On the existence of finitely presented intersection-saturated groups

Enric Ventura

Departament de Matemàtiques
Universitat Politècnica de Catalunya

Groups in Madrid 2023

ICMAT

(joint work with J. Delgado and M. Roy)

October 26th, 2023.

Outline

1. Our main results
2. Free-times-free-abelian groups
3. Realizable / unrealizable k-configurations
4. The free case
5. Open questions
6. Quotient-saturated groups

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Free groups

It is well known that subgroups of free groups are free ...

$$H \leqslant \mathbb{F}_n \quad \Rightarrow \quad H \text{ is free}$$

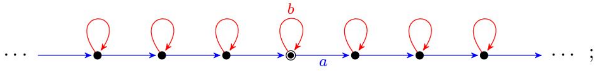
but not necessarily of rank $\leq n$.

Example

Consider $\mathbb{F}_2 = \langle x, y \mid \, \rangle$ and the normal closure of x,

$$\ll x \gg = \langle \ldots, y^2xy^{-2}, yxy^{-1}, x, y^{-1}xy, y^{-2}xy^2, \ldots \rangle.$$

Looking at its Stallings graph



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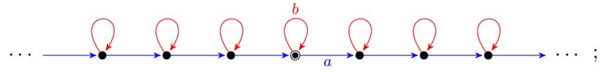

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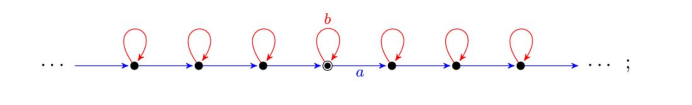

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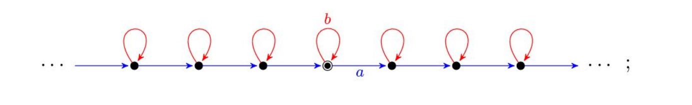

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The Howson property

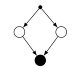
Definition

A group G is Howson if, for any finitely generated $H, K \leqslant_{fg} G$, the intersection $H \cap K$ is, again, finitely generated.

Theorem (Howson, 1954)

Free groups are Howson.

In other words... the configuration



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Out of $2^3 = 8$ possible such configurations this is the only one forbidden in free groups.

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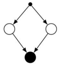

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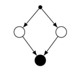

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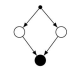

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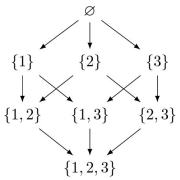
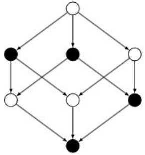
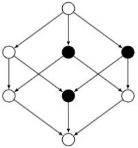
Using this convention, what about the following 3-configurations?







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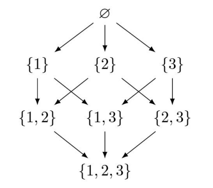

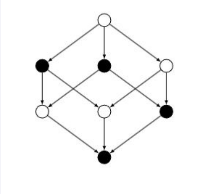

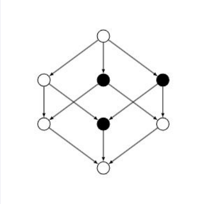

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Formal definitions

Definition

A (intersection) k-configuration is a map $\chi\colon \mathcal{P}([k]) \setminus \{\emptyset\} \to \{0,1\}$. If $\mathcal{I} = (1)\chi^{-1}$ is the support of χ, we write $\chi = \chi_{\mathcal{I}}$. Notation:

- $\mathbf{0} = \chi_{\emptyset}$ is the *zero-configuration;*
- $\mathbf{1} = \chi_{\mathcal{P}([k])\setminus\{\emptyset\}}$ is the *one-configuration;*
- $\chi_{\mathcal{I}}$ is an *almost-zero* k-configuration if $\mathcal{I} = \{I\}$.

Definition

A k-configuration χ is realizable in a group G if there exists subgroups $H_1, \ldots, H_k \leq G$ such that, for every $\emptyset \neq I \subseteq [k]$, $H_I = \cap_{i \in I} H_i$ if f.g. $\Leftrightarrow (I)\chi = 0$. Note that $H_{I \cup J} = H_I \cap H_J$.

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A group G is intersection-saturated if every k-configuration is realizable in G.

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Free-times-free-abelian groups

$$\mathbb{G} = \mathbb{F}_n \times \mathbb{Z}^m = \langle x_1, \ldots, x_n, t_1, \ldots, t_m \mid [x_i, t_j] = 1,\ [t_i, t_k] = 1 \rangle.$$

Normal form: $\forall g \in \mathbb{G}$, $g = w(x_1, \ldots, x_n) t_1^{a_1} \cdots t_m^{a_m} = wt^{\mathbf{a}}$, where $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m$. This way, $(ut^{\mathbf{a}})(vt^{\mathbf{b}}) = uvt^{\mathbf{a}+\mathbf{b}}$.

Observation

These groups sit in a split short exact sequence; and, for $H \leqslant \mathbb{G}$,

$$1 \to \mathbb{Z}^m \overset{\iota}{\hookrightarrow} \mathbb{G} \overset{\pi}{\twoheadrightarrow} \mathbb{F}_n \to 1,$$

$$1 \to L_H = H \cap \mathbb{Z}^m \hookrightarrow H \twoheadrightarrow H\pi \to 1.$$

Moreover, H is finitely generated $\Leftrightarrow$ $H\pi$ is so.

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Free-times-free-abelian groups

Proposition (Delgado–V. '13)

Every subgroup $H \leqslant \mathbb{G}$ admits a (computable) basis

$$H = \langle u_1 t^{\mathbf{a_1}}, u_2 t^{\mathbf{a_2}}, \ldots, u_r t^{\mathbf{a_r}}; t^{\mathbf{b_1}}, \ldots, t^{\mathbf{b_s}} \rangle,$$

where $\{u_1, \ldots, u_r\}$ is a free-basis for $H\pi$, $\mathbf{a_1}, \ldots, \mathbf{a_r} \in \mathbb{Z}^m$, $0 \leq r \leq \infty$, $\mathbf{b_1}, \ldots, \mathbf{b_s} \in \mathbb{Z}^m$ is an abelian-basis for $L_H = H \cap \mathbb{Z}^m$, and $0 \leq s \leq m$.

Proposition (Moldavanski)

The groups $F_n \times \mathbb{Z}^m$, $n \geq 2$, $m \geq 1$, are not Howson.

Question

Are them intersection-saturated?... ... no... but collectively yes ...

Theorem (Delgado–Roy–V. '22)

- *The set of configs realizable in $\mathbb{F}_n \times \mathbb{Z}^m$ increases strictly with m;*
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where $\{u_1, \ldots, u_r\}$ is a free-basis for $H\pi$, $\mathbf{a_1}, \ldots, \mathbf{a_r} \in \mathbb{Z}^m$, $0 \leq r \leq \infty$, $\mathbf{b_1}, \ldots, \mathbf{b_s} \in \mathbb{Z}^m$ is an abelian-basis for $L_H = H \cap \mathbb{Z}^m$, and $0 \leq s \leq m$.

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The groups $F_n \times \mathbb{Z}^m$, $n \geq 2$, $m \geq 1$, are not Howson.

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Are them intersection-saturated?... ... no... but collectively yes ...

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- *The set of configs realizable in $\mathbb{F}_n \times \mathbb{Z}^m$ increases strictly with m;*
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Theorem (Delgado–V. '13)

There is an algorithm which, on input (a set of generators for) $H, K \leqslant_{fg} \mathbb{G}$, decides whether $H \cap K$ is f.g. and, if so, computes a basis for it.

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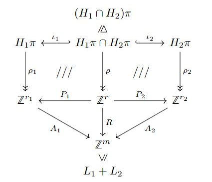
Given (basis for) subgroups $H_1, H_2 \leqslant_{fg} \mathbb{G} = \mathbb{F}_n \times \mathbb{Z}^m$, consider



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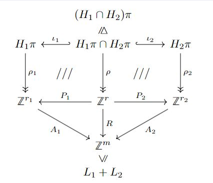

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Consider $\mathbb{G} = \mathbb{F}_4 \times \mathbb{Z}^2 = \langle x_1, x_2, x_3, x_4 \mid - \rangle \times \langle t_1, t_2 \mid [t_1, t_2] \rangle$, $M' = \langle x_1, x_2, t^{(1,0)} \rangle$, $M'' = \langle x_3, x_4, t^{(0,1)} \rangle$, *and the respective subgroups*

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Let $H_1', \ldots, H_k' \leqslant \mathbb{G}' = \mathbb{F}_{n'} \times \mathbb{Z}^{m'}$ *and* $H_1'', \ldots, H_k'' \leqslant \mathbb{G}'' = \mathbb{F}_{n''} \times \mathbb{Z}^{m''}$ *be* $k \geq 2$ *subgroups of* G' *and* G''*, resp. Write* $r' = \operatorname{rk}\left(\bigcap_{j=1}^{k} H_j'\pi\right)$, $r'' = \operatorname{rk}\left(\bigcap_{j=1}^{k} H_j''\pi\right)$*, and consider* $\langle H_1', H_1''\rangle, \ldots, \langle H_k', H_k''\rangle \leqslant \mathbb{G}' \circledast \mathbb{G}'' = (\mathbb{F}_{n'} * \mathbb{F}_{n''}) \times (\mathbb{Z}^{m'} \oplus \mathbb{Z}^{m''})$*. Then, if* $\min(r', r'') \neq 1$*:*

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Again, not true without the hypothesis $\min(r', r'') \neq 1$.

Free-times-free-abelian groups

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Outline

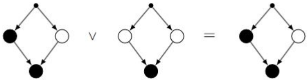
1. Our main results
2. Free-times-free-abelian groups
3. Realizable / unrealizable *k*-configurations
4. The free case
5. Open questions
6. Quotient-saturated groups

Positive results

Definition

Define the join of two k-configurations χ and χ' as

$$\begin{array}{rcl} \chi \vee \chi' \colon \mathcal{P}([k]) \setminus \{\varnothing\} & \to & \{0,1\} \\ I & \mapsto & \begin{cases} 0 & \text{if } (I)\chi = (I)\chi' = 0, \\ 1 & \text{otherwise.} \end{cases} \end{array}$$



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Let χ' (resp. χ'') be k-config. realized by $H_1', \ldots, H_k' \leqslant \mathbb{G}' = \mathbb{F}_{n'} \times \mathbb{Z}^{m'}$ (resp. $H_1'', \ldots, H_k'' \leqslant \mathbb{G}'' = \mathbb{F}_{n''} \times \mathbb{Z}^{m''}$) with $r_I' = \operatorname{rk}\left(\bigcap_{i \in I} H_i' \pi\right) \neq 1$ (resp. $r_I'' \neq 1$) $\forall\ I \subseteq [k]$ with $|I| \geq 2$. Then, $\chi' \vee \chi''$ is realizable in $\mathbb{G}' \circledast \mathbb{G}'' = \mathbb{F}_{n'+n''} \times \mathbb{Z}^{m'+m''}$ by $H_1 = \langle H_1', H_1'' \rangle, \ldots, H_k = \langle H_k', H_k'' \rangle$, again satisfying $r_I \neq 1\ \forall\ I \subseteq [k]$ with $|I| \geq 2$.

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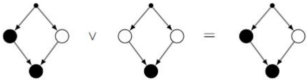

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The k-config. $\chi_{[k]}$ is realizable in $\mathbb{F}_n \times \mathbb{Z}^{k-1}$.

(Sketch of proof)

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\begin{aligned}
H_1 &= \langle x, y; t^{e_2}, \ldots, t^{e_{k-1}} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^{k-1}, \\
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Corollary

Any almost-zero k-config. χ_{I_0} is realizable in $\mathbb{F}_n \times \mathbb{Z}^{|I_0|-1}$ by subgroups $H_1, \ldots, H_k$ further satisfying $\operatorname{rk}\left(\bigcap_{i \in I} H_i \pi\right) \neq 1$, for every $\emptyset \neq I \subseteq [k]$.

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Theorem (Delgado–Roy–V. '22)

Every k-configuration $\chi_{\mathcal{I}}$ is realizable in $\mathbb{F}_n \times \mathbb{Z}^m$, for $n \geq 2$ and $m \geq \sum_{I \in \mathcal{I}} (|I| - 1)$.

(proof)

- *Decompose* $\chi_{\mathcal{I}} = \chi_{I_1} \vee \cdots \vee \chi_{I_r}$, *where* $\mathcal{I} = \{I_1, \ldots, I_r\}$;
- *realize each* χ_{I_j} *in* $\mathbb{F}_2 \times \mathbb{Z}^{|I_j|-1}$, $j = 1, \ldots, r$;
- *put together in a strongly complementary way.* □

Example

Consider $\chi = \chi_{\mathcal{I}}$, *where* $\mathcal{I} = \{\{1\}, \{2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. *Let us realize it in* $\mathbb{F}_2 \times \mathbb{Z}^m$ *for* $m = 0 + 1 + 2 + 2 = 5$. *Decomposing* χ, *we have*

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Positive results

Example (cont.)

In $\mathbb{F}_2 = \langle x, y \mid - \rangle$ *take the freely independent words* $u_j = y^{-j}xy^j \in \mathbb{F}_2$, $j \in \mathbb{Z}$. *Let* $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}, \mathbf{e_5}\}$ *be the canonical basis for* $\mathbb{Z}^5$. *Realize:*

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And note that $\operatorname{rk}\left(\bigcap_{i \in I} H'_i \pi\right) \neq 1$, $\operatorname{rk}\left(\bigcap_{i \in I} H''_i \pi\right) \neq 1$, $\operatorname{rk}\left(\bigcap_{i \in I} H'''_i \pi\right) \neq 1$, *and* $\operatorname{rk}\left(\bigcap_{i \in I} H''''_i \pi\right) \neq 1$. *Therefore, we can realize* χ *by the following subgroups*

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- $\chi_{\{2,3,4\}}$ *as* $H''''_1 = \{1\}$, $H''''_2 = \langle u_4, u_5; t^{\mathbf{e_5}} \rangle$, $H''''_3 = \langle u_4, u_5; t^{\mathbf{e_4}} \rangle$, $H''''_4 = \langle u_4, u_5 t^{\mathbf{e_4}}; t^{\mathbf{e_5}-\mathbf{e_4}} \rangle$, *all inside* $G'''' = \langle u_4, u_5; t^{\mathbf{e_4}}, t^{\mathbf{e_5}} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^5$.

And note that $\operatorname{rk}\left(\bigcap_{i\in I} H'_i \pi\right) \neq 1$, $\operatorname{rk}\left(\bigcap_{i\in I} H''_i \pi\right) \neq 1$, $\operatorname{rk}\left(\bigcap_{i\in I} H'''_i \pi\right) \neq 1$, *and* $\operatorname{rk}\left(\bigcap_{i\in I} H''''_i \pi\right) \neq 1$. *Therefore, we can realize* χ *by the following subgroups*

Positive results

Example (cont.)

$$\begin{aligned} H_1 &= \langle \ldots, u_{-2}, u_{-1}, u_2, u_3; t^{\mathbf{e}_3} \rangle, \\ H_2 &= \langle u_0, u_1, u_4, u_5; t^{\mathbf{e}_5} \rangle, \\ H_3 &= \langle u_0, u_1 t^{\mathbf{e}_1}, u_2, u_3, u_4, u_5; t^{\mathbf{e}_2}, t^{\mathbf{e}_4} \rangle, \\ H_4 &= \langle u_2, u_3 t^{\mathbf{e}_2}, u_4, u_5 t^{\mathbf{e}_4}; t^{\mathbf{e}_3 - \mathbf{e}_2}, t^{\mathbf{e}_5 - \mathbf{e}_4} \rangle. \end{aligned}$$

of $G' \circledast G'' \circledast G''' \circledast G'''' \leqslant \mathbb{F}_2 \times \mathbb{Z}^5$. □

Corollary

$\mathbb{F}_2 \times \left(\bigoplus_{\aleph_0} \mathbb{Z} \right)$ *is intersection-saturated.*

Theorem (Delgado–Roy–V. '22)

There exist finitely presented intersection-saturated groups.

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Positive results

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(Proof 1)

- *Consider Thomson's group F;*
- *it is well know to be finitely presented and to contain $\bigoplus_{\aleph_0} \mathbb{Z}$;*
- *therefore, $\mathbb{F}_2 \times F$ is intersection-saturated.* □
- *(Need to take $\mathbb{F}_2 \times$ because F does not contain $\mathbb{F}_2$.)*

(Proof 2)

- *Consider $G = \left(\bigoplus_{\aleph_0} \mathbb{Z}\right) \rtimes_\alpha \mathbb{Z}$, where α is the automorphism given by right translation of generators;*
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An obstruction

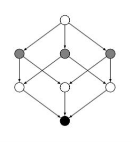
Lemma

Let $H_1, \ldots, H_k \leqslant \mathbb{G} = \mathbb{F}_n \times \mathbb{Z}^m$. Suppose that, for $\emptyset \neq I, J \subseteq [k]$, H_I and H_J are f.g. whereas $H_{I \cup J} = H_I \cap H_J$ is not. Then, $\exists i \in I, \exists j \in J$ s.t. $L_i = H_i \cap \mathbb{Z}^m$ and $L_j = H_j \cap \mathbb{Z}^m$ both have rank strictly smaller than m.

Proposition

Let χ be a k-config. and $\emptyset \neq I_1, \ldots, I_r \subseteq [k]$ be $r \geq 2$ subsets s.t. $\forall j \in [r]$, $(I_1 \cup \cdots \cup \widehat{I_j} \cup \cdots \cup I_r)\chi = 0$, but $(I_1 \cup \cdots \cup I_r)\chi = 1$. Then χ is not realizable in $\mathbb{F}_n \times \mathbb{Z}^{r-2}$.

Corollary

The 3-configurations  *are not realizable in $\mathbb{F}_n \times \mathbb{Z}$.*

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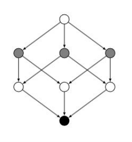

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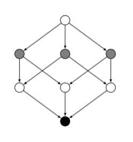
The 3-configurations  *are not realizable in $\mathbb{F}_n \times \mathbb{Z}$.*

An obstruction

Proposition

The k-configuration $\chi_{[k]}$ is realizable in $\mathbb{F}_n \times \mathbb{Z}^{k-1}$, but not in $\mathbb{F}_n \times \mathbb{Z}^{k-2}$.
Hence, the set of configurations realizable in $\mathbb{F}_n \times \mathbb{Z}^m$ increases strictly with m.

Outline

1. Our main results
2. Free-times-free-abelian groups
3. Realizable / unrealizable k-configurations
4. The free case
5. Open questions
6. Quotient-saturated groups

More on configurations

Definition

Let χ be a k-config. and let $i \in [k]$. Its restriction to $\widehat{i} = [k] \setminus \{i\}$ is the $(k-1)$-configuration

$$\begin{array}{rcl} \chi_{|\widehat{i}} \colon \mathcal{P}([k] \setminus \{i\}) \setminus \{\varnothing\} & \to & \{0,1\} \\ I & \mapsto & (I)\chi \, . \end{array}$$

Definition

Given two k-configurations χ, χ' and $\delta \in \{0,1\}$, we define

$$\begin{array}{rcl} \chi \boxplus_\delta \chi' \colon \mathcal{P}([k+1]) \setminus \{\varnothing\} & \to & \{0,1\} \\ I & \mapsto & \begin{cases} (I)\chi & \text{if } k+1 \notin I, \\ (I \setminus \{k+1\})\chi' & \text{if } \{k+1\} \subsetneq I, \\ \delta & \text{if } \{k+1\} = I, \end{cases} \end{array}$$

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More on cofigurations

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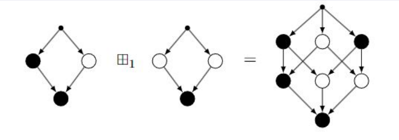

Definition

Let χ be a k-configuration, and $i \in [k]$. The index i is said to be 0-monochromatic (in χ) if $(I)\chi = 0\ \forall I \subseteq [k]$ containing i; i.e., if $\chi = \chi_{|\hat{i}} \boxplus_0 \mathbf{0}$. Similarly, the index i is said to be 1-monochromatic (in χ) if $\chi = \chi_{|\hat{i}} \boxplus_1 \mathbf{1}$.

Lemma

If a k-configuration χ is realizable in $\mathbb{F}_n$ with $n \geq 2$, then the $(k+1)$-configurations $\chi \boxplus_0 \mathbf{0}$, $\chi \boxplus_1 \mathbf{1}$, $\chi \boxplus_0 \chi$, and $\chi \boxplus_1 \chi$ are also realizable in $\mathbb{F}_n$.

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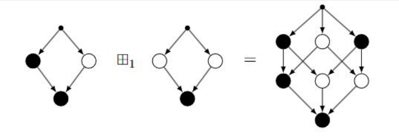

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If a k-configuration χ is realizable in $\mathbb{F}_n$ with $n \geq 2$, then the $(k+1)$-configurations $\chi \boxplus_0 \mathbf{0}$, $\chi \boxplus_1 \mathbf{1}$, $\chi \boxplus_0 \chi$, and $\chi \boxplus_1 \chi$ are also realizable in $\mathbb{F}_n$.

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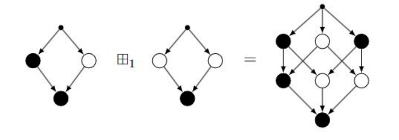

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Characterization for the free case

(Proof)

Let $\mathbb{F}_2 * \mathbb{F}_{\aleph_0} \simeq W * U = \langle w_1, w_2, \ldots \rangle * \langle u, v \rangle \leqslant \mathbb{F}_n$*, and take* $H_1, \ldots, H_k \leqslant W \leqslant \mathbb{F}_n$ *realizing* χ*. Now, in order to realize:*

- $\chi \boxplus_0 \mathbf{0}$, *take* $\widetilde{H}_1 = H_1, \ldots, \widetilde{H}_k = H_k$, *and* $\widetilde{H}_{k+1} = \{1\}$;
- $\chi \boxplus_1 \mathbf{1}$, *take* $\widetilde{H}_1 = H_1 * \langle u, v \rangle, \ldots, \widetilde{H}_k = H_k * \langle u, v \rangle$ *and* $\widetilde{H}_{k+1} = \ll u \gg_U$: $\widetilde{H}_1, \ldots, \widetilde{H}_k$ *realize* $\chi \vee \mathbf{0} = \chi$ *and, for every* $i \neq k+1$, $\widetilde{H}_{k+1} \cap \widetilde{H}_i = \widetilde{H}_{k+1}$ *which is non-f.g.;*
- $\chi \boxplus_0 \chi$, *take* $\widetilde{H}_1 = H_1, \ldots, \widetilde{H}_k = H_k$, *and* $\widetilde{H}_{k+1} = \mathbb{F}_n$;
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Definition

A k*-configuration* χ *is said to be* **Howson** *if, for every* $\emptyset \neq I, J \subseteq [k]$, $(I)\chi = (J)\chi = 0 \;\Rightarrow\; (I \cup J)\chi = 0$.

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Definition

A k-configuration χ is said to be Howson if, for every $\emptyset \neq I, J \subseteq [k]$, $(I)\chi = (J)\chi = 0 \ \Rightarrow \ (I \cup J)\chi = 0$.

Characterization for the free case

(Proof)

Let $\mathbb{F}_2 * \mathbb{F}_{\aleph_0} \simeq W * U = \langle w_1, w_2, \ldots \rangle * \langle u, v \rangle \leqslant \mathbb{F}_n$, *and take* $H_1, \ldots, H_k \leqslant W \leqslant \mathbb{F}_n$ *realizing* χ. *Now, in order to realize:*

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Theorem (Delgado–Roy–V., '22)

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For $\Leftarrow$, we will do induction on the cardinal of the support of χ, say s (regardless of its size k).

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1. Our main results
2. Free-times-free-abelian groups
3. Realizable / unrealizable k-configurations
4. The free case
5. Open questions
6. Quotient-saturated groups

Open questions

Question

Can we characterize the k-configurations realizable in $\mathbb{F}_n \times \mathbb{Z}^m$, for each particular m? At least find an algorithm to decide whether a given χ is realizable in a given abelian dimension m.

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Is there a finitely presented intersection-saturated group G which does not contain $\mathbb{F}_2 \times \mathbb{Z}^m$, for some $m \in \mathbb{N}$?

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Characterize the realizable configurations in your favorite group G.

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Taking configurations over $\mathbb{N} = \{0, 1, 2, \ldots\}$ instead of $\{0, 1\}$ ..., is any k-configuration realizable in $\mathbb{F}_n$, $n \geqslant 2$, if and only if it does not violate Hanna Neumann inequality?

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Quotient-saturated groups

Definition

Let $\Gamma = (V, E, \iota, \tau, c)$ be a colored DAG, and let G be a group. We say that Γ is realizable in G if $\exists\, N_v \trianglelefteq G$ for $v \in V$, in such a way that:

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(iii) *for any vertex v, the quotient group $G_v = G/N_v$ is finitely presented if and only if $c(v) = 0$.*

A group G is said to be quotient-saturated if every finite colored DAG is realizable in G.

Remark

We want to attach normal subgroups of G to the vertices $v \in V\Gamma$ in such a way that directed paths precisely model inclusions.
Alternatively, we want to attach quotients of G to the vertices $v \in V\Gamma$ in such a way that directed paths precisely model projections…
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A group G is said to be quotient-saturated if every finite colored DAG is realizable in G.

Remark

We want to attach normal subgroups of G to the vertices $v \in V\Gamma$ in such a way that directed paths precisely model inclusions. Alternatively, we want to attach quotients of G to the vertices $v \in V\Gamma$ in such a way that directed paths precisely model projections... and colors precisely model f.p./non-f.p. character.

Quotient-saturated groups

Definition

*Let $\Gamma = (V, E, \iota, \tau, c)$ be a colored DAG, and let G be a group. We say that Γ is **realizable** in G if $\exists\, N_v \trianglelefteq G$ for $v \in V$, in such a way that:*

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Theorem (Delgado–Roy–V.)

Let G be a hyperbolic group. Any non-elementary, finitely presented subgroup $D \leqslant G$ is quotient-saturated.

Corollary

Any non-elementary hyperbolic group G is quotient-saturated.

Corollary

Any non-abelian free group $\mathbb{F}_n$, $n \geqslant 2$, is quotient-saturated.

Corollary

Non-elementary, finitely presented, non quotient-saturated groups D do not embed in any hyperbolic group G.

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GRACIAS

THANKS