Finding the equations satisfied by a given element in the free group

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Equations in Groups and Complexity

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(joint work with A. Rosenmann)

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Outline

1. Equations, dependence, dependence closure
2. Main results
3. Stallings graphs
4. Back to equations

Outline

Equations

Definition

Let G be a group, and $H \leqslant G$. An H-equation is an element $w(X) \in H * \langle X \rangle \simeq H * \mathbb{Z}$ (usually written $w(X) = 1$). It has the form

$$w(X) = h_0 X^{\epsilon_1} h_1 \cdots h_{d-1} X^{\epsilon_d} h_d,$$

where $h_0, \ldots, h_d \in H$, $\epsilon_1, \ldots, \epsilon_d = \pm 1$, and, for $i = 1, \ldots, d-1$, $h_i = 1$ implies $\epsilon_i = \epsilon_{i+1}$. The integer $d \geqslant 0$ is called the degree of $w(X)$. Further, $w(X)$ is balanced if $\epsilon_1 + \cdots + \epsilon_d = 0$.

Definition

An element $g \in G$ is a solution of $w(X)$ if $w(g) = h_0 g^{\epsilon_1} h_1 \cdots h_{n-1} g^{\epsilon_n} h_n = 1$ in G.

Example

For $h \neq 1$, the H-eq. $X^2 h X^{-2} = h$ (meaning $h^{-1} X 1 X h X^{-1} 1 X^{-1} = 1$) is a balanced equation of degree 4, having $g \in G$ as a solution $\Leftrightarrow$ $g^2 \in Cen_G(h)$.

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There are many results concerning equations in different families of groups...

Theorem (Makanin/Razborov)

There is an algorithm which, given an equation over a free group F_r, decides whether it has a solution in F_r, or not. In the affirmative case, one can give a finite description of the set of all such solutions.

We are interested in the dual problems:

Problem

Given $H \leqslant_{fg} G$ and $g \in G$, does g satisfy some non-trivial H-equation $w(X) = 1$? In the affirmative case, find/describe them all.

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Given $H \leqslant_{fg} G$, describe the set of all elements $g \in G$ satisfying some non-trivial H-equation (say, 'algebraic' over H).

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- *If $g \in H$ then g is dependent on H (satisfying $g^{-1}X = 1$).*
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In general, $\operatorname{dep}(H)$ is not necessarily a subgroup of G.

In the free group $G = F_{\{a,b\}}$, let $H = \langle a^2, b^2 \rangle$. Both $a, b \in \operatorname{dep}(H)$ (satisfying the H-equations $a^{-2}X^2 = 1$ and $b^{-2}X^2 = 1$, resp.), but $ab \notin \operatorname{dep}(H)$ (since $\{a^2, b^2, ab\}$ is a freely independent set). □

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For $H \leqslant G$, define $H_0 \leqslant H_1 \leqslant H_2 \leqslant \cdots$ as $H_0 = H$ and $H_i = \mathrm{Dep}(H_{i-1}) = \mathrm{Dep}^i(H)$, $i \geqslant 1$. The dependence closure of H is $\widehat{\mathrm{Dep}}(H) = \cup_{i \geqslant 0} H_i \leqslant G$. Of course, $\widehat{\mathrm{Dep}}(H)$ is the smallest dependence-closed subgroup containing H.

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For $H \leqslant G$, define $H_0 \leqslant H_1 \leqslant H_2 \leqslant \cdots$ as $H_0 = H$ and $H_i = \mathrm{Dep}\,(H_{i-1}) = \mathrm{Dep}^{\,i}(H)$, $i \geqslant 1$. The dependence closure of H is $\widehat{\mathrm{Dep}}\,(H) = \cup_{i \geqslant 0} H_i \leqslant G$. Of course, $\widehat{\mathrm{Dep}}\,(H)$ is the smallest dependence-closed subgroup containing H.

Outline

Main results

Theorem (A)

Let $F(A)$ be a free group. There is an algorithm which, on input a (set of generators for a) subgroup $H \leqslant_{\mathrm{fg}} F(A)$, it computes finitely many elements $g_1, \ldots, g_t \in F(A)$ dependent on H such that $\mathrm{dep}_{F(A)}(H) = Hg_1H \cup \cdots \cup Hg_tH$.

Theorem (B)

Let $F(A)$ be a free group. There is an algorithm which, on input $H \leqslant_{\mathrm{fg}} F(A)$ and $g \in F(A)$, decides whether g is dependent on H and, in case it is, it computes $m \geqslant 1$ many non-trivial H-equations $w_1(X), \ldots, w_m(X) \in H * \langle X \rangle$ such that $w_1(g) = \cdots = w_m(g) = 1$ and $\ker \varphi_g = \ll w_1(X), \ldots, w_m(X) \gg$.

Theorem (C)

If $H \leqslant_{fg} F(A)$ then $\widehat{\mathrm{Dep}}(H)$ is again f.g. and computable (in particular, $H_0 \leqslant H_1 \leqslant \cdots \leqslant \widehat{\mathrm{Dep}}(H)$ stabilizes in finitely many steps).

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A proof using Nielsen transformations

A first proof is easy using classical results...

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Given $H \leqslant G$ and $g \in G$, consider the morphism $\varphi_g \colon H * \langle X \rangle \to G$, $h \mapsto h$, $X \mapsto g$. Then, $w(X)\varphi_g = w(g)$ and so, $\{w(X) \mid w(g) = 1\} = \ker \varphi_g \trianglelefteq H * \langle X \rangle$.

Proof Thm. B.

- Compute a free basis $\{h_1, \ldots, h_r\}$ for H.
- Consider the morphism

$$\begin{array}{rccc} \varphi_g \colon & H * \langle X \rangle & \to & F(A). \\ & h_1 & \mapsto & h_1 \\ & & \cdots & \\ & h_r & \mapsto & h_r \\ & X & \mapsto & g \end{array}$$

- Since $\mathrm{Im}(\varphi_g) = \langle h_1, \ldots, h_r, g \rangle = \langle H, g \rangle$, we deduce that $\mathrm{rk}(\mathrm{Im}(\varphi_g)) \leqslant r + 1$, say $\mathrm{rk}(\mathrm{Im}(\varphi_g)) = r + 1 - m$, for $m \geqslant 0$, and there

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Outline

1. Equations, dependence, dependence closure
2. Main results
3. Stallings graphs
4. Back to equations

Stallings automata

Definition

A Stallings automaton over A is a finite A-graph (V, E, q_0), such that:

1- *it is connected,*

2- *it is trim, (no vertex of degree 1 except possibly q_0),*

3- *it is deterministic (no two edges with the same label go out of (or into) the same vertex).*

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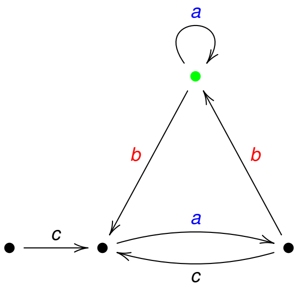

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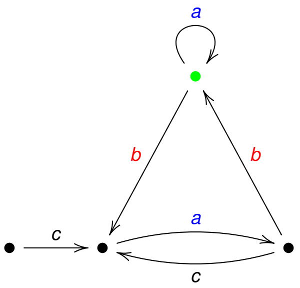

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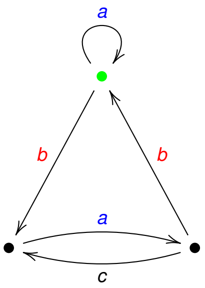

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of $F(A)$ and Stallings automata:

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$\pi(\mathcal{A}, q_0)$ and $L(\mathcal{A})$

Definition

Given $\mathcal{A} = (V, E, q_0)$, its fundamental group and its language are:
$\pi(\mathcal{A}, q_0) = \{$ *closed paths at* q_0 *mod. cancel.* $\} \simeq F_{1-|V\mathcal{A}|+|E\mathcal{A}|}$,
$L(\mathcal{A}) = \{$ *labels of closed paths at* $q_0 \} \leqslant F(A)$.

Proposition

For every Stallings automaton $\mathcal{A} = (V, E, q_0)$, and every maximal tree T, the group $L(\mathcal{A})$ is free with free basis

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where $T[p, q]$ denotes the geodesic in T from p to q, and $\ell(\gamma) \in F(A)$ stands for the label of the path γ. Thus, $\mathrm{rk}(L(\mathcal{A})) = 1 - |V| + |E|$.

Corollary

The 'label' morphism $\ell\colon \pi(\mathcal{A}, q_0) \twoheadrightarrow L(\mathcal{A}) \leqslant F(A)$, $\gamma \mapsto \ell(\gamma)$, is onto; and injective when $\mathcal{A}$ is a Stallings automaton.

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Constructing the automaton from the subgroup

Given generators $\{g_1, \ldots, g_n\}$ for $H \leqslant F(A)$ (as reduced words), construct the flower automaton, denoted $\mathcal{F}(\{g_1, \ldots, g_n\})$.

Clearly, $\mathcal{F}(\{g_1, \ldots, g_n\})$ is trim, and $L(\mathcal{F}(\{g_1, \ldots, g_n\})) = H$,

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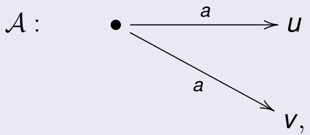

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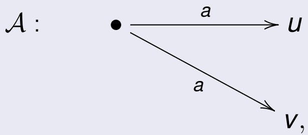

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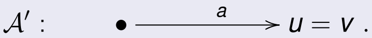

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Note that it induces an epimorphism $\varphi\colon \pi(\mathcal{A}, q_0) \twoheadrightarrow \pi(\mathcal{A}', q_0)$, which is an isomorphism (of free groups) iff the folding is open.

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If $\mathcal{A} \overset{\varphi}{\rightsquigarrow} \mathcal{A}'$ is a Stallings folding then $L(\mathcal{A}) = L(\mathcal{A}')$; also, $\varphi\ell = \ell$.

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Definition

This operation, $\mathcal{A} \overset{\varphi}{\rightsquigarrow} \mathcal{A}'$, is called an elementary Stallings folding. It is said to be open if $u \neq v$ and closed if $u = v$.
Note that it induces an epimorphism $\varphi\colon \pi(\mathcal{A}, q_0) \twoheadrightarrow \pi(\mathcal{A}', q_0)$, which is an isomorphism (of free groups) iff the folding is open.

Lemma (Stallings)

If $\mathcal{A} \overset{\varphi}{\rightsquigarrow} \mathcal{A}'$ is a Stallings folding then $L(\mathcal{A}) = L(\mathcal{A}')$; also, $\varphi\ell = \ell$.

Given a f.g. subgroup $H = \langle g_1, \ldots, g_n \rangle \leqslant F_A$ (we assume the g_i are reduced words), do the following:

1- *draw the flower automaton, $\mathcal{F} = \mathcal{F}(\{g_1, \ldots, g_n\})$,*

2- *perform successive foldings until obtaining a Stallings automaton, denoted Γ_H,*

$$\mathcal{F} \rightsquigarrow \mathcal{A}_1 \rightsquigarrow \cdots \rightsquigarrow \mathcal{A}_t = \Gamma_H.$$

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Local confluence

It can be shown that

Proposition

The automaton Γ_H does not depend on the sequence of foldings.

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The automaton Γ_H does not depend on the generators of H.

Theorem

The following is a well defined bijection:

$$\begin{array}{rcl} \{\text{f.g. subgroups of } F_A\} & \longleftrightarrow & \{\text{Stallings automata}\} \\ H & \to & \Gamma_H \\ L(\mathcal{A}) & \leftarrow & \mathcal{A} \end{array}$$

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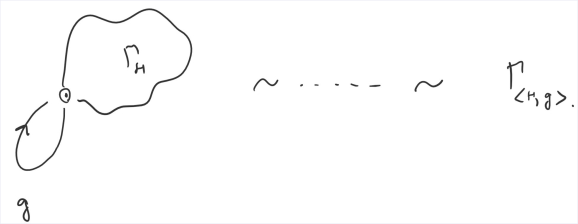
Outline

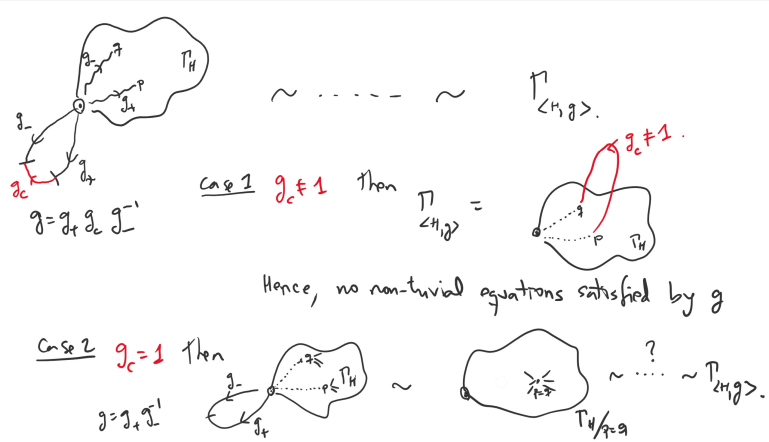
An easy free factor result

Proposition (Miasnikov–V.–Weil, 07; Rosenmann, 01)

Let $H \leqslant F$ be free groups, and $g \in F$. The following are equivalent:
(a) the morphism $\varphi_g \colon H * \langle X \rangle \to F$ is injective;
(b) $\ker(\varphi_g) = 1$, i.e., no nontrivial equation satisfied by g;
(c) H is a proper free factor of $\langle H, g \rangle$;
(d) H is contained in a proper free factor of $\langle H, g \rangle$.
If, in addition, H is f.g., then these are further equivalent to:
(e) $\mathrm{rk}(\langle H, g \rangle) = \mathrm{rk}(H) + 1$;
(f) $\mathrm{rk}(\langle H, g \rangle) > \mathrm{rk}(H)$.

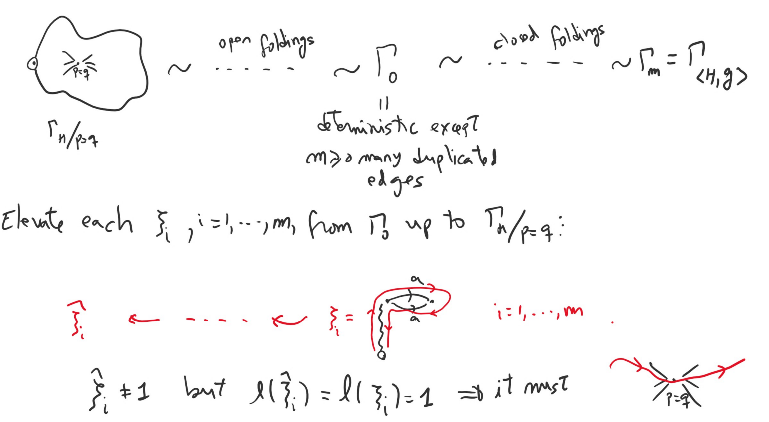


Folding down to $\Gamma_{\langle H,g \rangle}$



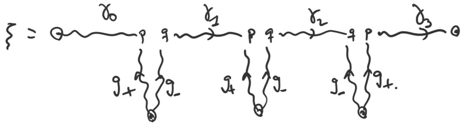
Elevating the elementary paths ξ_i

Fold $\Gamma_H/(p=q)$ down to $\Gamma_{\langle H,g\rangle}$ doing first the open foldings, and the closed ones at the end. Choose a maximal tree T in Γ_0 and



Getting the equation

Looking at each such $\widehat{\xi}$ in Γ_H, it is a closed path with several (≥ 1) $p-q$ and/or $q-p$ discontinuities:



$\ell(\xi) = \gamma_0\gamma_1\gamma_2\gamma_3$

$$1 = \gamma_0\gamma_1\gamma_2\gamma_3 = (\gamma_0 g_+^{-1})(g_+ g_-^{-1})(g_- \gamma_1 g_+^{-1})(g_+ g_-^{-1})(g_- \gamma_2 g_-^{-1})(g_- g_+^{-1})(g_+ \gamma_3)$$

$$\gamma_0 g_+^{-1} = h_0 \in H,\quad g_+ g_-^{-1} = g,\quad g_-\gamma_1 g_+^{-1} = h_1 \in H,\quad g_+ g_-^{-1} = g,\quad g_-\gamma_2 g_-^{-1} = h_2 \in H,\quad g_- g_+^{-1} = g^{-1},\quad g_+\gamma_3 = h_3 \in H$$

Hence, g is a solution of $w(X) = h_0 X h_1 X h_2 X^{-1} h_4$.

We have them all

Collect equations $w_1(X), \ldots, w_m(X)$ from the $m \geqslant 0$ closed foldings above and...

Claim

$\ll w_1(X), \ldots, w_m(X) \gg = \ker \varphi_g$.

Proof.

From the pair of edges at the i-th closed folding, choose a primary and a secondary one, $\{e_1^i, e_2^i\}$, with $e_2^i \notin ET$ (of course, $\ell(e_1^i) = \ell(e_2^i)$).
Let $w(X)$ be an H-equation s.t. $w(g) = 1$; let us show that $w(X) \in \ll w_1(X), \ldots, w_m(X) \gg$.
It determines a closed path $\widehat{\xi}$ with discontinuities in Γ_H, which projects down to a closed path ξ in Γ_0.
Let's do induction on the number of visits to secondary edges: □

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We have them all

$w(X) = h_0 X h_1 X h_2 X^{-1} h_3$

$\hat{\xi} = \circ \xrightarrow{h_0} \circ \xrightarrow{g_+} p \quad q \xrightarrow{g_-^{-1}} \circ \xrightarrow{h_2} \circ \xrightarrow{g_+} p \quad q \xrightarrow{g_-^{-1}} \circ \xrightarrow{h_2} \circ \xrightarrow{g_-} q \quad p \xrightarrow{g_+^{-1}} \circ \xrightarrow{h_3} \circ$

project down to Γ_0, ξ, and $\ell(\xi) = \ell(\hat{\xi}) = 1$ because $w(g) = 1$.

- if ξ visits no secondary edge $\Rightarrow$ it is a closed path in $\Gamma_{\langle H, g \rangle} \leq \Gamma_H$ reading 1 $\Downarrow$

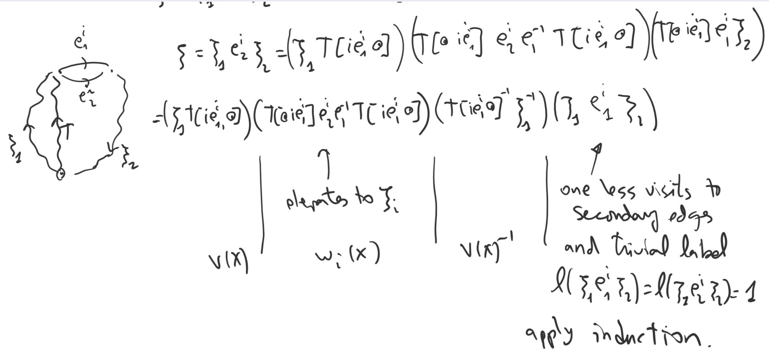
w(X) was the trivial equation. $\Leftarrow$ $\hat{\xi} = 1$ $\Leftarrow$ $\xi = 1$

- otherwise, look at the first visit to a secondary edge, say

$\xi = \xi_1 e_2^i \xi_2$ (with ξ_1 visiting no secondaries)

We have them all

We have the following decomposition and apply induction:



THANKS