuples).

Theorem (Dahmani, Girardel, 2010)

For a hyperbolic group G, $\mathrm{Aut}(G)$ is O.D. (also for tuples).

Theorem (Kharlampovich–V., 2012)

For G torsion-free relatively hyperbolic with abelian parabolic subgroups, $\mathrm{Aut}(G)$ is O.D. (also for tuples).

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Theorem (Collins, Zieschang, 1984)

Let $G_1, \ldots, G_n$ be freely indecomposable groups with $\mathrm{Aut}(G_i)$ being O.D. Then, its free product $G = G_1 * G_2 * \cdots * G_n$ has $\mathrm{Aut}(G)$ O.D.

Theorem (Levitt–Vogtman, 2000)

For a surface group G, $\mathrm{Aut}(G)$ is O.D. (also for tuples).

Theorem (Dahmani, Girardel, 2010)

For a hyperbolic group G, $\mathrm{Aut}(G)$ is O.D. (also for tuples).

Theorem (Kharlampovich–V., 2012)

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Outline

Finding orbit undecidable subgroups

Proposition (Bogopolski–Martino–V., 2008)

Let F be a group, and let $A \leqslant B \leqslant \mathrm{Aut}(F)$ and $u \in F$ be such that $B \cap Stab(u) = 1$. Then, A is O.D. $\Rightarrow$ $MP(A, B)$ solvable.

Proof. *Given $\varphi \in B \leq \mathrm{Aut}(F)$, let $w = u\varphi$ and*

$$\{\phi \in B \mid u\phi = w\} = (B \cap Stab(u)) \cdot \varphi = \{\varphi\}.$$

So, u can be mapped to w by somebody in A $\Leftrightarrow$ $\varphi \in A$. □

Let F be a group, and let $A \leqslant B \leqslant \mathrm{Aut}(F)$ and $u \in F$ be such that $B \cap Stab^*(u) = 1$. Then, $A \cdot \mathrm{Inn}(F)$ is O.D. $\Rightarrow$ $MP(A, B)$ solvable.

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Proof. *By Mihailova's construction, for every group $U = \langle a_1, a_2 \mid r_1, \ldots, r_m \rangle$ with unsolvable word problem, the finitely generated subgroup*

$$\begin{aligned} A &= \langle (a_1, a_1), (a_2, a_2), (r_1, 1), \ldots, (r_m, 1) \rangle \\ &= \{(v, w) \mid v =_U w\} \leqslant F_2 \times F_2 \end{aligned}$$

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Finding orbit undecidable subgroups

For free groups

Corollary (Bogopolski–Martino–V., 2008)

$\mathrm{Aut}(F_r)$ contains f.g. orbit undecidable subgroups, for $r \geqslant 3$.

Proof. *Take the copy B of $F_2 \times F_2$ in $\mathrm{Aut}(F_3)$ via the embedding*

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Finding orbit undecidable subgroups

For free abelian groups

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Proof. *Consider* $F_2 \simeq \langle P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rangle \leq_{24} GL_2(\mathbb{Z})$.

- $Stab(1,0) = \{M \mid (1,0)M = (1,0)\} = \{ \begin{pmatrix} 1 & 0 \\ n & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z}\}$.
- $\langle P, Q \rangle \cap Stab(1,0) = \langle \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \rangle$.
- *Choose a free subgroup* $F_2 \simeq \langle P', Q' \rangle \leq \langle P, Q \rangle$ *such that* $\langle P', Q' \rangle \cap Stab(1,0) = \{I\}$ *and consider*

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Finding orbit undecidable subgroups

- *Note that $B \simeq F_2 \times F_2$.*
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Every finitely generated subgroup of $GL_2(\mathbb{Z})$ is O.D.

Question

Does there exist an orbit undecidable subgroup of $GL_3(\mathbb{Z})$?

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Finding orbit undecidable subgroups

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- *Write* $u = (1, 0, 1, 0)$. *By construction,* $B \cap Stab(u) = \{Id\}$.
- *Take* $A \leq B \simeq F_2 \times F_2$ *with unsolvable membership problem.*
- *By previous Proposition,* $A \leqslant \mathrm{GL}_4(\mathbb{Z})$ *is orbit undecidable.*
- *Similarly for* $A \leqslant \mathrm{GL}_d(\mathbb{Z})$, $d \geqslant 4$. $\square$

Proposition (Bogopolski–Martino–V., 2008)

Every finitely generated subgroup of $GL_2(\mathbb{Z})$ *is O.D.*

Question

Does there exist an orbit undecidable subgroup of $GL_3(\mathbb{Z})$ *?*

Outline

Connection to semidirect products

Observation (Bogopolski–Martino–V., 2008)

Let F be f.g., and $A \leqslant_{\mathrm{fg}} \mathrm{Aut}(F)$. If $A \ltimes F$ has solvable CP, then $A \cdot \mathrm{Inn}(F) \leqslant \mathrm{Aut}(F)$ is orbit decidable.

Proof. *$G = A \ltimes F$ contains elements $(\alpha, x) \in A \times F$ operated like*

$$(\alpha_1, x_1) \cdot (\alpha_2, x_2) = (\alpha_1\alpha_2, (x_1\alpha_2)x_2)$$

$$(\alpha, x)^{-1} = (\alpha^{-1}, x^{-1}\alpha^{-1}).$$

For $x_1, x_2 \in F \leqslant G$, we have $x_1 \sim_G x_2 \iff \exists (\alpha, x) \in A \ltimes F$ s.t.

$$\begin{aligned}(Id, x_2) &= (\alpha, x)^{-1} \cdot (Id, x_1) \cdot (\alpha, x) \\ &(\alpha^{-1}, x^{-1}\alpha^{-1}) \cdot (\alpha, (x_1\alpha)x) \\ &(Id, x^{-1}(x_1\alpha)x).\end{aligned}$$

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Connection to semidirect products

In fact, for the free and free abelian cases (among others), the converse is also true after "erasing the relations from A":

Let F be a group, $\alpha_1, \ldots, \alpha_m \in \mathrm{Aut}(F)$, and consider $A = \langle \alpha_1, \ldots, \alpha_m \rangle \leqslant \mathrm{Aut}(F)$ and the semidirect product $G = F_m \ltimes_{\alpha_1,\ldots,\alpha_m} F$.

Theorem (Bogopolski–Martino–V., 2008)

Let F be $\mathbb{Z}^d$ or F_r. Then $G = F_m \ltimes_{\alpha_1,\ldots,\alpha_m} F$ has solvable CP if and only if $A \cdot \mathrm{Inn}(F) = \langle \alpha_1, \ldots, \alpha_m \rangle \cdot \mathrm{Inn}(F) \leqslant \mathrm{Aut}(F)$ is orbit decidable.

This comes from a more general result:

- replace F to any group with solvable *TCP*,
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Let

$$1 \longrightarrow F \stackrel{\alpha}{\longrightarrow} G \stackrel{\beta}{\longrightarrow} H \longrightarrow 1$$

be an algorithmic short exact sequence of groups such that

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Then,

$$CP(G) \text{ is solvable} \iff A_G = \left\{ \begin{array}{rcl} \gamma_g \colon F & \to & F \\ x & \mapsto & g^{-1}xg \end{array} \,\middle|\, g \in G \right\}$$

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Twisted conjugacy

Definition

For $\varphi \in \mathrm{End}(F)$, two elements $u, v \in F$ are said to be φ-twisted conjugated, denoted $u \sim_\varphi v$, if $v = (g\varphi)^{-1}ug$ for some $g \in F$.

Definition

The twisted conjugacy problem for F, denoted $TCP(F)$: "Given $\varphi \in \mathrm{Aut}(F)$ and $u, v \in F$ decide whether $u \sim_\varphi v$".

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$TCP(\mathbb{Z}^d)$ is solvable.

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Is $TCP(F_r)$ solvable for endomorphisms ?

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Double-$TCP(F_r)$ is unsolvable for $r \geqslant 28$.

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Proof. *$CP(G)$ splits into two subproblems:*

- given $u, v \in F$ decide whether they are conjugate in G: this is orbit decidability of $A_G \leq Aut(F)$.

- given $g, g' \in G \setminus F$ decide whether they are conjugate in G; Let us solve this using (i), (ii) and (iii):

- *check whether $g\beta, g'\beta$ are conjugate in H; if not, g, g' are not conjugate in G either.*
- *Otherwise, compute $u \in G$ such that $(u\beta)^{-1}(g\beta)(u\beta) = g'\beta$.*
- *Changing g to g^u, we can assume $g\beta = g'\beta \neq 1_H$. Compute $f \in F$ such that $g' = gf$.*
- *Compute the centralizer of $g\beta \neq 1$ in H, and preimages $y_1, \ldots, y_t$ in G: $C_H(g\beta) = \langle g\beta\rangle(y_1\beta) \sqcup \cdots \sqcup \langle g\beta\rangle(y_t\beta)$.*
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- *check whether $g\beta, g'\beta$ are conjugate in H; if not, g, g' are not conjugate in G either.*
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- *All possible conjugators from g to g' in G commute with $g\beta = g'\beta$ in H, so they are of the form $g^r y_i x$, for some $r \in \mathbb{Z}$, $i = 1, \ldots, t$ and $x \in F$. Now,*

$$(x^{-1} y_i^{-1} g^{-r})\, g\, (g^r y_i x) = x^{-1}(y_i^{-1} g y_i) x = x^{-1} g p_i x$$

and

$$x^{-1} g p_i x = gf \iff \begin{array}{l} g^{-1} x^{-1} g p_i x = f \\ (x\psi_g)^{-1} p_i x = f \\ f \sim_{\psi_g} p_i, \end{array}$$

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Outline

Positive applications

For free abelian-by-free groups: $1 \to \mathbb{Z}^d \to G \to F_m \to 1.$

Corollary

$\mathbb{Z}^d$-by-$\mathbb{Z}$ groups have solvable conjugacy problem.

Corollary (Bogopolski–Martino–V., 2008)

If $\Gamma = \langle M_1, \ldots, M_m \rangle$ is of finite index in $GL_d(\mathbb{Z})$ then $\mathbb{Z}^d \rtimes_{M_1,\ldots,M_m} F_m$ has solvable conjugacy problem.

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**Corollary (Bogopolski–Martino–Maslakova–V. 2006
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Corollary (González-Meneses–V., 2008)

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Negative applications

Theorem (Miller, 70's)

There exist free-by-free groups (more precisely $F_3 \rtimes F_{14}$) with unsolvable conjugacy problem.

Theorem (Bogopolski–Martino–Maslakova–V., 2006)

There exist $\mathbb{Z}^4$-by-free groups (more precisely $\mathbb{Z}^4$-by-F_{14}) with unsolvable conjugacy problem.

Theorem (Burillo–Matucci–V., 2012)

There exists a Thompson-by-free group with unsolvable conjugacy problem.

Question

Does there exist a $\mathbb{Z}^3$-by-free group with unsolvable conjugacy problem ?

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Those orbit undecidable examples $\Gamma \leqslant \mathrm{GL}_4(\mathbb{Z})$ came from Mihailova's construction, so they are not finitely presented...

Proposition (Sunic–V.)

For $d \geqslant 6$, $\mathrm{GL}_d(\mathbb{Z})$ contains f.g., orbit undecidable, free, subgroups.

Proof. *Let $d \geqslant 6$.*

- *Since $d-2 \geqslant 4$, there exists $\langle g_1, \ldots, g_m \rangle = \Gamma \leqslant \mathrm{GL}_{d-2}(\mathbb{Z})$ being orbit undecidable.*
- *Let $F_m = \langle f_1, \ldots, f_m \rangle$, and choose matrices $s_1, \ldots, s_m \in \mathrm{GL}_2(\mathbb{Z})$ such that $\langle s_1, \ldots, s_m \rangle \simeq F_m$.*
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In summary,

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There exist automaton groups (i.e. self-similar groups generated by finite self-similar sets) with unsolvable conjugacy problem.

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THANKS