Universitat Politècnica de Catalunya
Facultat de Matemàtiques i Estadística

Master in Advanced Mathematics and Mathematical Engineering

Master's thesis

Free by Free-Abelian Groups: Isomorphisms and Algorithmic Questions

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January, 2026

Primer de tot, m'agradaria agrair-vos a vosaltres, Enric i Jordi, per haver-me ajudat tant al llarg d'aquest últim any. La Teoria de Grups sempre ha estat l'àrea de les matemàtiques que més m'ha apassionat, i ha estat un veritable plaer haver pogut aprendre tant de vosaltres, tant a l'assignatura d'Àlgebra no Commutativa com al llarg d'aquest treball de fi de màster. Em fa molta il·lusió començar el doctorat i poder continuar aprenent al vostre costat durant els propers quatre anys.

També m'agradaria agrair al Dr. Francesc Fité, amb qui vaig tenir per primera vegada l'oportunitat de descobrir el fascinant món de la teoria de grups, i per això t'estic molt agraït.

A la meva família, perquè sempre heu cregut en mi al llarg de tots aquests anys; mai hauria arribat fins on soc si no fos per vosaltres. Mama i Papa, us estic molt agraït per tot el que heu fet per mi. També moltes gràcies a tu, Arnau, que sempre has estat al meu costat i ets el millor germà del món!

A tu, Joana, per estimar-me tant i haver-me donat sempre tant de suport; m'has fet créixer molt com a persona. Em sap greu no haver pogut passar gaire temps al teu costat aquest últim mes a causa de tanta feina amb el projecte!

I finalment, a tots els meus amics, que, tot i haver estat molt enfeinat aquest últim semestre, sempre que he estat amb vosaltres he pogut oblidar-me una mica de tanta feina i passar'm-ho molt bé!

Abstract

We study the isomorphism problem and the automorphism group of free-by-free-abelian (FBFA) groups, that is, groups of the form $F_n \rtimes \mathbb{Z}^m$, generalizing free-by-cyclic groups. We study the class of full FBFA groups, for which the isomorphism problem admits an explicit description, and show that isomorphic full FBFA groups have conjugate images of the induced outer action in $\mathrm{Out}(F_n)$. We study the subgroup of automorphisms preserving F_n, which coincides with the full automorphism group in the full case. Finally, in the case $n = 2$, we show that two FBFA groups are isomorphic if and only if their induced outer images are conjugate in $\mathrm{Out}(F_2)$, deducing that the isomorphism problem is algorithmically decidable, and we compute the automorphism group.

Keywords

Free-by-free-abelian groups; Automorphism groups; Isomorphism problem; Free groups.

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Introduction

A free-by-cyclic group is a group of the form $M_\phi = F_n \rtimes_\phi \mathbb{Z}$, where F_n denotes the free group of rank $n \geq 1$ and $\phi \in \mathrm{Aut}(F_n)$. In geometric group theory, the family of free-by-cyclic groups has been deeply studied as important examples, and although many things are known for them, they remain a source of open problems, such as the subgroup membership problem, which asks whether a given element belongs to a specified subgroup. For example, the authors in [3] studied the outer automorphism group $\mathrm{Out}(M_\phi)$ under certain technical conditions on ϕ, and used these results in order to compute $\mathrm{Out}(M_\phi)$ in the case $n = 1, 2$. Using these, they were able to characterize the isomorphism problem through the result below:

Theorem (Bogopolski–Martino–Ventura)**.** *Let $\psi, \phi \in \mathrm{Aut}(F_2)$. The groups $F_2 \rtimes_\psi \mathbb{Z}$ and $F_2 \rtimes_\phi \mathbb{Z}$ are isomorphic if and only if $[\psi]$ and $[\phi]^{\pm 1}$ are conjugate in $\mathrm{Out}(F_2)$.*

Since the conjugacy problem in $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$ is algorithmically decidable (see [4]), the above result implies that the isomorphism problem for the subfamily of free-by-cyclic groups with $n = 2$ is algorithmically decidable. Some other decidability problems have been studied, for example in [2] the same authors proved that free-by-cyclic groups have solvable conjugacy problem.

In this written work, we will study a generalization of the free-by-cyclic groups, the so-called free by free-abelian (FBFA) groups, that is groups of the form $M_\alpha := F_n \rtimes_\alpha \mathbb{Z}^m$ where $n, m \geq 1$ are integers and $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ a group homomorphism. Note that the case $m = 1$ corresponds to the free-by-cyclic groups. The case $n = 1$ is straightforward to analyse and differs substantially from the other cases, therefore we will suppose throughout the whole written work that $n \geq 2$.

Building on the work in [3], we study both the automorphism group $\mathrm{Aut}(M_\alpha)$ and the isomorphism problem for the family of FBFA groups, which can be formulated as the follows:

Problem. Let $M_\alpha = F_{n_1} \rtimes_\alpha \mathbb{Z}^m$ and $M_\beta = F_{n_2} \rtimes_\beta \mathbb{Z}^{m_2}$ be two FBFA groups, and suppose that they are given by the following "data": integers $m_1, m_2 \geq 1$ and $n_1, n_2 \geq 2$, automorphisms[1] $\alpha_1, ..., \alpha_{m_1} \in \mathrm{Aut}(F_n)$ commuting pair-wise and $\beta_1, ..., \beta_{m_2} \in \mathrm{Aut}(F_n)$ also commuting pair-wise. The *isomorphism problem for the family of FBFA groups* is the algorithmic problem of deciding, given such data, whether $M_\alpha \cong M_\beta$.

The isomorphism problem for the family of FBFA groups is, in general, very difficult, and little is known even for the subfamily of free-by-cyclic groups. We note that some of the ideas and techniques used in this work are adapted from those in [3], while others are entirely new. Indeed, our study of $\mathrm{Aut}(M_\alpha)$ will largely be independent of the isomorphism problem, and consequently our approach to the latter will differ substantially.

More precisely, in Section 1 we first define and study the general structure of the group M_α. We then introduce the induced group homomorphism $[\alpha] : \mathbb{Z}^m \longrightarrow \mathrm{Out}(F_n)$ and observe that the isomorphism type of M_α depends only on $[\alpha]$, up to a change of basis in $\mathbb{Z}^m$ and conjugation in $\mathrm{Out}(F_n)$. Next, we examine the abelianization M_α^{ab} and define the subfamily of *full FBFA groups* as those M_α for which the preabelianization of the torsion subgroup in M_α^{ab} is a finitely generated free group. We then show that being full is equivalent to having $r_\alpha := \sum_{i=1}^m \mathrm{rank}_{\mathbb{Z}}(\alpha_i^{\mathrm{ab}} - \mathrm{Id}_m) = n$, so that one can easily determine whether a given M_α is full. Full FBFA groups behave much better than the general case, which makes them easier to study. In particular, isomorphisms for the subfamily of full FBFA groups can be described very explicitly, as the following result shows:

[1] These correspond to the images of the canonical vectors of $\mathbb{Z}^{m_1}$ and $\mathbb{Z}^{m_2}$ by α and β, respectively.

Theorem. *Let $M_\alpha = F_{n_1} \rtimes_\alpha \mathbb{Z}^{m_1}$ and $M_\beta = F_{n_2} \rtimes_\beta \mathbb{Z}^{m_2}$ be such that M_α is full. Then $M_\alpha \cong M_\beta$ if and only if $n_1 = n_2 =: n$, $m_1 = m_2 =: m$ and there exist $P \in \mathrm{GL}_m(\mathbb{Z})$ and $\chi \in \mathrm{Aut}(F_n)$ satisfying the following commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}^m & \xrightarrow{[\alpha]} & \mathrm{Out}(F_n) \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \gamma_{[\chi]}} \\ \mathbb{Z}^m & \xrightarrow{[\beta]} & \mathrm{Out}(F_n) \end{array}$$

In such a case, $\mathrm{Im}[\alpha]$ *is conjugate to* $\mathrm{Im}[\beta]$ *in* $\mathrm{Out}(F_n)$.

In particular, when M_α is full, the isomorphism type of $\mathrm{Im}[\alpha]$ is an invariant of M_α. We remark that the situation described above is very similar to that of the closely related family of (free-abelian)-by-free groups, that is, groups of the form $\mathbb{Z}^m \rtimes F_n$, which has been studied in the literature (see [5]). For this family, the isomorphism problem has been shown to be undecidable (see [8]). We observe that in the non-full case the above result no longer holds, and we will present an explicit example illustrating this phenomenon. Under additional technical assumptions on this subgroup, one can prove a partial converse: if $\mathrm{Im}[\alpha]$ and $\mathrm{Im}[\beta]$ are conjugate in $\mathrm{Out}(F_n)$, then $M_\alpha \cong M_\beta$. The most significant instance of this phenomenon occurs for the subfamily of full FBFA groups for which $\mathrm{Im}[\alpha]$ is torsion-free:

Corollary. *Let $\alpha, \beta : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be group homomorphisms such that M_α and M_β are full, and suppose that* $\mathrm{Im}[\alpha]$ *is torsion free. Then $M_\alpha \cong M_\beta$ if and only if* $\mathrm{Im}[\alpha]$ *and* $\mathrm{Im}[\beta]$ *are conjugate subgroups of* $\mathrm{Out}(F_n)$.

We conclude this section by studying the centre $Z(M_\alpha)$ of M_α and showing that $Z(M_\alpha) \cong \mathrm{Ker}[\alpha]$. In particular, the rank of $\mathrm{Ker}[\alpha]$ is an isomorphism invariant of M_α.

In Section 2 we study the subgroup of $\mathrm{Aut}(M_\alpha)$ consisting of automorphisms that preserve F_n, which we denote by $\mathrm{Aut}_{F_n}(M_\alpha)$. This subgroup is particularly relevant because, when M_α is full, it coincides with the entire automorphism group. We then study the normalizer of $\mathrm{Im}[\alpha]$ in $\mathrm{Out}(F_n)$, denoted by $\mathrm{N}(\mathrm{Im}[\alpha])$, and its subgroup $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$, formed by those $[\psi]$ in the normalizer such that the corresponding automorphism of $\mathrm{Im}[\alpha] \cong \mathbb{Z}^m/\mathrm{Ker}[\alpha]$ induced by conjugation lifts to an automorphism of $\mathbb{Z}^m$. Next, we prove that there is a short exact sequence of groups

$$1 \longrightarrow S_\alpha \longrightarrow \mathrm{Aut}_{F_n}(M_\alpha) \longrightarrow \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}} \longrightarrow 1,$$

where $S_\alpha \leq \mathrm{GL}_m(\mathbb{Z})$ is the subgroup formed by those matrices A such that the rows of $A - \mathrm{Id}_m$ belong to $\mathrm{Ker}[\alpha]$, and $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$ denotes the corresponding pre-image subgroup in $\mathrm{Aut}(F_n)$. Afterwards, we study when this short exact sequence splits, yielding an isomorphism

$$\mathrm{Aut}_{F_n}(M_\alpha) \cong S_\alpha \rtimes \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}},$$

and we identify conditions on $\mathrm{Im}[\alpha]$ that force such a splitting. We also use these results to study the inner automorphism group and show that $\mathrm{Inn}(M_\alpha) \cong \pi^{-1}\,\mathrm{Im}[\alpha]$. We finish the section by briefly studying $\mathrm{Out}_{F_n}(M_\alpha) := \mathrm{Aut}_{F_n}(M_\alpha)/\mathrm{Inn}(M_\alpha)$, which coincides with $\mathrm{Out}(M_\alpha)$ when M_α is full, and most of the results for $\mathrm{Aut}_{F_n}(M_\alpha)$ can be adapted to $\mathrm{Out}_{F_n}(M_\alpha)$.

In Section 3 we restrict our attention to the case $n = 2$, that is, when $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^m$. Since abelianization induces an isomorphism $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$, this case is significantly easier to study than the general one, as the structure of $\mathrm{Out}(F_n)$ is considerably more complicated for $n \geq 3$. Accordingly, in

Subsection 3.1 we examine the abelian subgroups of $\mathrm{GL}_2(\mathbb{Z})$, which are relatively simple since $\mathrm{GL}_2(\mathbb{Z})$ is virtually free, and classify them up to conjugacy for our purposes. In Subsection 3.2 we classify M_α into 12 cases based on the conjugacy class of $\mathrm{Im}[\alpha]$ in $\mathrm{GL}_2(\mathbb{Z})$. Up to isomorphism, each M_α is of the form $\mathbb{Z}^{m-k} \times (F_2 \rtimes \mathbb{Z}^k)$ for $k = 0, 1, 2$. Using this classification and the results from Section 1, we then prove:

Theorem. *Let $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^{m_1}$ and $M_\beta = F_2 \rtimes_\beta \mathbb{Z}^{m_2}$. Then $M_\alpha \cong M_\beta$ if and only if $m_1 = m_2$ and $\mathrm{Im}[\alpha]$ and $\mathrm{Im}[\beta]$ are conjugate in $\mathrm{GL}_2(\mathbb{Z})$.*

Moreover, using the decomposition $\mathrm{GL}_2(\mathbb{Z}) \cong D_4 *_{D_2} D_6$ as an amalgamated product of finite groups (cf. [6, Example 5.2, Ch. I]), together with the fact that the conjugacy problem for finitely generated subgroups in such amalgams is algorithmically decidable (cf. [11, Corollary 5.17]), we deduce the following:

Corollary. *The isomorphism problem for the family of FBFA groups with $n = 2$ is algorithmically decidable.*

We note that the above results generalize those previously obtained for free-by-cyclic groups by Bogopolski, Martino, and Ventura [3]. Furthermore, our techniques may be of independent interest: when applied to the free-by-cyclic case, they yield a considerably simpler proof, which could be useful in developing more general arguments.

Finally, in Subsection 3.3 we compute the automorphism group of M_α in the case $n = 2$. In order to make the arguments easier, we suppose that M_α is of the form given in the classification of Subsection 3.2. In the cases where M_α is full, the computation easily follows from the results in Section 2, and we obtain the following result:

Proposition. Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_2)$ be a group homomorphism, and suppose that M_α is full. Then $[\mathrm{N}(\mathrm{Im}[\alpha]) : \mathrm{C}(\mathrm{Im}[\alpha])] \leq 2$ and there exists a group homomorphism $s : \pi^{-1}\mathrm{N}(\mathrm{Im}[\alpha]) \longrightarrow \mathrm{GL}_m(\mathbb{Z})$ with kernel $\pi^{-1}\mathrm{C}(\mathrm{Im}[\alpha])$ such that

$$\mathrm{Aut}(M_\alpha) \cong S_\alpha \rtimes \pi^{-1}\mathrm{N}(\mathrm{Im}[\alpha])$$

with $\psi \in \pi^{-1}\mathrm{N}(\mathrm{Im}[\alpha])$ acting on $A \in S_\alpha$ via $\psi^{-1}A\psi = s_\psi^{-1}As_\psi$.

We conclude by computing $\mathrm{Aut}(M_\alpha)$ in the cases where M_α is not full. These cases are by far the most difficult to analyse, since not every automorphism preserves the subgroup F_2, and the structure of $\mathrm{Aut}(M_\alpha)$ is substantially more complicated than in the full case. Our general strategy is to show that any automorphism of M_α can be composed, on either the left or the right, with certain well-understood automorphisms of M_α so that the resulting automorphism preserves F_2. Once this reduction is achieved, we may apply our analysis of $\mathrm{Aut}_{F_2}(M_\alpha)$ from Section 2.

Future Work

The results obtained in this work naturally give rise to several questions that deserve further investigation. Of the remaining tasks, two appear to be fairly straightforward and naturally follow from the material we have covered in this work. First, it remains to compute $\mathrm{Out}(M_\alpha)$ in the case $n = 2$. This computation can be carried out using the analysis of $\mathrm{Out}(M_\alpha)$ developed in Section 2, together with our explicit description of $\mathrm{Aut}(M_\alpha)$ for $n = 2$ given in Subsection 3.3. We have chosen not to include this computation, as it would largely repeat arguments already used in the study of $\mathrm{Aut}(M_\alpha)$. As a consequence, this would yield a generalization of the results on outer automorphism groups of free-by-cyclic groups in the case $n = 2$ obtained in [3].

Second, the algorithm developed in [11] for the subgroup conjugacy problem in amalgams of finite groups is rather involved, relying on a generalized Stallings folding method, and lies beyond the scope of this work. Since we are only concerned with the conjugacy problem for abelian subgroups of $\mathrm{GL}_2(\mathbb{Z})$, whose structure is relatively simple as described in Subsection 3.1, it would be of interest to develop a simpler and more computationally efficient algorithm that, given finite generating sets, determines whether two such abelian subgroups are conjugate. Taking into account the structure of the abelian subgroups of $\mathrm{GL}_2(\mathbb{Z})$ studied in Subsection 3.1, it appears that this problem can be reduced to the conjugacy problem in $\mathrm{GL}_2(\mathbb{Z})$, which admits a straightforward algorithm as discussed in [4].

Having discussed these two more straightforward problems, we now turn to related directions that are considerably more challenging, and whose resolution would likely require extending our methods or developing new techniques. First, we have seen that the structure of the group M_α depends heavily on $\mathrm{Im}[\alpha]$, which is a finitely generated abelian subgroup of $\mathrm{Out}(F_n)$. Since our study of FBFA groups in the case $n = 2$ relied crucially on the structure of the abelian subgroups of $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$, we expect that a further study of the finitely generated abelian subgroups of $\mathrm{Out}(F_n)$, together with the techniques and results developed in this work, could yield insights for the general case. Some work has already been done in this direction: for $n > 2$, a finite abelian subgroup of $\mathrm{Out}(F_n)$ has order at most 2^n (see [1]), and the rank of a free abelian subgroup of $\mathrm{Out}(F_n)$ is at most $2n - 3$ (see [14]). Moreover, the abelian subgroups of $\mathrm{Out}(F_n)$ have been classified up to finite index in [7].

We have seen that the isomorphism problem for FBFA groups with $n = 2$ is algorithmically decidable, which naturally raises the question of the general case. It is known (see [8]) that there is no algorithm to decide whether two groups of the form $\mathbb{Z}^4 \rtimes F_{15}$ are isomorphic, so that in particular the isomorphism problem for the closely related family of (free-abelian)-by-free groups is algorithmically undecidable. This result relies on the fact that $\mathrm{GL}_4(\mathbb{Z})$ contains a subgroup isomorphic to $F_2 \times F_2$, which in turn contains a finitely generated subgroup with an unsolvable membership problem via a construction due to Mihailova (see [12]). Since $\mathrm{Out}(F_n)$ is generally more complicated than $\mathrm{GL}_n(\mathbb{Z})$, we conjecture that the isomorphism problem for FBFA groups is algorithmically undecidable in the general case.

The isomorphism problem for full FBFA groups also deserves attention, as we have a strong characterization of isomorphism in this case thanks to Theorem 1.15. In this setting, it is not clear whether the problem is algorithmically decidable, and further work on the subgroup conjugacy problem for finitely generated abelian subgroups of $\mathrm{Out}(F_n)$ could provide valuable insight.

As we have computed the automorphism group for the case $n = 2$, and analysed, in the general case, the automorphisms that leave F_n invariant, several related questions remain to be explored. For instance, one could study the subgroup of fixed points of an automorphism of M_α, as well as various decidability problems for $\mathrm{Aut}(M_\alpha)$. It would also be interesting to understand $\mathrm{Aut}(M_\alpha)$ in the general case, along with homomorphisms between FBFA groups.

Moreover, classical decision problems for the family of FBFA groups also deserve further attention. For instance, the subgroup membership problem, that is, deciding algorithmically whether a given element belongs to a subgroup, remains open and would require a detailed study of the subgroups of M_α, which we have not addressed in this work. Other natural questions include intersection problems, the conjugacy problem, which, as noted, is already solved for free-by-cyclic groups, and various other decidability-related questions.

Notation

We summarize below some general notation, terminology, and conventions used throughout the written work; more specific notation and terminology is introduced in the parts of the written work where it is first used.

We use lowercase boldface Latin font ($\mathbf{a}, \mathbf{b}, \mathbf{c}, ...$) to denote elements in $\mathbb{Z}^m$, the free abelian group of rank $m \geq 1$, and uppercase Latin font ($A, B, C, ...$) to denote matrices in $\mathrm{GL}_m(\mathbb{Z})$, the general linear group of degree m over the integers. Words in the free group F_n of rank $n \geq 2$ will usually be denoted by ($\omega, x, y, u, v, ...$), while integers will usually be denoted by other lowercase Latin letters ($a, b, c, ...$). Lowercase Greek letters ($\alpha_i, \beta_j, \psi, \phi, \chi, ...$) are used to denote automorphisms of F_n, whereas uppercase Greek letters ($\Psi, \Phi, \Theta, ...$) are used to denote isomorphisms or automorphisms of semidirect products $F_n \rtimes \mathbb{Z}^m$. Furthermore, functions are assumed to act on the right; that is, we denote by $(x)f$ (or simply xf) the image of the element x by a function f, and we denote by fg the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$. The $m \times m$ identity matrix will be denoted by Id_m, the identity in groups related to automorphisms by Id, and the identity in both F_n and M_α by 1.

Given a group G and an element $g \in G$, we shall write γ_g to denote the inner automorphism of G given by right conjugation by g, $h\gamma_g = g^{-1}hg$ for $h \in G$. We extend this pattern to the ambient semidirect products of the form $F_n \rtimes \mathbb{Z}^m$, where inner automorphisms will be denoted by Γ_g instead of γ_g in order to avoid any possible confusion when conjugating by an element of F_n. As usual, we denote by $\mathrm{Aut}(G)$ the group of automorphisms of G, $\mathrm{Inn}(G)$ the group of inner automorphisms of G, and $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ the outer automorphism group of G. If H is a normal subgroup of G, we will denote by $g \cdot H$ the class of g in the quotient group G/H. An exception to this will be the quotient $\mathrm{Out}(F_n)$, where the class of an automorphism $\psi \in \mathrm{Aut}(F_n)$ will be denoted by $[\psi]$.

1. Free by Free-Abelian Groups

Let $\mathbb{Z}^m$ be the free abelian group of rank $m \geq 1$ given by the presentation

$$\mathbb{Z}^m = \langle t_1, ..., t_m \mid [t_i, t_j] = 1 \quad \forall i, j = 1, ..., m \rangle.$$

For every $\mathbf{a} = (a_1, ..., a_m) \in \mathbb{Z}^m$ we will use the notation $t^{\mathbf{a}} = t_1^{a_1} \cdots t_m^{a_m}$; in particular $t_i = t^{\mathbf{e_i}}$ where $\mathbf{e_i}$ denotes the i-th canonical generator of $\mathbb{Z}^m$ and $t^{\mathbf{a}} \cdot t^{\mathbf{b}} = t^{\mathbf{a}+\mathbf{b}}$ if $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$. We also consider F_n the free group of rank $n \geq 2$ given by the presentation

$$F_n = \langle x_1, ..., x_n \mid - \rangle.$$

Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. We will write $\alpha^{\mathbf{a}}$ for the image of $\mathbf{a} \in \mathbb{Z}^m$ under α; in particular $\alpha^{\mathbf{a}} \cdot \alpha^{\mathbf{b}} = \alpha^{\mathbf{a}+\mathbf{b}}$ if $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$. Note that such a homomorphism is uniquely determined by the images $\alpha_i := \alpha^{\mathbf{e_i}}$, $i = 1, ..., m$, of the standard basis vectors $\mathbf{e_i}$, that is, by m automorphisms $\alpha_1, ..., \alpha_m \in \mathrm{Aut}(F_n)$ which commute pairwise. We now consider the semi-direct product $M_\alpha = F_n \rtimes_\alpha \mathbb{Z}^m$, which we will sometimes also denote by $M_\alpha = F_n \rtimes_{\alpha_1,...,\alpha_m} \mathbb{Z}^m$. We say that a group G is **free by free-abelian** (FBFA) if it is isomorphic to a group M_α of this form. [2] We will assume that FBFA groups are given in this form; that is, a group M_α which is characterized by the following "data": two integers $m \geq 1$, $n \geq 2$, and a group homomorphism $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$. The group M_α admits the following group presentation:

$$M_\alpha = \left\langle \begin{array}{l|ll} t_1, ..., t_m & t_i t_j = t_j t_i & \forall\, i, j \in [1, m] \\ x_1, ..., x_n & t_j^{-1} x_i t_j = x_i \alpha_j & \forall\, i \in [1, n],\ j \in [1, m] \end{array} \right\rangle.$$

We note that $\{t^{\mathbf{a}} x \mid \mathbf{a} \in \mathbb{Z}^m,\ x \in F_n\}$ is a set of normal forms for the elements of M_α, and that the product of two such elements and the inverse are given by

$$t^{\mathbf{a}} x \cdot t^{\mathbf{b}} y = t^{\mathbf{a}+\mathbf{b}} \cdot \left(x \alpha^{\mathbf{b}} \cdot y \right), \quad (t^{\mathbf{a}} x)^{-1} = x^{-1} t^{-\mathbf{a}} = t^{-\mathbf{a}} (x \alpha^{-\mathbf{a}})^{-1}.$$

It is easy to see that $\langle t_1, ..., t_m \rangle$ and $\langle x_1, ..., x_n \rangle$ are subgroups of M_α naturally isomorphic to $\mathbb{Z}^m$ and F_n, respectively, which we may henceforth identify. The elements $t_1, ..., t_m$ are usually called the stable letters. Using this identification we have that F_n is a normal subgroup of M_α, and there is an isomorphism $M_\alpha / F_n \cong \mathbb{Z}^m$ mapping a class $t^{\mathbf{a}} x \cdot F_n \in M_\alpha / F_n$ to $t^{\mathbf{a}} \in \mathbb{Z}^m$. In particular, if $\mathbf{a_1}, ..., \mathbf{a_k} \in \mathbb{Z}^m$ and $y_1, ..., y_k \in F_n$, then by projecting onto the quotient $M_\alpha / F_n \cong \mathbb{Z}^m$ we can see that

$$(t^{\mathbf{a_1}} y_1) \cdot (t^{\mathbf{a_2}} y_2) \cdots (t^{\mathbf{a_k}} y_k) = t^{\mathbf{a_1} + ... + \mathbf{a_k}} y$$

for some $y \in F_n$. Recall that a group is said to be torsion-free if it has no non-trivial elements of finite order. In our case, we have the following result:

Lemma 1.1. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. The group M_α is torsion-free.*

Proof. Suppose that $t^{\mathbf{a}} \omega \in M_\alpha$ is an element such that $(t^{\mathbf{a}} \omega)^n = 1$ for some integer $n > 0$. By projecting this element onto the quotient $M_\alpha / F_n \cong \mathbb{Z}^m$ we can see that $(t^{\mathbf{a}} \omega)^n = t^{n\mathbf{a}} x = 1$ for some $x \in F_n$. In particular, we must have $\mathbf{a} = 0$ and so $\omega^n = 1$, which implies that $\omega = 1$ because F_n has no elements of finite order; this finishes the proof. □

[2] We intentionally exclude from the family the degenerated cases corresponding to $n = 1$, that is, groups of the form $\mathbb{Z} \rtimes_\alpha \mathbb{Z}^m$ with $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(\mathbb{Z}) \cong C_2$ a group homomorphism. This case is straightforward to analyse; these are isomorphic to either $\mathbb{Z}^{m+1}$ or $(\mathbb{Z} \rtimes_\beta \mathbb{Z}) \times \mathbb{Z}^{m-1}$, where β denotes the inversion. Note that these groups contain a free abelian subgroup of index ≤ 2, hence they cannot be isomorphic to a FBFA group.

Observation 1.2. More generally, one may similarly show that the semi-direct product of two torsion-free groups is also torsion-free.

The structure of the group M_α is determined by the homomorphism $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$. In fact, we will see later on that the structure of M_α only depends on the classes $[\alpha_1], ..., [\alpha_m] \in \mathrm{Out}(F_n) := \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$, and for this purpose we introduce the following notation:

Notation 1.3. Given a group homomorphism $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$, we will denote by

$$[\alpha] : \mathbb{Z}^m \longrightarrow \mathrm{Out}(F_n)$$

the group homomorphism obtained by composing α with the natural projection $\pi : \mathrm{Aut}(F_n) \longrightarrow \mathrm{Out}(F_n)$.

Observation 1.4. Both $\mathrm{Im}\,\alpha$ and $\mathrm{Im}[\alpha]$ are finitely generated abelian subgroups (of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$, respectively).

Both $\mathrm{Im}\,\alpha$ and $\mathrm{Im}[\alpha]$ are small in the following sense. If $n > 2$, then any finite abelian subgroup of $\mathrm{Aut}(F_n)$ or $\mathrm{Out}(F_n)$ has order at most 2^n (see [1]). In the case $n = 2$, the order is bounded by 4 for $\mathrm{Aut}(F_2)$ and by 6 for $\mathrm{Out}(F_2)$ (see Section 3.1). Moreover, for $n \geq 2$, the rank of a free abelian subgroup of $\mathrm{Aut}(F_n)$ is at most $2n-2$, while the rank of a free abelian subgroup of $\mathrm{Out}(F_n)$ is at most $2n-3$ (see [14]). This will prove crucial in the study of M_α in the case $n = 2$ in Section 3, since in this case $\mathrm{Im}[\alpha]$ is small enough that the analysis of M_α becomes much simpler. We also note that the abelian subgroups of $\mathrm{Out}(F_n)$ have been classified up to finite index in [7].

We will now prove that the isomorphism type of M_α only depends on the homomorphism $[\alpha]$, so that when studying the isomorphism type of M_α one may replace $\alpha_1, ..., \alpha_m$ for another set of (commuting) automorphism $\alpha'_1, ..., \alpha'_m \in \mathrm{Aut}(F_n)$ such that $[\alpha_i] = [\alpha'_i]$ for $i = 1, ..., m$. More precisely, we have the following result:

Proposition 1.5. *Let $\alpha, \beta : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be two group homomorphisms and suppose there exist $P \in \mathrm{GL}_m(\mathbb{Z})$ and $\chi \in \mathrm{Aut}(F_n)$ satisfying the following commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}^m & \xrightarrow{[\alpha]} & \mathrm{Out}(F_n) \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \gamma_{[\chi]}} \\ \mathbb{Z}^m & \xrightarrow{[\beta]} & \mathrm{Out}(F_n) \end{array}$$

Equivalently, if $\mathbf{p_i} \in \mathbb{Z}^m$ is the i-th row of P, then $[\beta^{\mathbf{p_i}}] = [\chi]^{-1}[\alpha_i][\chi]$ for $i = 1, ..., m$. In this a case, $M_\alpha \cong M_\beta$.

Proof. Let us write $s_1, ..., s_m$ for the stable letters in M_β, and consider the following map defined on the generators of M_α

$$\begin{aligned} \Psi : M_\alpha &\longrightarrow M_\beta \\ x_i &\longmapsto x_i\chi \\ t_j &\longmapsto s^{\mathbf{p_j}}\omega_j \end{aligned}$$

where $\omega_i \in F_n$ is the unique word such that $\chi^{-1}\alpha_i\chi = \beta^{\mathbf{p_i}}\gamma_{\omega_i}$ for $i = 1, ..., m$. In order to show that this defines a group homomorphism, we must show that the relations of M_α are preserved i.e. they are sent to the identity of M_β. Firstly, for every $i = 1, ..., n$ and $j = 1, ..., m$ we have that

$$\begin{aligned} t_j^{-1}x_i t_j (x_i\alpha_j)^{-1} &\mapsto \omega_j^{-1}s^{-\mathbf{p_j}} \cdot x_i\chi \cdot s^{\mathbf{p_j}}\omega_j (x_i\alpha_j\chi)^{-1} \\ &= \omega_j^{-1} \cdot x_i\chi\beta^{\mathbf{p_j}} \cdot \omega_j (x_i\alpha_j\chi)^{-1} \\ &= x_i\chi\beta^{\mathbf{p_j}}\gamma_{\omega_j}(x_i\alpha_j\chi)^{-1} \\ &= x_i\alpha_j\chi(x_i\alpha_j\chi)^{-1} \\ &= 1. \end{aligned}$$

On the other hand, for every $i, j = 1, ..., m$ we have that

$$t_i t_j t_i^{-1} t_j^{-1} \mapsto s^{\mathbf{p_i}}\omega_i s^{\mathbf{p_j}}\omega_j (s^{\mathbf{p_i}}\omega_i)^{-1}(s^{\mathbf{p_j}}\omega_j)^{-1}$$

In order to see that the above expression is the identity in M_β, we must show that

$$s^{\mathbf{p_i}}\omega_i s^{\mathbf{p_j}}\omega_j = s^{\mathbf{p_j}}\omega_j s^{\mathbf{p_i}}\omega_i \iff s^{\mathbf{p_i}+\mathbf{p_j}} \cdot \omega_i\beta^{\mathbf{p_j}} \cdot \omega_j = s^{\mathbf{p_j}+\mathbf{p_i}} \cdot \omega_j\beta^{\mathbf{p_i}} \cdot \omega_i \iff \omega_i\beta^{\mathbf{p_j}} \cdot \omega_j = \omega_j\beta^{\mathbf{p_i}} \cdot \omega_i$$

We recall that $Z(F_n) = 1$ because $n \geq 2$, hence we have an isomorphism $F_n \cong \mathrm{Inn}(F_n)$ given by $u \mapsto \gamma_u$, $u \in F_n$. Therefore, in order to see that the above two words coincide, it is enough to see that the inner automorphisms of F_n defined by them coincide; hence we calculate

$$\begin{aligned} \gamma_{\omega_i\beta^{\mathbf{p_j}}\cdot\omega_j} &= \gamma_{\omega_i\beta^{\mathbf{p_j}}}\gamma_{\omega_j} \\ &= \beta^{-\mathbf{p_j}}\gamma_{\omega_i}\beta^{\mathbf{p_j}}\gamma_{\omega_j} \\ &= \beta^{-\mathbf{p_j}}(\beta^{-\mathbf{p_i}}\chi^{-1}\alpha_i\chi)\beta^{\mathbf{p_j}}(\beta^{-\mathbf{p_j}}\chi^{-1}\alpha_j\chi) \\ &= \beta^{-(\mathbf{p_i}+\mathbf{p_j})}\chi^{-1}\alpha_i\alpha_j\chi. \end{aligned}$$

By performing the same calculation with i and j interchanged, we obtain $\gamma_{\omega_j\beta^{\mathbf{p_i}}\cdot\omega_i} = \beta^{-(\mathbf{p_j}+\mathbf{p_i})}\chi^{-1}\alpha_j\alpha_i\chi$. Since $\alpha_i\alpha_j = \alpha_j\alpha_i$, it follows that $\gamma_{\omega_i\beta^{\mathbf{p_j}}\cdot\omega_j} = \gamma_{\omega_j\beta^{\mathbf{p_i}}\cdot\omega_i}$, and hence $\omega_i\beta^{\mathbf{p_j}} \cdot \omega_j = \omega_j\beta^{\mathbf{p_i}} \cdot \omega_i$, as desired. Thus, we have shown that Ψ is a well-defined group homomorphism.

Now, let $\mathbf{a} \in \mathbb{Z}^m$ and $\omega \in F_n$; by composing Ψ with the projection onto the quotient $M_\beta/F_n = \mathbb{Z}^m$ we may easily see that $(t^{\mathbf{a}}\omega)\Psi = s^{\mathbf{a}P}x$ for some $x \in F_n$. In particular, if $(t^{\mathbf{a}}\omega)\Psi = s^{\mathbf{a}P}x = 1$ then we must have $\mathbf{a}P = 0$, which implies $\mathbf{a} = 0$. Now, $\omega\Psi = \omega\chi = 1$ implies that $\omega = 1$, thus showing that the kernel of Ψ is trivial. Lastly, since χ is an automorphism of F_n we have that $x_1, ..., x_n$ belong to the image of Ψ, which implies that $s^{\mathbf{p_1}}, ..., s^{\mathbf{p_m}}$ belong to the image of Ψ and, because $\mathbf{p_1}, ..., \mathbf{p_m}$ are a basis of $\mathbb{Z}^m$, we get that $s_1, ..., s_m$ are also in the image of Ψ; this proves that Ψ is an isomorphism. □

Proposition 1.5 not only implies that the isomorphism type of M_α depends on $[\alpha]$, but also that it remains unchanged after replacing α for $P\alpha\gamma_{[\chi]}$ with $P \in \mathrm{GL}_m(\mathbb{Z})$ and $\chi \in \mathrm{Aut}(F_n)$. In other words, neither a change of basis on $\mathbb{Z}^m$ nor replacing $\alpha_1, ..., \alpha_m$ for their images $\chi^{-1}\alpha_1\chi, ..., \chi^{-1}\alpha_m\chi$ under a conjugation of $\mathrm{Aut}(F_n)$ changes the isomorphism type of M_α. Our next goal is to show that, if the homomorphism α satisfies a certain technical condition, then the condition in Proposition 1.5 characterizes the isomorphism class of M_α, that is, any FBFA group isomorphic to M_α arises precisely in this manner.

To formulate this technical condition, we first need to study the abelianization of M_α, which is equal to

$$M_\alpha^{\mathrm{ab}} = \left\langle t_1, \dots, t_m \;\middle|\; t_i t_j = t_j t_i \;\; \forall i,j \in [1,m] \right\rangle \;\oplus\; \left\langle x_1, \dots, x_n \;\middle|\; \begin{array}{l} x_i x_j = x_j x_i \;\; \forall i,j \in [1,n] \\ x_i = x_i \alpha_j \;\; \forall i \in [1,n],\, j \in [1,m] \end{array} \right\rangle$$
$$= \mathbb{Z}^m \;\oplus\; \mathbb{Z}^n \Big/ \sum_{i=1}^{m} \mathrm{Im}(\alpha_i^{\mathrm{ab}} - \mathrm{Id}_n),$$

where α_i^{ab} denotes the induced automorphism on the abelianization $F_n^{\mathrm{ab}} \cong \mathbb{Z}^n$. The free rank of M_α^{ab} is an invariant of M_α which depends on the homomorphism α and can be computed explicitly from $\alpha_1, ..., \alpha_m$. More precisely, we make the following definition:

Definition 1.6. Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. We define the rank of α as

$$r_\alpha := \mathrm{rank}_{\mathbb{Z}} \sum_{i=1}^{m} \mathrm{Im}(\alpha_i^{\mathrm{ab}} - \mathrm{Id}),$$

and note that $0 \leq r_\alpha \leq n$.

The number r_α is easily computed using basic linear algebra. Moreover, if T_α denotes the torsion of M_α^{ab}, then there is an isomorphism

$$M_\alpha^{\mathrm{ab}} \cong \mathbb{Z}^{m+n-r_\alpha} \oplus T_\alpha.$$

In particular, the free rank of the abelianization of M_α, that is, the Betti number of M_α, is equal to $m+n-r_\alpha$.

Observation 1.7. If there is an isomorphism $F_n \rtimes_\alpha \mathbb{Z}^m \cong F_n \rtimes_\beta \mathbb{Z}^m$, then it follows from the corresponding isomorphism between the abelianizations that $r_\alpha = r_\beta$.

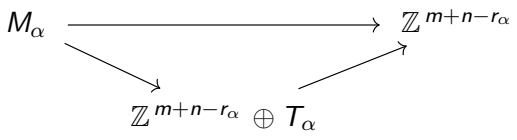
Let us denote by H_α the preabelianization of T_α in M_α, and note that H_α is a characteristic subgroup of M_α. In particular, it is also a normal subgroup of M_α which is contained in F_n, hence it is also a normal subgroup of F_n. We have an isomorphism $F_n/H_\alpha \cong \mathbb{Z}^{n-r_\alpha}$ and, using the fact that a non-trivial normal subgroup of F_n is of finite index if and only if it is finitely generated, we can deduce the following result:

Lemma 1.8. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism and let H_α and r_α be as before. If $r_\alpha = n$ then $H_\alpha = F_n$; otherwise, H_α is an non-finitely generated free group.*

We also have an isomorphism $M_\alpha/H_\alpha \cong \mathbb{Z}^{m+n-r_\alpha}$, and the following lemma characterizes H_α in terms of this property:

Lemma 1.9. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism and let H_α and r_α be as before. Then H_α is the unique normal subgroup of M_α with quotient isomorphic to $\mathbb{Z}^{m+n-r_\alpha}$.*

Proof. Suppose that H is a normal subgroup of M_α such that $M_\alpha/H \cong \mathbb{Z}^{m+n-r_\alpha}$. The projection $M_\alpha \longrightarrow M_\alpha/H \cong \mathbb{Z}^{m+n-r_\alpha}$ factors through $M_\alpha^{\mathrm{ab}} \cong \mathbb{Z}^{m+n-r_\alpha} \oplus T_\alpha$, hence we have a commutative diagram



We now observe that $H_\alpha \mapsto T_\alpha \mapsto \mathbf{0}$, because for any morphism between an abelian group and a torsion-free abelian group, the torsion must map to $\mathbf{0}$. In particular, this implies that $H_\alpha \subseteq H$. By the third isomorphism theorem we have that

$$(M_\alpha/H_\alpha)/(H/H_\alpha) \cong M_\alpha/H \cong \mathbb{Z}^{m+n-r_\alpha},$$

hence the quotient of $M_\alpha/H_\alpha \cong \mathbb{Z}^{m+n-r_\alpha}$ by H/H_α is isomorphic to $\mathbb{Z}^{m+n-r_\alpha}$, and this is only possible if $H/H_\alpha = \mathbf{0}$ i.e. $H = H_\alpha$; this finishes the proof. □

An immediate consequence of these two last results is the following:

Lemma 1.10. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism and r_α be as before. Then F_n is the unique normal subgroup of M_α with quotient isomorphic to $\mathbb{Z}^m$ if and only if $r_\alpha = n$.*

Proof. We have already dealt with the if implication in Lemmas 1.8 and 1.9. For the "only if" implication, if $r_\alpha \neq n$ we could choose the subgroup of M_α^{ab} obtained by considering $m-1$ direct summands from $\mathbb{Z}^m$ and a direct summand from $\mathbb{Z}^{n-r_\alpha}$, and its preimage would be a normal subgroup of M_α with quotient isomorphic to $\mathbb{Z}^m$ which is different from F_n; this finishes the proof. □

The group M_α is significantly easier to study when $r_\alpha = n$. In this case, Lemma 1.8 implies that $H_\alpha = F_n$ is a characteristic subgroup of M_α, meaning that every automorphism of M_α must leave F_n invariant. As we will discuss in Section 2, this considerably simplifies the study of $\mathrm{Aut}(M_\alpha)$. Moreover, Lemma 1.8 also implies that the property that r_α attains its maximum possible value is an isomorphism invariant of the family of FBFA groups in the following sense: if $F_n \rtimes \mathbb{Z}^m \cong F_{n'} \rtimes_\beta \mathbb{Z}^{m'}$, then $r_\alpha = n$ if and only if $r_\beta = n'$. Furthermore, we will see later on that thanks to Lemma 1.10 the isomorphism problem for groups of the form $M_\alpha = F_n \rtimes \mathbb{Z}^m$ becomes significantly easier when we restrict our attention to those with $r_\alpha = n$. Summarizing all that we have discussed, we then make the following definition:

Definition 1.11. Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. We say that $M_\alpha = F_n \rtimes_\alpha \mathbb{Z}^m$ is **full** if M_α satisfies any of the following equivalent definitions:

(a) M_α^{ab} is the direct sum of $\mathbb{Z}^m$ and a finite abelian group,

(b) F_n is the unique normal subgroup of M_α with quotient isomorphic to $\mathbb{Z}^m$,

(c) the preimage in M_α of the torsion subgroup of M_α^{ab} is a finitely generated free group,

(d) the preimage in M_α of the torsion subgroup of M_α^{ab} is F_n,

(e) $r_\alpha = \mathrm{rank}_{\mathbb{Z}} \sum_{i=1}^m \mathrm{Im}(\alpha_i^{\mathrm{ab}} - \mathrm{Id}_n) = n$.

Observation 1.12. The property of being full is an isomorphism invariant for the family of FBFA groups.

Observation 1.13. If M_α is full, then F_n is a characteristic subgroup of M_α, that is, every automorphism of M_α leaves F_n invariant.

Let $\beta : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism and let $M_\beta = F_n \rtimes_\beta \mathbb{Z}^m$. We will study the isomorphisms $\Psi : M_\alpha \longrightarrow M_\beta$ such that $F_n\Psi = F_n$, if one exists at all. This will be of great use for two reasons. The first one is that, if both M_α and M_β are full and there exists an isomorphism $\Psi : M_\alpha \longrightarrow M_\beta$, then we must have $M_\beta/F_n\Psi \cong \mathbb{Z}^m$ since an isomorphism preserves the isomorphism type of quotients, and this implies that $F_n\Psi = F_n$ by one of the equivalent conditions in Definition 1.11. Therefore, we will

get a nice characterization of when $M_\alpha \cong M_\beta$ if these are full, which will be crucial in order to solve the isomorphism problem in the case $n = 2$ in Section 3.2. The second reason is that, in the particular case when $\alpha = \beta$, we get a description of the automorphisms $\Psi \in \mathrm{Aut}(M_\alpha)$ such that $F_n\Psi = F_n$. The set of these automorphisms forms a subgroup of $\mathrm{Aut}(M_\alpha)$ which we will study in depth in Section 2, and which coincides with $\mathrm{Aut}(M_\alpha)$ if M_α is full. More precisely, we have the following result:

Lemma 1.14. *Let $\alpha, \beta : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be group homomorphisms, and suppose that $M_\alpha \cong M_\beta$. The isomorphisms $\Psi : M_\alpha \longrightarrow M_\beta$ such that $F_n\Psi = F_n$ are precisely the maps of the form*

$$\begin{aligned} \Psi : M_\alpha &\longrightarrow M_\beta \\ x_i &\longmapsto x_i\chi \\ t_j &\longmapsto s^{\mathbf{p_j}}\omega_j \end{aligned}$$

with $P \in \mathrm{GL}_m(\mathbb{Z})$ and $\chi \in \mathrm{Aut}(F_n)$ satisfying the commutative diagram in Proposition 1.5, $\mathbf{p_i} \in \mathbb{Z}^m$ the i-th row of P and $\omega_i \in F_n$ the unique word such that $\beta^{\mathbf{p_i}}\gamma_{\omega_i} = \chi^{-1}\alpha_i\chi$ for $i = 1, ..., m$.

Proof. It is clear by Proposition 1.5 that every map Ψ of this form is an isomorphism such that $F_n\Psi = F_n$. Reciprocally, suppose that $\Psi : M_\alpha \longrightarrow M_\beta$ is an isomorphism such that $F_n\Psi = F_n$. In particular, the restriction of Ψ to F_n gives an automorphism $\chi \in \mathrm{Aut}(F_n)$ such that $x_i\Psi = x_i\chi$ for $i = 1, ..., n$. Now, we can write $t_j\Psi = s^{\mathbf{p_j}}\omega_j$ for certain $\mathbf{p_j} \in \mathbb{Z}^m$ and $\omega_j \in F_n$ for $j = 1, ..., m$. The composition of Ψ with the projection $M_\beta \longrightarrow M_\beta/F_n = \mathbb{Z}^m$ is a surjective map whose image is generated by $s^{\mathbf{p_1}}, ..., s^{\mathbf{p_m}}$, hence $\mathbf{p_1}, ..., \mathbf{p_m}$ must be a basis of $\mathbb{Z}^m$, and the matrix P having these vectors as rows belongs to $\mathrm{GL}_m(\mathbb{Z})$. Now, if $x \in F_n$ is any word, we have the relation $t_j^{-1}xt_j = x\alpha_j$ in M_α for $j = 1, ..., m$ and, because Ψ must preserve these relations, an analogous calculation to those done in the proof of Proposition 1.5 shows that

$$t_j^{-1}xt_j(x\alpha_j)^{-1} \mapsto x\chi\beta^{\mathbf{p_j}}\gamma_{\omega_j}(x\alpha_j\chi)^{-1} = 1$$

Since this is true for all $x \in F_n$, we deduce that $\chi\beta^{\mathbf{p_i}}\gamma_{\omega_i} = \alpha_j\chi$; this finishes the proof. □

We are now fully prepared to prove the main theorem concerning the isomorphism problem for the family of full FBFA groups. Specifically, if M_α is full, then the condition in Proposition 1.5 characterizes the isomorphism class of M_α, that is, any FBFA group isomorphic to M_α arises precisely in this manner. More precisely, we have the following result:

Theorem 1.15. *Let $M_\alpha = F_{n_1} \rtimes_\alpha \mathbb{Z}^{m_1}$ and $M_\beta = F_{n_2} \rtimes_\beta \mathbb{Z}^{m_2}$ be such that M_α is full. Then $M_\alpha \cong M_\beta$ if and only if $n_1 = n_2 =: n$, $m_1 = m_2 =: m$ and there exist $P \in \mathrm{GL}_m(\mathbb{Z})$ and $\chi \in \mathrm{Aut}(F_n)$ satisfying the following commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}^m & \xrightarrow{[\alpha]} & \mathrm{Out}(F_n) \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \gamma_{[\chi]}} \\ \mathbb{Z}^m & \xrightarrow{[\beta]} & \mathrm{Out}(F_n) \end{array}$$

Equivalently, if $\mathbf{p_j} \in \mathbb{Z}^m$ is the i-th row of P, then $[\beta^{\mathbf{p_j}}] = [\chi]^{-1}[\alpha_j][\chi]$ for $j = 1, ..., m$.

Proof. The if implication is clear from Proposition 1.5. For the only if implication, suppose that M_α is full and that $M_\alpha \cong M_\beta$. In particular, there is an isomorphism $F_{n_1} = H_\alpha \cong H_\beta$ between the preimages of the torsion in the respective abelianizations, hence H_β is a finitely generated free group and M_β is

also full by Definition 1.11. In particular, the isomorphism $F_{n_1} = H_\alpha \cong H_\beta = F_{n_2}$ implies that $n_1 = n_2$. We also have an isomorphism $\mathbb{Z}^{m_1} \oplus T_\alpha \cong M_\alpha^{ab} \cong M_\beta^{ab} \cong \mathbb{Z}^{m_2} \oplus T_\beta$, which implies that $m_1 = m_2$. Now, let $\Psi : M_\alpha \longrightarrow M_\beta$ be an isomorphism. Note that $F_n\Psi$ is a normal subgroup of M_β with quotient isomorphic to $\mathbb{Z}^m$, hence $F_n\Psi = F_n$. In particular, Ψ must be of the form described in Lemma 1.14, which means that there exist $P \in \mathrm{GL}_m(\mathbb{Z})$ and $\chi \in \mathrm{Aut}(F_n)$ such that $[\beta^{\mathbf{p_i}}] = [\chi]^{-1}[\alpha_i][\chi]$ for $i = 1, ..., m$ or, equivalently, that P and χ satisfy the commutative diagram from the statement of the theorem; this finishes the proof. □

The above theorem will be of great importance in Section 3.2 when solving the isomorphism problem for M_α in the case $n = 2$. Moreover, a very important consequence of this theorem is the following:

Corollary 1.16. *Let $\alpha, \beta : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be group homomorphisms such that $M_\alpha \cong M_\beta$ are full. Then $\mathrm{Im}[\alpha]$ and $\mathrm{Im}[\beta]$ are conjugate subgroups of $\mathrm{Out}(F_n)$.*

Proof. Applying Theorem 1.15 we see that there is a basis $\mathbf{p_1}, ..., \mathbf{p_m}$ of $\mathbb{Z}^m$ and an automorphism $\chi \in \mathrm{Aut}(F_n)$ such that $[\beta^{\mathbf{p_i}}] = [\chi]^{-1}[\alpha_i][\chi]$ for $i = 1, ..., m$. In particular, the subgroups $\langle [\beta^{\mathbf{p_1}}], ..., [\beta^{\mathbf{p_m}}] \rangle$ and $\mathrm{Im}[\alpha] = \langle [\alpha_1], ..., [\alpha_m] \rangle$ are conjugate to each other in $\mathrm{Out}(F_n)$. However, the first subgroup coincides with $\mathrm{Im}[\beta]$ because $\mathbf{p_1}, ..., \mathbf{p_m}$ is a basis of $\mathbb{Z}^m$; this finishes the proof. □

Observation 1.17. In particular, if $M_\alpha \cong M_\beta$ are full then $\mathrm{Im}[\alpha] \cong \mathrm{Im}[\beta]$. Since these are finitely generated abelian groups, it follows from the structure theorem for finitely generated abelian groups that they have the same free rank and isomorphic torsion subgroups.

Observation 1.18. Corollary 1.16 does not hold in general, that is, if we do not require the FBFA groups to be full. The following example, taken from [3], illustrates this. Consider the group

$$G = \langle s, t \mid t^{-3}st^2st^{-1}s^{-1}ts^{-2}ts \rangle$$

On one hand, using Tietze transformations we see there is an isomorphism

$$\begin{aligned} G &\cong \langle s_0, s_1, s_2, s_3, t \mid s_1 = t^{-1}s_0t,\ s_2 = t^{-1}s_1t,\ s_3 = t^{-1}s_2t,\ s_3s_1s_2^{-1}s_1^{-1}s_0 = 1 \rangle \\ &\cong \langle s_0, s_1, s_2, t \mid t^{-1}s_0t = s_1,\ t^{-1}s_1t = s_2,\ t^{-1}s_2t = s_0^{-1}s_1^2s_2s_1^{-1} \rangle \\ &\cong F_3 \rtimes_\phi \mathbb{Z}, \end{aligned}$$

where ϕ is the automorphism of $F_3 = \langle a, b, c \rangle$ given by $a \mapsto b$, $b \mapsto c$, $c \mapsto a^{-1}b^2cb^{-1}$. On the other hand, also using Tietze transformations we see there is an isomorphism

$$\begin{aligned} G &\cong \langle t_{-2}, t_{-1}, t_0, t_1, s \mid t_{-1} = s^{-1}t_{-2}s,\ t_0 = s^{-1}t_{-1}s,\ t_1 = s^{-1}t_0s,\ t_0^{-3}t_2^{-1}t_{-1}^{-2}t_1t_1 = 1 \rangle \\ &\cong \langle t_{-2}, t_{-1}, t_0, s \mid s^{-1}t_{-2}s = t_{-1},\ s^{-1}t_{-1}s = t_0,\ s^{-1}t_0s = t_1^{-1}t_{-2}^{-2}t_0^3 \rangle \\ &\cong F_3 \rtimes_\psi \mathbb{Z}, \end{aligned}$$

where ψ is the automorphism of $F_3 = \langle a, b, c \rangle$ given by $a \mapsto b$, $b \mapsto c$, $c \mapsto b^{-1}ab^{-2}c^3$. In particular, we obtain an isomorphism $F_3 \rtimes_\phi \mathbb{Z} \cong F_3 \rtimes_\psi \mathbb{Z}$. Now observe that $\phi^{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ and $\psi^{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}$ are elements of $\mathrm{GL}_3(\mathbb{Z})$ of infinite order, since both matrices are non-diagonalizable over $\mathbb{C}$. Moreover, they are not conjugate in $\mathrm{GL}_3(\mathbb{Z})$, as they have determinants -1 and 1, respectively. Since abelianization induces a group homomorphism $\mathrm{Out}(F_3) \longrightarrow \mathrm{GL}_3(\mathbb{Z})$, it follows that $[\phi]$ and $[\psi]$ are elements of infinite order in $\mathrm{Out}(F_3)$. Furthermore, the above determinant obstruction implies that $[\phi]$ is not conjugate in $\mathrm{Out}(F_3)$ to either $[\psi]$ or $[\psi]^{-1}$. Consequently, the cyclic subgroups $\langle [\phi] \rangle, \langle [\psi] \rangle \cong \mathbb{Z}$ are not conjugate in

$\mathrm{Out}(F_3)$, despite the fact that the corresponding semidirect products $F_3 \rtimes_\phi \mathbb{Z}$ and $F_3 \rtimes_\psi \mathbb{Z}$ are isomorphic. Not surprisingly, computing $r_\phi = \mathrm{rank}_{\mathbb{Z}}(\phi^{\mathrm{ab}} - \mathrm{Id}_3) = 2$ and $r_\psi = \mathrm{rank}_{\mathbb{Z}}(\psi^{\mathrm{ab}} - \mathrm{Id}_3) = 2$ shows that both of these FBFA groups are non-full.

It is natural to ask whether the converse of Corollary 1.16 holds, and this is what we will discuss next. Suppose that M_α and M_β are full and that $\mathrm{Im}[\alpha]$ is conjugate to $\mathrm{Im}[\beta]$ in $\mathrm{Out}(F_n)$, so that there exists $[\chi] \in \mathrm{Out}(F_n)$ and $\mathbf{q_i} \in \mathbb{Z}^m$ such that $[\chi]^{-1}[\alpha_i][\chi] = [\beta^{\mathbf{q_i}}]$ for $i = 1, \dots, m$, and $[\beta^{\mathbf{q_1}}], \dots, [\beta^{\mathbf{q_m}}]$ generate $\mathrm{Im}[\beta]$. This means that the classes $\mathbf{q_1} + \mathrm{Ker}[\beta], \dots, \mathbf{q_m} + \mathrm{Ker}[\beta]$ generate the quotient group $\mathbb{Z}^m/\mathrm{Ker}[\beta]$. Therefore, there exists a matrix $P \in \mathrm{GL}_m(\mathbb{Z})$ such that P and χ satisfy the commutative diagram from Theorem 1.15, if and only if the classes $\mathbf{q_1} + \mathrm{Ker}[\beta], \dots, \mathbf{q_m} + \mathrm{Ker}[\beta]$ lift to a basis $\mathbf{p_1}, \dots, \mathbf{p_m}$ of $\mathbb{Z}^m$, i.e. this basis satisfies that $\mathbf{p_i} - \mathbf{q_i} \in \mathrm{Ker}[\alpha]$ for $i = 1, \dots, m$. In general, such a matrix P is not guaranteed to exist, which suggests that the converse of Corollary 1.16 may not necessarily be true, although we have not been able to find a definitive proof or counterexample.

Under certain additional conditions on $\mathrm{Im}[\alpha]$, the converse to Corollary 1.16 can be shown to be true. For example, one may easily verify that the converse holds if $\mathrm{Im}[\alpha] \cong C_r$ with $r = 2, 3, 4, 6$. A remarkable fact is that the converse also holds when $\mathrm{Im}[\alpha]$ is torsion-free, that is, when it is a finitely generated free abelian group, or equivalently, when $\mathrm{Ker}[\alpha]$ is a direct summand of $\mathbb{Z}^m$. More precisely, we have the following result:

Corollary 1.19. *Let $\alpha, \beta : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be group homomorphisms such that M_α and M_β are full, and suppose that $\mathrm{Im}[\alpha]$ is torsion free. Then $M_\alpha \cong M_\beta$ if and only if $\mathrm{Im}[\alpha]$ and $\mathrm{Im}[\beta]$ are conjugate subgroups of $\mathrm{Out}(F_n)$.*

Proof. The only if implication is clear by Corollary 1.16. For the if implication, suppose that $\mathrm{Im}[\alpha]$ is conjugate to $\mathrm{Im}[\beta]$ in $\mathrm{Out}(F_n)$. In particular, we have that $\mathrm{Im}[\beta] \cong \mathrm{Im}[\alpha]$ is also torsion free, hence these are isomorphic to $\mathbb{Z}^k$ with $0 \leq k \leq m$ an integer. Since $\mathrm{Im}[\alpha] \cong \mathbb{Z}^m/\mathrm{Ker}[\alpha]$ is torsion free, we have that $\mathrm{Ker}[\alpha]$ is a direct summand of $\mathbb{Z}^m$, hence we may choose a basis $\mathbf{a_1}, \dots, \mathbf{a_m}$ of $\mathbb{Z}^m$ such that $[\alpha^{\mathbf{a_1}}], \dots, [\alpha^{\mathbf{a_k}}]$ is a free basis of $\mathrm{Im}[\alpha]$ and $\mathbf{a_{k+1}}, \dots, \mathbf{a_m}$ is a free basis of $\mathrm{Ker}[\alpha]$. Let $[\chi] \in \mathrm{Out}(F_n)$ be such that $[\chi]^{-1}\,\mathrm{Im}[\alpha][\chi] = \mathrm{Im}[\beta]$, so that conjugation by $[\chi]$ induces an isomorphism $\mathrm{Im}[\alpha] \longrightarrow \mathrm{Im}[\beta]$. In particular, if we write $[\chi]^{-1}[\alpha^{\mathbf{a_i}}][\chi] = [\beta^{\mathbf{b_i}}]$ with $\mathbf{b_i} \in \mathbb{Z}^m$ for $i = 1, \dots, k$, then $[\beta^{\mathbf{b_1}}], \dots, [\beta^{\mathbf{b_k}}]$ forms a free basis for $\mathrm{Im}[\beta] \cong \mathbb{Z}^m/\mathrm{Ker}[\beta]$. We may extend these to a basis $\mathbf{b}_1, \dots, \mathbf{b}_m$ of $\mathbb{Z}^m$ such that $\mathbf{b}_{k+1}, \dots, \mathbf{b}_m$ form a free basis of $\mathrm{Ker}[\beta]$. Now let $A, B \in \mathrm{GL}_m(\mathbb{Z})$ be the matrices whose i-th rows are $\mathbf{a}_i$ and $\mathbf{b}_i$, respectively, and let $P := A^{-1}B \in \mathrm{GL}_m(\mathbb{Z})$. A straightforward computation shows that if $\mathbf{p_i}$ is the i-th row of P, then $[\chi]^{-1}[\alpha_i][\chi] = [\beta^{\mathbf{p_i}}]$, which implies that $M_\alpha \cong M_\beta$ by Proposition 1.5; this finishes the proof. □

There are other certain conditions under which the converse of Corollary 1.16 can be shown to be true. For example, we will see in Section 3.2 that the converse is true in the case $n = 2$. In fact, we will see that both implications are true even if we don't require M_α and M_β to be full. For this purpose, we will have to understand better the structure of $\mathrm{Im}[\alpha] \cong \mathbb{Z}^m/\mathrm{Ker}[\alpha]$ which, according to the structure theorem for finitely generated abelian groups, is isomorphic to $\mathbb{Z}^{m-\ell} \oplus \bigoplus_{i=1}^{\ell} \mathbb{Z}/d_i\mathbb{Z}$ with ℓ a positive integer equal to the rank of $\mathrm{Ker}[\alpha]$ and $d_1 \mid d_2 \mid \cdots \mid d_\ell$ positive integers. One problem is that the isomorphism provided by the structure theorem of finitely generated abelian groups is non-explicit, which makes it harder to study $\mathrm{Im}[\alpha]$ and how it relates to the structure of M_α. For this purpose, we introduce the following result, which shows that every M_α is isomorphic to an M_β such that $\mathrm{Im}[\alpha] = \mathrm{Im}[\beta]$ and the isomorphism provided by the structure theorem of finitely generated abelian groups is made explicit. Consequently, the study of FBFA

groups may be reduced to groups given in this form. This reduction will be used in Subsection 3.3, where we study the automorphism group of M_α in the case $n = 2$.

Lemma 1.20. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. Then there exists a group homomorphism $\beta : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ such that $M_\alpha \cong M_\beta$, $\mathrm{Im}[\alpha] = \mathrm{Im}[\beta]$ and*

$$\mathrm{Ker}[\beta] = \langle d_1\mathbf{e_1}, ..., d_\ell\mathbf{e_\ell}\rangle$$

where $d_1 \mid d_2 \mid \cdots \mid d_\ell$ are positive integers and ℓ is the rank of $\mathrm{Ker}[\alpha]$. In particular, we also have that

$$\mathrm{Im}[\beta] \cong \mathbb{Z}^m/\langle d_1\mathbf{e_1}, ..., d_\ell\mathbf{e_\ell}\rangle = \mathbb{Z}^{m-\ell} \oplus \bigoplus_{i=1}^{\ell} \mathbb{Z}/d_i\mathbb{Z}$$

Proof. Let $A \in M_{\ell\times m}(\mathbb{Z})$ be a matrix whose rows form a basis for $\mathrm{Ker}[\alpha]$. Putting A into smith normal form, there are $Q \in \mathrm{GL}_\ell(\mathbb{Z})$ and $P \in \mathrm{GL}_m(\mathbb{Z})$ such that $D := QAP = \mathrm{diag}(d_1, ..., d_\ell) \in M_{\ell\times m}(\mathbb{Z})$, where $d_1 \mid d_2 \mid \cdots \mid d_\ell$ are positive integer. Now, we let $\beta = P^{-1}\alpha$ and note that $\mathrm{Im}[\alpha] = \mathrm{Im}[\beta]$. Observe that we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^m & \xrightarrow{[\alpha]} & \mathrm{Out}(F_n) \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathbb{Z}^m & \xrightarrow{[\beta]} & \mathrm{Out}(F_n) \end{array}$$

Applying Proposition 1.5 we get that $M_\alpha \cong M_\beta$. Finally, observe that

$$\mathrm{Ker}[\beta] = \mathrm{Ker}[P^{-1}\alpha] = \mathrm{Ker}[\alpha] \cdot P = \mathrm{row}(A) \cdot P = \mathrm{row}(AP) = \mathrm{row}(Q^{-1}D) = \mathrm{row}(D),$$

where row denotes the row space of a matrix, i.e., the linear subspace generated by its rows; this finishes the proof. □

Observation 1.17 together with Theorem 1.15 imply that the rank of $\mathrm{Ker}[\alpha]$ is an isomorphism invariant within the family of full FBFA groups. We will next show that this remains true in general. To this end, we study the centre $Z(M_\alpha)$ of the group M_α, that is, the set of elements that commute with every element of M_α. The centre is a normal subgroup of M_α and, moreover, a characteristic subgroup. In particular, the isomorphism type of $Z(M_\alpha)$ is an isomorphism invariant of M_α. We also note that understanding the structure of the centre will be important when computing the automorphism group of FBFA cyclic groups in Section 3.3, since any automorphism of a group restricts to an automorphism of its centre. We have the following result:

Proposition 1.21. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism, then there is an isomorphism $Z(M_\alpha) \cong \mathrm{Ker}[\alpha]$. More precisely,*

$$Z(M_\alpha) = \{t^{\mathbf{a}}\omega^{-1} \mid \mathbf{a} \in \mathrm{Ker}[\alpha],\ \omega \in F_n,\ \gamma_\omega = \alpha^{\mathbf{a}}\},$$

and for each such element of $Z(M_\alpha)$ the word ω belongs to $\cap_{i=1}^m \mathrm{Fix}(\alpha_i)$.

Proof. Observe that $t^{\mathbf{a}}\omega^{-1}$ belongs to $Z(M_\alpha)$ if and only it commutes with each $t^{\mathbf{b}} \in \mathbb{Z}^m$ and $x \in F_n$. Now, we observe that

$$t^{\mathbf{a}}\omega^{-1} \cdot x = x \cdot t^{\mathbf{a}}\omega^{-1} \iff t^{\mathbf{a}}\omega^{-1}x = t^{\mathbf{a}}(x\alpha^{\mathbf{a}})\omega^{-1} \iff x\gamma_\omega = x\alpha^{\mathbf{a}}$$

is true for all $x \in F_n$ if and only if $\gamma_\omega = \alpha^{\mathbf{a}}$ and, in particular, $\mathbf{a} \in \mathrm{Ker}[\alpha]$. Then, for each $\mathbf{b} \in \mathbb{Z}^m$ we have that

$$\gamma_{\omega\alpha^{\mathbf{b}}} = \alpha^{-\mathbf{b}}\gamma_\omega\alpha^{\mathbf{b}} = \alpha^{-\mathbf{b}}\alpha^{\mathbf{a}}\alpha^{\mathbf{b}} = \alpha^{\mathbf{a}} = \gamma_\omega,$$

which implies that $\omega\alpha^{\mathbf{b}} = \omega$ and, in particular, ω belongs to $\cap_{i=1}^m \mathrm{Fix}(\alpha_i)$. Hence, we have that

$$t^{\mathbf{b}} \cdot t^{\mathbf{a}}\omega^{-1} = t^{\mathbf{a}}(\omega\alpha^{\mathbf{b}})^{-1}t^{\mathbf{b}} = t^{\mathbf{a}}\omega^{-1} \cdot t^{\mathbf{b}}.$$

Thus, we have shown that $t^{\mathbf{a}}\omega^{-1}$ belongs to $Z(M_\alpha)$ if and only if $\gamma_\omega = \alpha^{\mathbf{a}}$. Finally, consider the map

$$\begin{aligned} f : Z(M_\alpha) &\longrightarrow \mathrm{Ker}[\alpha] \\ t^{\mathbf{a}}\omega^{-1} &\longmapsto \quad \mathbf{a} \end{aligned}$$

It is a group homomorphism because

$$f(t^{\mathbf{a_1}}\omega_1^{-1} \cdot t^{\mathbf{a_2}}\omega_2^{-1}) = f(t^{\mathbf{a_1}+\mathbf{a_2}}(\omega_2\omega_1)^{-1}) = \mathbf{a_1} + \mathbf{a_2} = f(t^{\mathbf{a_1}}\omega_1^{-1}) + f(t^{\mathbf{a_2}}\omega_2^{-1}).$$

It is injective because if $f(t^{\mathbf{a}}\omega^{-1}) = \mathbf{a} = \mathbf{0}$ then $\gamma_\omega = 1$, hence $\omega = 1$ and $t^{\mathbf{a}}\omega^{-1} = 1$. It is also surjective because if $\mathbf{a} \in \mathrm{Ker}[\alpha]$ then $\alpha^{\mathbf{a}} \in \mathrm{Inn}(F_n)$ and there is $\omega \in F_n$ such that $\gamma_\omega = \alpha^{\mathbf{a}}$, and so $t^{\mathbf{a}}\omega^{-1}$ belongs to the centre and its image is $\mathbf{a}$. □

Observation 1.22. One may easily check that $\mathbb{Z}^m \cap Z(M_\alpha) \cong \mathrm{Ker}\,\alpha = \{\mathbf{a} \in \mathbb{Z}^m \mid \alpha^{\mathbf{a}} = \mathrm{Id}\}$.

Observation 1.23. An immediate consequence of Proposition 1.21 is that the number $\ell_\alpha := \mathrm{rank}_{\mathbb{Z}}\,\mathrm{Ker}[\alpha]$ is an isomorphism invariant for the family of FBFA groups M_α. More precisely, if $F_{n_1} \rtimes_\alpha \mathbb{Z}^{m_1} \cong F_{n_2} \rtimes_\beta \mathbb{Z}^{m_2}$ then $\ell_\alpha = \ell_\beta$.

More can be said about the set of words $\omega \in F_n$ such that $t^{\mathbf{a}}\omega^{-1}$ belongs to $Z(M_\alpha)$. Namely, we will see next that the set of these words is either trivial, or forms a cyclic subgroup of F_n. More precisely, we have the following:

Lemma 1.24. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. Then either $\mathrm{Ker}[\alpha] = \mathrm{Ker}\,\alpha$ or there is a non-zero vector $\mathbf{v} \in \mathbb{Z}^m$ such that $\mathrm{Ker}[\alpha] = \mathrm{Ker}\,\alpha \oplus \langle\mathbf{v}\rangle$. Equivalently, the quotient group $\mathrm{Ker}[\alpha]/\,\mathrm{Ker}\,\alpha$ is either trivial or isomorphic to $\mathbb{Z}$.*

Proof. Consider the group homomorphism $\mathrm{Ker}[\alpha] \longrightarrow F_n$ mapping $\mathbf{a} \in \mathrm{Ker}[\alpha]$ to the word $\omega \in F_n$ such that $\alpha^{\mathbf{a}} = \gamma_\omega$; its kernel is precisely $\mathrm{Ker}\,\alpha$. Its image, which is isomorphic to $\mathrm{Ker}[\alpha]/\,\mathrm{Ker}\,\alpha$, is an abelian subgroup of F_n, hence its either trivial or isomorphic to $\mathbb{Z}$. □

Observation 1.25. In the case where $\mathrm{Ker}[\alpha] = \mathrm{Ker}\,\alpha$ we get that $Z(M_\alpha) \subset \mathbb{Z}^m$. In the case where $\mathrm{Ker}[\alpha] = \mathrm{Ker}\,\alpha \oplus \langle\mathbf{v}\rangle$ with $\mathbf{v} \neq 0$, we get that there is a surjective morphism $\tau : \mathrm{Ker}[\alpha] \longrightarrow \mathbb{Z}$, given by the projection of $\mathrm{Ker}[\alpha]$ onto $\langle\mathbf{v}\rangle$, and a word $\omega \in F_n$ given by $\gamma_\omega := \alpha^{-\mathbf{v}}$ such that every element of $Z(M_\alpha)$ is of the form $t^{\mathbf{a}}\omega^{\mathbf{a}\tau}$ with $\mathbf{a} \in \mathrm{Ker}[\alpha]$.

Observation 1.26. In view of Lemma 1.24, we can choose a basis of $\mathrm{Ker}[\alpha]$ consisting of non-zero vectors $\mathbf{v_1}, ..., \mathbf{v_{\ell-1}} \in \mathrm{Ker}\,\alpha$ and $\mathbf{v} \in \mathrm{Ker}[\alpha]$. Applying the proof of Lemma 1.20 to this basis, we get a group M_β satisfying the conclusions of the lemma with either $\mathrm{Ker}\,\beta = \mathrm{Ker}[\beta]$ or $\mathrm{Ker}\,\beta = \langle d_1\mathbf{e_1}, ..., d_{\ell-1}\mathbf{e_{\ell-1}}\rangle$. In other words, $\beta_1, ..., \beta_{\ell-1}$ are automorphisms of F_n of orders $d_1, ..., d_{\ell-1}$ respectively, and β_ℓ satisfies either $\beta_\ell^{d_\ell} = \mathrm{Id}$ or $\beta_\ell^{d_\ell} = \gamma_\omega$ with $\omega \in F_n$, $\omega \neq 1$. Moreover, we have that $\mathbb{Z}^{m-\ell} \cong \langle \alpha_{\ell+1}, ..., \alpha_m\rangle \leq \mathrm{Aut}(F_n)$.

We close this section by noting that, since the free rank of $\mathrm{Im}[\alpha]$ is equal to $m - \ell_\alpha$, it follows from Observation 1.4 and the discussion following it that $m - \ell_\alpha \leq 2n - 3$. This bound will play a crucial role in Section 3, where we focus on the case $n = 2$. In this situation, the inequality forces $\ell_\alpha = m - 1$ or $\ell_\alpha = m$, which significantly simplifies the structure of M_α. Although this conclusion follows from the previously discussed results in [14], we will prove it in the case $n = 2$ for completeness.

2. The Automorphism Group

Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism, and consider

$$\mathrm{Aut}_{F_n}(M_\alpha) := \{\Psi \in \mathrm{Aut}(M_\alpha) \mid F_n\Psi \leq F_n\}$$

to be the subgroup of $\mathrm{Aut}(M_\alpha)$ formed by those automorphisms of M_α which leave F_n invariant.

Lemma 2.1. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism, and let $\Psi \in \mathrm{Aut}(M_\alpha)$. Then $F_n\Psi \leq F_n$ if and only if $F_n\Psi = F_n$.*

Proof. Since Ψ is an automorphism, it preserves the quotients, hence $M_\alpha/F_n\Psi \cong M_\alpha/F_n \cong \mathbb{Z}^m$. By the third isomorphism theorem, we have that

$$\mathbb{Z}^m/(F_n/F_n\Psi) \cong ((M_\alpha/F_n\Psi)/(F_n/F_n\Psi) \cong M_\alpha/F_n \cong \mathbb{Z}^m,$$

which implies that $F_n/F_n\Psi = \{1\}$ i.e. $F_n\Psi = F_n$. □

One of the main reasons to study this subgroup is that it actually coincides with the automorphism group for the family of full FBFA groups:

Observation 2.2. If M_α is full then Observation 1.13 implies that $\mathrm{Aut}(M_\alpha) = \mathrm{Aut}_{F_n}(M_\alpha)$.

The other main reason is that, in the case $n = 2$, the study of this subgroup will allow us to understand $\mathrm{Aut}(M_\alpha)$ even if M_α is not full. The general strategy will be to see that every $\Psi \in \mathrm{Aut}(M_\alpha)$ can be multiplied either by the left or the right by certain automorphisms of M_α, which are well understood, such that the resulting automorphism leaves F_2 invariant i.e. belongs to $\mathrm{Aut}_{F_2}(M_\alpha)$. The general form of the automorphisms in $\mathrm{Aut}_{F_n}(M_\alpha)$ follows from a previous result:

Lemma 2.3. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. The automorphisms in $\mathrm{Aut}_{F_n}(M_\alpha)$ are precisely the maps of the form*

$$\begin{aligned} \Psi : M_\alpha &\longrightarrow M_\alpha \\ x_i &\longmapsto x_i\psi \\ t_j &\longmapsto t^{\mathbf{a_j}}\omega_j \end{aligned}$$

where $\psi \in \mathrm{Aut}(F_n)$, the vectors $\mathbf{a_1}, ..., \mathbf{a_m} \in \mathbb{Z}^m$ are the rows of an invertible matrix $A \in \mathrm{GL}_m(\mathbb{Z})$ and $\omega_1, ..., \omega_m \in F_n$ are words such that $\alpha^{\mathbf{a_i}}\gamma_{\omega_i} = \psi^{-1}\alpha_i\psi$ for $i = 1, ..., m$.

Proof. Follows immediately from Lemmas 2.1 and 1.14 by setting $\alpha = \beta$. □

Observation 2.4. Every $\Psi \in \mathrm{Aut}_{F_n}(M_\alpha)$ is uniquely determined by a pair (ψ, A) such that $\psi \in \mathrm{Aut}(F_n)$, $A \in \mathrm{GL}_m(\mathbb{Z})$ and $[\alpha^{\mathbf{a_i}}] = [\psi]^{-1}[\alpha_i][\psi]$ for $i = 1, ..., m$. In fact, $\psi = \Psi|_{F_n}$ and A is the matrix of the automorphism induced by Ψ on the quotient $M_\alpha/F_n \cong \mathbb{Z}^m$ with respect to the basis $t_1, ..., t_m$. We also note that the words $\omega_i \in F_n$ can be uniquely recovered from the equality $\gamma_{\omega_i} = \alpha^{-\mathbf{a_i}}\psi^{-1}\alpha_i\psi$.

Observation 2.5. Let $\Psi, \Phi \in \mathrm{Aut}_{F_n}(M_\alpha)$ be represented by (ψ, A) and (ϕ, B). Using Observation 2.4 we see that the composition $\Psi\Phi$ is represented by $(\psi\phi, AB)$. In particular, $\mathrm{Aut}_{F_n}(M_\alpha)$ can be identified with a certain subgroup of $\mathrm{Aut}(F_n) \times \mathrm{GL}_m(\mathbb{Z})$.

Our next objective is to better understand this subgroup of $\mathrm{Aut}(F_n) \times \mathrm{GL}_m(\mathbb{Z})$. Observation 2.4 shows that every $\Psi \in \mathrm{Aut}_{F_n}(M_\alpha)$ is uniquely represented by a pair (ψ, A), where $\psi \in \mathrm{Aut}(F_n)$ and $A \in \mathrm{GL}_m(\mathbb{Z})$ satisfy the commutative diagram of Proposition 1.5 with $\alpha = \beta$. We will reinterpret this diagram in terms of automorphisms of $\mathrm{Im}[\alpha] \cong \mathbb{Z}^m/\mathrm{Ker}[\alpha]$ induced by conjugation by elements of the normalizer of $\mathrm{Im}[\alpha]$, and which lift to automorphisms of $\mathbb{Z}^m$.

We denote by $\mathrm{N}(\mathrm{Im}[\alpha])$ and $\mathrm{C}(\mathrm{Im}[\alpha])$ the normalizer and centralizer of $\mathrm{Im}[\alpha]$ in $\mathrm{Out}(F_n)$, respectively. Every $[\psi] \in \mathrm{N}(\mathrm{Im}[\alpha])$ induces by conjugation and automorphism $\Lambda_{[\psi]}$ of $\mathrm{Im}[\alpha]$. We have a morphism

$$\begin{array}{ccc} \Lambda : \mathrm{N}(\mathrm{Im}[\alpha]) & \longrightarrow & \mathrm{Aut}(\mathrm{Im}[\alpha]) \\ [\psi] & \longmapsto & \Lambda_{[\psi]} \end{array}$$

whose kernel is equal to $\mathrm{C}(\mathrm{Im}[\alpha])$. Hence, the quotient $\mathrm{N}(\mathrm{Im}[\alpha])/\mathrm{C}(\mathrm{Im}[\alpha])$ is canonically isomorphic to $\mathrm{Im}\,\Lambda$, the subgroup of $\mathrm{Aut}(\mathrm{Im}[\alpha])$ formed by those automorphisms of $\mathrm{Im}[\alpha]$ coming from conjugation by elements of $\mathrm{N}(\mathrm{Im}[\alpha])$.

Observation 2.6. Note that if $\Psi \in \mathrm{Aut}_{F_n}(M_\alpha)$ is represented by (ψ, A) then

$$[\psi]^{-1}\langle [\alpha_1], ..., [\alpha_m]\rangle [\psi] = \langle [\alpha^{\mathbf{a_1}}], ..., [\alpha^{\mathbf{a_m}}]\rangle = \langle [\alpha_1], ..., [\alpha_m]\rangle$$

because the rows $\mathbf{a_1}, ..., \mathbf{a_m}$ of A form a basis of $\mathbb{Z}^m$. In particular, we have that $[\psi] \in \mathrm{N}(\mathrm{Im}[\alpha])$.

Therefore, every $\Psi \in \mathrm{Aut}_{F_n}(M_\alpha)$ is uniquely represented by a pair (ψ, A) such that $[\psi] \in \mathrm{N}(\mathrm{Im}[\alpha])$, $A \in \mathrm{GL}_m(\mathbb{Z})$ with rows $\mathbf{a_1}, ..., \mathbf{a_m} \in \mathbb{Z}^m$ and $[\psi]^{-1}[\alpha_i][\psi] = [\alpha^{\mathbf{a_i}}]$ for $i = 1, ..., m$. We may reinterpret this last condition in the following way:

Definition 2.7. Given $[\psi] \in \mathrm{N}(\mathrm{Im}[\alpha])$ we say that a matrix $A \in \mathrm{GL}_m(\mathbb{Z})$ **lifts** $\Lambda_{[\psi]}$, or simply that A lifts $[\psi]$, if these fit into a commutative diagram of the form

$$\begin{array}{ccc} \mathbb{Z}^m & \xrightarrow{\quad A \quad} & \mathbb{Z}^m \\ \downarrow{\scriptstyle [\alpha]} & & \downarrow{\scriptstyle [\alpha]} \\ \mathrm{Im}[\alpha] & \xrightarrow{\quad \Lambda_{[\psi]} \quad} & \mathrm{Im}[\alpha] \end{array}$$

Equivalently, if $\mathbf{a_1}, ..., \mathbf{a_m}$ are the rows of A then $[\alpha^{\mathbf{a_i}}] = [\psi]^{-1}[\alpha_i][\psi]$ for $i = 1, ..., m$.

If we canonically identify $\mathrm{Im}[\alpha] = \mathbb{Z}^m/\mathrm{Ker}[\alpha]$ via the first isomorphism theorem, then this condition is equivalent to having a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^m & \xrightarrow{\quad A \quad} & \mathbb{Z}^m \\ \downarrow & & \downarrow \\ \mathbb{Z}^m/\mathrm{Ker}[\alpha] & \xrightarrow{\quad \Lambda_{[\psi]} \quad} & \mathbb{Z}^m/\mathrm{Ker}[\alpha] \end{array}$$

where the vertical arrows denote the canonical projection to the quotient. In other words, in this situation we say that the automorphism $\Lambda_{[\psi]}$ of $\mathbb{Z}^m/\mathrm{Ker}[\alpha]$ is "lifted" to an automorphism of $\mathbb{Z}^m$ via the matrix A.

An automorphism of a quotient of $\mathbb{Z}^m$ does not necessarily lift to an automorphism of $\mathbb{Z}^m$; for example, the automorphism $\mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ sending $1 \mapsto 2$ does not lift to an automorphism of $\mathbb{Z}$. For this reason,

given $[\psi] \in \mathrm{N}(\mathrm{Im}[\alpha])$, it is not necessarily true that there exists a matrix $A \in \mathrm{GL}_m(\mathbb{Z})$ lifting $\Lambda_{[\psi]}$. In particular, given $\psi \in \pi^{-1}(\mathrm{N}(\mathrm{Im}[\alpha]))$ (the preimage of $\mathrm{N}(\mathrm{Im}[\alpha])$ in $\mathrm{Aut}(F_n)$), there need not exist an automorphism $\Psi \in \mathrm{Aut}_{F_n}(M_\alpha)$ and a matrix $A \in \mathrm{GL}_m(\mathbb{Z})$ such that Ψ is represented by the pair (ψ, A). Equivalently, the morphism $\mathrm{Aut}_{F_n}(M_\alpha) \to \mathrm{N}(\mathrm{Im}[\alpha])$ mapping $(\psi, A) \mapsto [\psi]$ need not be surjective, and the following definition describes the image of this morphism:

Definition 2.8. Given $[\psi] \in \mathrm{N}(\mathrm{Im}[\alpha])$, we say that $\Lambda_{[\psi]}$ **lifts**, or simply that $[\psi]$ lifts, if there is a matrix $A \in \mathrm{GL}_m(\mathbb{Z})$ lifting $\Lambda_{[\psi]}$. We then define

$$\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}} := \{[\psi] \in \mathrm{N}(\mathrm{Im}[\alpha]) \mid \Lambda_{[\psi]} \text{ lifts}\}$$

to be the set of elements $[\psi]$ in the normalizer of $\mathrm{Im}[\alpha]$ such that $\Lambda_{[\psi]}$ lifts.

Observation 2.9. The set $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$ is a subgroup of $\mathrm{N}(\mathrm{Im}[\alpha])$ which contains $\mathrm{C}(\mathrm{Im}[\alpha])$ as a normal subgroup. Indeed, if $[\psi] \in \mathrm{C}(\mathrm{Im}[\alpha])$ then $\Lambda_{[\psi]} = \mathrm{Id}$ is lifted by the identity matrix $\mathrm{Id}_m \in \mathrm{GL}_m(\mathbb{Z})$, hence $[\psi]$ belongs to $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$; in particular it is non-empty. Moreover, if $[\psi], [\phi] \in \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$ are lifted by $A, B \in \mathrm{GL}_m(\mathbb{Z})$, respectively, then one may easily check that $[\psi][\phi]^{-1}$ is lifted by AB^{-1}, hence $[\psi][\phi]^{-1} \in \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$. Finally, $\mathrm{C}(\mathrm{Im}[\alpha])$ is normal in $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$, since it is normal in $\mathrm{N}(\mathrm{Im}[\alpha])$.

Notation 2.10. We denote by $\pi^{-1}\,\mathrm{Im}[\alpha]$, $\pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha])$, $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$ and $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$ the respective preimage subgroups in $\mathrm{Aut}(F_n)$.

We are now ready to describe the subgroup of $\mathrm{Aut}(F_n) \times \mathrm{GL}_m(\mathbb{Z})$ to which $\mathrm{Aut}_{F_n}(M_\alpha)$ is isomorphic. On its own, this isomorphism is essentially a notational shift rather than a new result. However, it clarifies the structure of $\mathrm{Aut}_{F_n}(M_\alpha)$ and provides a convenient notation which is easier to work with. The statement is as follows:

Proposition 2.11. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. We have an isomorphism*

$$\mathrm{Aut}_{F_n}(M_\alpha) \cong \{(\psi, A) \mid \psi \in \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathit{lift}},\ A \in \mathrm{GL}_m(\mathbb{Z}),\ A \textit{ lifts } \Lambda_{[\psi]}\},$$

where the latter is a subgroup of $\mathrm{Aut}(F_n) \times \mathrm{GL}_m(\mathbb{Z})$.

Proof. Let us denote by X the set on the right. By Observation 2.4 every $\Psi \in \mathrm{Aut}_{F_n}(M_\alpha)$ is uniquely determined by a pair (ψ, A) with $\psi \in \mathrm{Aut}(F_n)$ and $A \in \mathrm{GL}_m(\mathbb{Z})$ such that $[\alpha^{\mathbf{a_i}}] = [\psi]^{-1}[\alpha_i][\psi]$ for $i = 1, ..., m$, where $\mathbf{a_i} \in \mathbb{Z}^m$ denotes the i-th row of A. We note that this last condition is equivalent to saying that $\psi \in \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$ and that A lifts $\Lambda_{[\psi]}$. Therefore, the map $\mathrm{Aut}_{F_n}(M_\alpha) \longrightarrow X$ sending each $\Psi \in \mathrm{Aut}_{F_n}(M_\alpha)$ to the corresponding pair $(\psi, A) \in X$ representing Ψ is a bijection. We can then translate the group structure from $\mathrm{Aut}_{F_n}(M_\alpha)$ to X using this bijection, so that $\mathrm{Aut}_{F_n}(M_\alpha) \cong X$. Finally, the operation on X is defined component-wise, as Observation 2.5 shows. □

From now on we will identify $\mathrm{Aut}_{F_n}(M_\alpha)$ with the group described in the prior lemma, and consider

$$S_\alpha := \{(\mathrm{Id}, A) \mid A \in \mathrm{GL}_m(\mathbb{Z}) \text{ lifts } \Lambda_{[\mathrm{Id}]} = \mathrm{Id}\}.$$

It is immediate to check that S_α is a normal subgroup of $\mathrm{Aut}_{F_n}(M_\alpha)$. Moreover, one may easily check that $A \in \mathrm{GL}_m(\mathbb{Z})$ lifts $\mathrm{Id} \in \mathrm{Aut}(\mathbb{Z}^m/\mathrm{Ker}[\alpha])$ if and only if $\mathbf{e_i}(A - \mathrm{Id}_m) \in \mathrm{Ker}[\alpha]$ for $i = 1, ..., m$, i.e., the rows of $A - \mathrm{Id}_m$ belong to $\mathrm{Ker}[\alpha]$. Therefore, there is an isomorphism

$$S_\alpha \cong \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid \mathbf{e_i}(A - \mathrm{Id}_m) \in \mathrm{Ker}[\alpha] \text{ for } i = 1, ..., m\} \leq \mathrm{GL}_m(\mathbb{Z})$$

which allows us to identify S_α with the above subgroup of $\mathrm{GL}_m(\mathbb{Z})$. The following observation is immediate from the above isomorphism, and will be widely used in Section 3.3; note that it also applies to the setting of Lemma 1.20:

Observation 2.12. If $\mathrm{Ker}[\alpha] = \langle d_1\mathbf{e_1}, \dots, d_m\mathbf{e_m}\rangle$ with $d_1, \dots, d_m \geq 0$ integers, then

$$S_\alpha = \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid A\mathbf{e_i}^\mathsf{T} \equiv \mathbf{e_i}^\mathsf{T} \pmod{d_i} \text{ for } i = 1, \dots, m\}.^{3}$$

The main reason for introducing S_α is the following. Suppose that $\psi \in \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$, so that there exists a certain $A \in \mathrm{GL}_m(\mathbb{Z})$ such that A lifts $\Lambda_{[\psi]}$. If $B \in \mathrm{GL}_m(\mathbb{Z})$ is another matrix that lifts $\Lambda_{[\psi]}$, then $(\psi, A)^{-1}(\psi, B) = (\mathrm{Id}, A^{-1}B)$, so that $A^{-1}B$ lifts $\Lambda_{[\mathrm{Id}]} = \mathrm{Id}$ i.e. $A^{-1}B \in S_\alpha$. This shows that the set of matrices in $\mathrm{GL}_m(\mathbb{Z})$ which lift $\Lambda_{[\psi]}$ is equal to the coset $A \cdot S_\alpha$. In other words, every element of $\mathrm{Aut}_{F_n}(M_\alpha)$ whose first coordinate is equal to ψ is of the form (ψ, AC) for a certain $C \in S_\alpha$. More precisely, we have the following theorem:

Theorem 2.13. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. Then there is a short exact sequence*

$$1 \longrightarrow S_\alpha \xrightarrow{\ \iota\ } \mathrm{Aut}_{F_n}(M_\alpha) \xrightarrow{\ \tau\ } \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{lift} \longrightarrow 1$$

where $\iota : S_\alpha \longrightarrow \mathrm{Aut}_{F_n}(M_\alpha)$ denotes the inclusion $A \mapsto (\mathrm{Id}, A)$ and $\tau : \mathrm{Aut}_{F_n}(M_\alpha) \longrightarrow \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{lift}$ the projection onto the first coordinate. Equivalently, there is an isomorphism $\mathrm{Aut}_{F_n}(M_\alpha)/S_\alpha \cong \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{lift}$.

Proof. Clearly ι and τ are injective and surjective group homomorphisms, respectively. Moreover, we have that $\mathrm{Ker}\,\tau$ is the set of elements (Id, A) such that $A \in \mathrm{GL}_m(\mathbb{Z})$ lifts Id; hence $\mathrm{Ker}\,\tau = S_\alpha = \mathrm{Im}\,\iota$ and this finishes the proof. □

Theorem 2.13 shows that $\mathrm{Aut}_{F_n}(M_\alpha)$ is an extension of $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$ by S_α. As we will see in Section 3.3, every FBFA group with $n = 2$ is isomorphic to a certain $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^m$ such that the short exact sequence in the above theorem splits, hence $\mathrm{Aut}_{F_2}(M_\alpha) = S_\alpha \rtimes \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$, which reduces the study of $\mathrm{Aut}_{F_2}(M_\alpha)$ to that of these two subgroups. For this purpose, we will use the following criterion to determine when the short exact sequence splits:

Proposition 2.14. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. Then the short exact sequence described in Theorem 2.13 splits if and only if there is a group homomorphism*

$$\begin{aligned} s : \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{lift} &\longrightarrow \mathrm{GL}_m(\mathbb{Z}) \\ \psi \quad &\longmapsto \quad s_\psi \end{aligned}$$

satisfying that s_ψ lifts $\Lambda_{[\psi]}$ for all $\psi \in \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{lift}$. In such a case, we have that

$$\mathrm{Aut}_{F_n}(M_\alpha) \cong S_\alpha \rtimes \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{lift}$$

where $\psi \in \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{lift}$ acts on $A \in S_\alpha$ via $\psi^{-1}A\psi = s_\psi^{-1}As_\psi$.

Proof. By definition, the short exact sequence splits if and only if there exists a group homomorphism $\bar{s} : \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}} \to \mathrm{Aut}_{F_n}(M_\alpha)$ such that $\bar{s}\tau$ is the identity on $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$. This forces $\bar{s}$ to be of the form $\psi\bar{s} = (\psi, s_\psi)$ with s a group homomorphism satisfying the properties in the proposition's statement; the result easily follows from here. □

Observation 2.15. For any $[\psi] \in \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$, the map $\Lambda_{[\psi]}$ depends only on the class $[\psi] \cdot \mathrm{C}(\mathrm{Im}[\alpha])$. In particular, if there exists a group homomorphism $\bar{s} : \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\,\mathrm{C}(\mathrm{Im}[\alpha]) \longrightarrow \mathrm{GL}_m(\mathbb{Z})$ such that the image of any class $[\psi]\cdot\mathrm{C}(\mathrm{Im}[\alpha])$ lifts $\Lambda_{[\psi]}$, then the composition of the natural map $\pi^{-1}(\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}) \longrightarrow \mathrm{GL}_m(\mathbb{Z})$ with $\bar{s}$ yields a group homomorphism s satisfying Proposition 2.14.

[3] If $d_i = 0$, the congruence $A\mathbf{e_i}^\mathsf{T} \equiv \mathbf{e_i}^\mathsf{T} \pmod 0$ is interpreted as exact equality $A\mathbf{e_i}^\mathsf{T} = \mathbf{e_i}^\mathsf{T}$. If $d_i = 1$, the congruence is automatically satisfied.

In the situation of Proposition 2.14, the subgroup $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$ is identified with the subgroup of $\mathrm{Aut}_{F_n}(M_\alpha)$ consisting of elements of the form (ψ, s_ψ), where $\psi \in \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}$. An immediate consequence of the above two results is the following:

Corollary 2.16. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. If $\mathrm{N}(\mathrm{Im}[\alpha]) = \mathrm{C}(\mathrm{Im}[\alpha])$ then*

$$\mathrm{Aut}_{F_n}(M_\alpha) \cong S_\alpha \times \pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha]).$$

Proof. We know that $\mathrm{C}(\mathrm{Im}[\alpha]) \leq \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}} \leq \mathrm{N}(\mathrm{Im}[\alpha])$, and our hypothesis implies that these are equalities. Moreover, for every $\psi \in \pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha])$ we have that $\Lambda_{[\psi]} = \mathrm{Id}$, hence $s_\psi = \mathrm{Id}_m$ lifts $\Lambda_{[\psi]}$. This provides a group homomorphism s satisfying the statement of Proposition 2.14; therefore we have an isomorphism $\mathrm{Aut}_{F_n}(M_\alpha) \cong S_\alpha \times \pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha])$, since in this case the induced action is the identity. □

In some cases, the isomorphism type of $\mathrm{Im}[\alpha]$ forces the short exact sequence of Theorem 2.13 to split. For example, it is straightforward to show using Proposition 2.14 that if $\mathrm{Im}[\alpha] \cong \mathbb{Z}$ or C_r with $r = 2, 3, 4, 6$, then the short exact sequence splits and $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}} = \mathrm{N}(\mathrm{Im}[\alpha])$. This will be very useful when studying the case $n = 2$ in Section 3.3, as we will show that any cyclic subgroup of $\mathrm{Out}(F_2)$ must be isomorphic to either $\mathbb{Z}$ or C_r with $r = 2, 3, 4, 6$. Another such situation, which is quite general, is described in the following corollary:

Corollary 2.17. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism and suppose that $\mathrm{Ker}[\alpha]$ is a direct summand of $\mathbb{Z}^m$ or, equivalently, that $\mathrm{Im}[\alpha]$ is torsion-free. Then $\mathrm{N}(\mathrm{Im}[\alpha]) = \mathrm{N}(\mathrm{Im}[\alpha])_{lift}$ and the short exact sequence described in Theorem 2.13 splits.*

Proof. If H is a direct summand of $\mathbb{Z}^m$, then there exists a subgroup H' of $\mathbb{Z}^m$ such that $\mathbb{Z}^m = H \oplus H'$, and we may identify the quotient $\mathbb{Z}^m/H$ with H'. We choose a basis $\mathbf{v_1}, ..., \mathbf{v_m}$ of $\mathbb{Z}^m$ such that $\mathbf{v_1}, ..., \mathbf{v_\ell}$ is a basis of H and $\mathbf{v_{\ell+1}}, ..., \mathbf{v_m}$ is a basis of H', with ℓ equal to the rank of H, and let $P \in \mathrm{GL}_m(\mathbb{Z})$ be the matrix with rows $\mathbf{v_1}, ..., \mathbf{v_m}$. If f is an automorphism of $Z^m/H = H'$, then we may extend it to an automorphism $\bar{f}$ of $\mathbb{Z}^m$ by defining $\bar{f}(\mathbf{v_i}) = \mathbf{v_i}$ for $i = 1, ..., \ell$, and note that $\bar{f}$ lifts f. If $A_f \in \mathrm{GL}_{m-\ell}(\mathbb{Z})$ is the matrix of f in the basis $\mathbf{v_{\ell+1}}, ..., \mathbf{v_m}$, then $B_f = P^{-1} \begin{pmatrix} \mathrm{Id}_\ell & 0 \\ 0 & A_f \end{pmatrix} P \in \mathrm{GL}_m(\mathbb{Z})$ is the matrix of $\bar{f}$ in the canonical basis $\mathbf{e_1}, ..., \mathbf{e_m}$. The matrix B_f lifts the automorphism $f \in \mathrm{Aut}(\mathbb{Z}^m/H)$, and provides a group homomorphism $B : \mathrm{Aut}(\mathbb{Z}^m/H) \longrightarrow \mathrm{GL}_m(\mathbb{Z})$, $f \mapsto B_f$, such that B_f always lifts $f \in \mathrm{Aut}(\mathbb{Z}^m/H)$.

We will now apply the previous discussion to $H = \mathrm{Ker}[\alpha]$, so that we have a group homomorphism $B : \mathrm{Aut}(\mathbb{Z}^m/H) \longrightarrow \mathrm{GL}_m(\mathbb{Z})$ such that B_f always lifts $f \in \mathrm{Aut}(\mathbb{Z}^m/H)$ We identify $\mathrm{Im}[\alpha] = \mathbb{Z}^m/\,\mathrm{Ker}[\alpha]$ via the first isomorphism theorem. In particular, for every $[\psi] \in \mathrm{N}(\mathrm{Im}[\alpha])$ we have that $B_{\Lambda_{[\psi]}} \in \mathrm{GL}_m(\mathbb{Z})$ lifts $\Lambda_{[\psi]}$, hence $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}} = \mathrm{N}(\mathrm{Im}[\alpha])$. Recall that we may identify the quotient $\mathrm{N}(\mathrm{Im}[\alpha])/\,\mathrm{C}(\mathrm{Im}[\alpha])$ with a subgroup of $\mathrm{Aut}(\mathrm{Im}[\alpha]) = \mathrm{Aut}(\mathbb{Z}^m/\,\mathrm{Ker}[\alpha])$, with a class $[\psi] \cdot \mathrm{C}(\mathrm{Im}[\alpha])$ being identified with $\Lambda_{[\psi]}$. Then the restriction of B to $\mathrm{N}(\mathrm{Im}[\alpha])/\,\mathrm{C}(\mathrm{Im}[\alpha])$ provides a group homomorphism $\mathrm{N}(\mathrm{Im}[\alpha])/\,\mathrm{C}(\mathrm{Im}[\alpha])$ satisfying the hypothesis from Observation 2.15; this finishes the proof. □

The following observation provides a generalization of Corollary 2.16:

Observation 2.18. Consider the following subgroup of $\mathrm{Aut}_{F_n}(M_\alpha)$

$$\{(\psi, A) \mid \psi \in \pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha]),\ A \in S_\alpha\} \cong \pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha]) \times S_\alpha.$$

One may easily check that it is in fact a normal subgroup of $\mathrm{Aut}_{F_n}(M_\alpha)$. A similar reasoning to the one used in Theorem 2.13 shows there is a short exact sequence of groups

$$1 \longrightarrow \pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha]) \times S_\alpha \longrightarrow \mathrm{Aut}_{F_n}(M_\alpha) \longrightarrow \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\,\mathrm{C}(\mathrm{Im}[\alpha]) \longrightarrow 1$$

where $\pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha]) \times S_\alpha \to \mathrm{Aut}_{F_n}(M_\alpha)$ denotes the inclusion and $\mathrm{Aut}_{F_n}(M_\alpha) \to \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\,\mathrm{C}(\mathrm{Im}[\alpha])$ the projection onto the first coordinate composed with the natural map $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}} \to \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\,\mathrm{C}(\mathrm{Im}[\alpha])$. However, characterizing when this short exact sequence splits is more delicate, and we leave the discussion at this point.

We now study the group of inner automorphisms of M_α, denoted as usual by $\mathrm{Inn}(M_\alpha)$, which is a normal subgroup of $\mathrm{Aut}(M_\alpha)$. We have an isomorphism $\mathrm{Inn}(M_\alpha) \cong M_\alpha/Z(M_\alpha)$ and the isomorphism type of $\mathrm{Inn}(M_\alpha)$ is an isomorphism invariant of M_α. Although the structure of $\mathrm{Inn}(M_\alpha)$ could already have been analysed at the end of Section 1, the notation introduced in Proposition 2.11 allows for a cleaner treatment. We start by observing the following:

Observation 2.19. It is immediate to check that every inner automorphism of M_α leaves F_n invariant, hence $\mathrm{Inn}(M_\alpha)$ is a normal subgroup of $\mathrm{Aut}_{F_n}(M_\alpha)$. If we consider $\Gamma_{t^a\omega} \in \mathrm{Inn}(M_\alpha)$ to be the conjugation by $t^a\omega \in M_\alpha$, then we have that

$$x_i\Gamma_{t^{\mathbf{a}}\omega} = \omega^{-1}t^{-\mathbf{a}}x_i t^{\mathbf{a}}\omega = \omega^{-1}\cdot x_i\alpha^{\mathbf{a}}\cdot\omega = x_i\alpha^{\mathbf{a}}\gamma_\omega,$$

$$t_j\Gamma_{t^{\mathbf{a}}\omega} = \omega^{-1}t^{-\mathbf{a}}t_j t^{\mathbf{a}}\omega = \omega^{-1}\cdot t_j\cdot\omega = t_j(\omega\alpha_j)^{-1}\omega,$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. This means that $\Gamma_{t^{\mathbf{a}}\omega}$ corresponds to $(\alpha^{\mathbf{a}}\gamma_\omega, \mathrm{Id}_m) \in \mathrm{Aut}_{F_n}(M_\alpha)$. Hence, we may identify

$$\mathrm{Inn}(M_\alpha) = \{(\alpha^{\mathbf{a}}\gamma_\omega, \mathrm{Id}_m) \mid \mathbf{a} \in \mathbb{Z}^m,\ \omega \in F_n\} \trianglelefteq \mathrm{Aut}_{F_n}(M_\alpha).$$

Observation 2.19 together with the fact that $\pi^{-1}\,\mathrm{Im}[\alpha] = \{\alpha^{\mathbf{a}}\gamma_\omega \mid \mathbf{a} \in \mathbb{Z}^m,\ \omega \in F_n\}$ immediately imply the following result:

Proposition 2.20. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. Then there is an isomorphism*

$$\mathrm{Inn}(M_\alpha) \cong \pi^{-1}\,\mathrm{Im}[\alpha],$$

where $\pi^{-1}\,\mathrm{Im}[\alpha]$ denotes the preimage subgroup in $\mathrm{Aut}(F_n)$.

Observation 2.21. Proposition 2.20 implies that the isomorphism type of $\pi^{-1}\,\mathrm{Im}[\alpha]$ is an isomorphism invariant of a FBFA group M_α. More precisely, if $F_n \rtimes_\alpha \mathbb{Z}^m \cong F_{n'} \rtimes_\beta \mathbb{Z}^{m'}$ then $\pi^{-1}\,\mathrm{Im}[\alpha] \cong \pi^{-1}\,\mathrm{Im}[\beta]$.

Proposition 2.22. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism, and consider the semidirect product $F_n \rtimes \mathrm{Im}\,\alpha$, where $\mathrm{Im}\,\alpha$ acts naturally on F_n. Then*

(i) If $\mathrm{Ker}[\alpha] = \mathrm{Ker}\,\alpha$, then $\mathrm{Im}[\alpha] \cong \mathrm{Im}\,\alpha$ and there is an isomorphism $\mathrm{Inn}(M_\alpha) \cong F_n \rtimes \mathrm{Im}\,\alpha$.

(ii) If $\mathrm{Ker}[\alpha] = \mathrm{Ker}\,\alpha \oplus \langle\mathbf{v}\rangle$ with $\mathbf{v} \in \mathbb{Z}^m$ non-zero, then $\mathrm{Inn}(M_\alpha) \cong (F_n \rtimes \mathrm{Im}\,\alpha)/\left\langle(\alpha^{\mathbf{v}}, \omega^{-1})\right\rangle$, where $\omega \in F_n$ is such that $\alpha^{\mathbf{v}} = \gamma_\omega$ and $(\alpha^{\mathbf{v}}, \omega^{-1})$ denotes the corresponding element in $F_n \rtimes \mathrm{Im}\,\alpha$.

Proof. We identify $\mathrm{Inn}(M_\alpha) = \pi^{-1}\,\mathrm{Im}[\alpha]$ using Proposition 2.20. Note that, by Lemma 1.24, either case (i) or (ii) must occur. We consider the map $f : F_n \rtimes \mathrm{Im}\,\alpha \longrightarrow \mathrm{Inn}(M_\alpha)$, which sends $(\alpha^{\mathbf{a}}, x) \mapsto \alpha^{\mathbf{a}}\gamma_x$ for $\mathbf{a} \in \mathbb{Z}^m$ and $x \in F_n$; it is immediate to check that f is a surjective group homomorphism. In case (i), since $\mathrm{Ker}[\alpha] = \mathrm{Ker}\,\alpha$, no non-trivial element of $\mathrm{Im}\,\alpha$ can be an inner automorphism of F_n. It follows that the kernel of f is trivial, and therefore $\mathrm{Inn}(M_\alpha) \cong F_n \rtimes \mathrm{Im}\,\alpha$.

In case (ii), suppose $(\alpha^{\mathbf{a}}, x) \mapsto \alpha^{\mathbf{a}}\gamma_x = \mathrm{Id}$. Then $\alpha^{\mathbf{a}} = \gamma_x^{-1}$ implies $\mathbf{a} \in \mathrm{Ker}[\alpha] = \mathrm{Ker}\,\alpha \oplus \langle\mathbf{v}\rangle$, so there exists an integer d such that $\gamma_x^{-1} = \alpha^{\mathbf{a}} = \alpha^{d\mathbf{v}} = \gamma_\omega^d$. This gives $(\alpha^{\mathbf{a}}, x) = (\alpha^{d\mathbf{v}}, \omega^{-d}) = (\alpha^{\mathbf{v}}, \omega^{-1})^d$, where the last equality holds because $\alpha^{\mathbf{v}} = \gamma_\omega$ fixes ω^{-1}. Reciprocally, we have $(\alpha^{\mathbf{v}}, \omega^{-1}) \mapsto \alpha^{\mathbf{v}}\gamma_\omega^{-1} = \mathrm{Id}$. This shows that the kernel of f is $\langle(\alpha^{\mathbf{v}}, \omega^{-1})\rangle$, and the first isomorphism theorem then gives the result. □

We consider $\mathrm{Out}(M_\alpha) = \mathrm{Aut}(M_\alpha)/\mathrm{Inn}(M_\alpha)$ the outer automorphism group M_α. We shall finish this section by studying the image of $\mathrm{Aut}_{F_n}(M_\alpha)$ in $\mathrm{Out}(M_\alpha)$, which we denote as follows:

Definition 2.23. Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. We will denote by $\mathrm{Out}_{F_n}(M_\alpha) := Aut_{F_n}(M_\alpha)/\mathrm{Inn}(M_\alpha)$ the image of $\mathrm{Aut}_{F_n}(M_\alpha)$ in $\mathrm{Out}(M_\alpha)$, and note that it is a subgroup of $\mathrm{Out}(M_\alpha)$.

Observation 2.24. Let $\Psi, \Phi \in \mathrm{Aut}(M_\alpha)$ be such that $[\Psi] = [\Phi]$ in $\mathrm{Out}(M_\alpha)$. Then one may easily check that Ψ leaves F_n invariant if and only if Φ does. Hence it makes sense to say that a class $[\Psi] \in \mathrm{Out}(M_\alpha)$ leaves F_n invariant if one (and so, any) representative of the class in $\mathrm{Aut}(M_\alpha)$ leaves F_n invariant. Then $\mathrm{Out}_{F_n}(M_\alpha)$ is the subgroup of $\mathrm{Out}(M_\alpha)$ formed by those $[\Psi] \in \mathrm{Out}(M_\alpha)$ leaving F_n invariant.

The main reason for studying this subgroup is that it actually coincides with the outer automorphism group for the family of full FBFA groups:

Observation 2.25. If M_α is full then Observation 2.2 implies that $\mathrm{Out}_{F_n}(M_\alpha) = \mathrm{Out}(M_\alpha)$.

Taking into account Observation 2.19, we have that $\mathrm{Out}_{F_n}(M_\alpha)$ admits an analogous description to that of $\mathrm{Aut}_{F_n}(M_\alpha)$, with the first coordinate now taken modulo $\pi^{-1}\,\mathrm{Im}[\alpha]$. More precisely, we have the following result, analogous to Proposition 2.11:

Proposition 2.26. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. There is an isomorphism of groups*

$$\mathrm{Out}_{F_n}(M_\alpha) \cong \{([\psi]_{\mathrm{Im}[\alpha]}, A) \mid [\psi]_{\mathrm{Im}[\alpha]} \in \mathrm{N}(\mathrm{Im}[\alpha])_{lift}/\mathrm{Im}[\alpha],\ A \in \mathrm{GL}_m(\mathbb{Z}) \text{ lifts } \Lambda_{[\psi]}\},$$

where $[\psi]_{\mathrm{Im}[\alpha]}$ denotes the class of $[\psi] \in \mathrm{N}(\mathrm{Im}[\alpha])_{lift}$ in the quotient $\mathrm{N}(\mathrm{Im}[\alpha])_{lift}/\mathrm{Im}[\alpha]$ and the operation in the latter group is defined component-wise.

Proof. Let us denote by X the set on the right. First of all, note that $\mathrm{Im}[\alpha] \subseteq \mathrm{C}(\mathrm{Im}[\alpha])$, hence $\Lambda_{[\psi]}$ does not depend on the chosen representative of the class $[\psi]_{\mathrm{Im}[\alpha]}$ and X is well-defined. It is immediate to check that X forms a group with the operation being defined component-wise. We have a surjective group homomorphism $\mathrm{Aut}_{F_n}(M_\alpha) \longrightarrow X$ mapping $(\psi, A) \longmapsto ([\psi]_{\mathrm{Im}[\alpha]}, A)$. Now, note that $([\psi]_{\mathrm{Im}[\alpha]}, A)$ is the trivial element if and only if $A = \mathrm{Id}_m$ and $[\psi] \in \mathrm{Im}[\alpha]$, that is $\psi = \alpha^{\mathbf{a}}\gamma_\omega$ for some $\mathbf{a} \in \mathbb{Z}^m$ and $\omega \in F_n$. Hence, the kernel is equal to $\mathrm{Inn}(M_\alpha)$ and we have a group isomorphism $\mathrm{Out}_{F_n}(M_\alpha) = \mathrm{Aut}_{F_n}(M_\alpha)/\mathrm{Inn}(M_\alpha) \cong X$; this finishes the proof. □

From now on we will identify $\mathrm{Out}_{F_n}(M_\alpha)$ with the group described in Proposition 2.26. Note that we can identify S_α with the normal subgroup of $\mathrm{Out}_{F_n}(M_\alpha)$ formed by those elements (Id, A) such that $A \in \mathrm{GL}_m(\mathbb{Z})$ lifts Id. We also obtain a short exact sequence for $\mathrm{Out}_{F_n}(M_\alpha)$, analogous to that for $\mathrm{Aut}_{F_n}(M_\alpha)$ given in Theorem 2.13:

Proposition 2.27. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. There is a short exact sequence*

$$1 \longrightarrow S_\alpha \longrightarrow \mathrm{Out}_{F_n}(M_\alpha) \longrightarrow \mathrm{N}(\mathrm{Im}[\alpha])_{lift}/\mathrm{Im}[\alpha] \longrightarrow 1,$$

where $S_\alpha \longrightarrow \mathrm{Out}_{F_n}(M_\alpha)$ denotes the inclusion and $\mathrm{Out}_{F_n}(M_\alpha) \longrightarrow \mathrm{N}(\mathrm{Im}[\alpha])_{lift}/\mathrm{Im}[\alpha]$ the projection onto the first component.

Proof. Clearly the inclusion $S_\alpha \longrightarrow \mathrm{Out}_{F_n}(M_\alpha)$, $A \mapsto (\mathrm{Id}, A)$, is an injective group homomorphism. Moreover, the projection $\mathrm{Out}_{F_n}(M_\alpha) \longrightarrow \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\mathrm{Im}[\alpha]$ onto the first coordinate is surjective and one may easily check that its kernel is equal to S_α; this finishes the proof. □

Proposition 2.27 shows that $\mathrm{Out}_{F_n}(M_\alpha)$ is an extension of $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\mathrm{Im}[\alpha]$ by S_α. We also have an analogous criterion, similar to that of Proposition 2.14, for when the short exact sequence for $\mathrm{Out}_{F_n}(M_\alpha)$ splits:

Proposition 2.28. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. Then the short exact sequence described in Proposition 2.27 splits if and only if there is a group homomorphism*

$$\begin{aligned} s : \mathrm{N}(\mathrm{Im}[\alpha])_{lift}/\mathrm{Im}[\alpha] &\longrightarrow \mathrm{GL}_m(\mathbb{Z}) \\ [\psi]_{\mathrm{Im}[\alpha]} &\longmapsto s_{[\psi]_{\mathrm{Im}[\alpha]}} \end{aligned}$$

satisfying that $s_{[\psi]_{\mathrm{Im}[\alpha]}}$ lifts $\Lambda_{[\psi]}$ for all $[\psi]_{\mathrm{Im}[\alpha]} \in \mathrm{N}(\mathrm{Im}[\alpha])_{lift}/\mathrm{Im}[\alpha]$. In such a case, we have that

$$\mathrm{Out}_{F_n}(M_\alpha) \cong S_\alpha \rtimes \mathrm{N}(\mathrm{Im}[\alpha])_{lift}/\mathrm{Im}[\alpha]$$

where $[\psi]_{\mathrm{Im}[\alpha]} \in \mathrm{N}(\mathrm{Im}[\alpha])_{lift}/\mathrm{Im}[\alpha]$ acts on $A \in S_\alpha$ via $[\psi]^{-1}_{\mathrm{Im}[\alpha]} A [\psi]_{\mathrm{Im}[\alpha]} = s^{-1}_{[\psi]_{\mathrm{Im}[\alpha]}} A s_{[\psi]_{\mathrm{Im}[\alpha]}}$.

Proof. The proof is analogous to that of Proposition 2.14. □

Assume that the conditions of Proposition 2.28 are satisfied. The subgroup $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\mathrm{Im}[\alpha]$ is identified with the subgroup of $\mathrm{Out}_{F_n}(M_\alpha)$ consisting of elements of the form $([\psi]_{\mathrm{Im}[\alpha]}, s_{[\psi]_{\mathrm{Im}[\alpha]}})$, where $[\psi]_{\mathrm{Im}[\alpha]} \in \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\mathrm{Im}[\alpha]$. Moreover, the composition of the natural projection $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha]) \longrightarrow \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\mathrm{Im}[\alpha]$ with the homomorphism s yields a group homomorphism $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha]) \longrightarrow \mathrm{GL}_m(\mathbb{Z})$ satisfying the hypothesis of Proposition 2.14. We therefore obtain the following consequence:

Observation 2.29. The splitting of the short exact sequence for $\mathrm{Out}_{F_n}(M_\alpha)$ in Proposition 2.27 implies the splitting of the short exact sequence for $\mathrm{Aut}_{F_n}(M_\alpha)$ in Theorem 2.13.

Moreover, under the situation of Observation 2.15 both short exact sequences split:

Observation 2.30. If there exists a group homomorphism $\bar{s} : \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\,\mathrm{C}(\mathrm{Im}[\alpha]) \longrightarrow \mathrm{GL}_m(\mathbb{Z})$ such that the image of any class $[\psi]\cdot\mathrm{C}(\mathrm{Im}[\alpha])$ lifts $\Lambda_{[\psi]}$, then the composition of the natural map $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\mathrm{Im}[\alpha] \longrightarrow \mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}}/\,\mathrm{C}(\mathrm{Im}[\alpha])$ with $\bar{s}$ yields a group homomorphism s satisfying Proposition 2.28.

Using the above observation, one may easily check that the proofs of Corollaries 2.16 and 2.17 can be easily be adapted to prove the following analogous results, which we state without a proof:

Proposition 2.31. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism. If $\mathrm{N}(\mathrm{Im}[\alpha]) = \mathrm{C}(\mathrm{Im}[\alpha])$ then*

$$\mathrm{Out}_{F_n}(M_\alpha) \cong S_\alpha \times \mathrm{C}(\mathrm{Im}[\alpha])/\mathrm{Im}[\alpha].$$

Proposition 2.32. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ be a group homomorphism and suppose that $\mathrm{Ker}[\alpha]$ is a direct summand of $\mathbb{Z}^m$ or, equivalently, that $\mathrm{Im}[\alpha]$ is torsion-free. Then $\mathrm{N}(\mathrm{Im}[\alpha]) = \mathrm{N}(\mathrm{Im}[\alpha])_{lift}$ and the short exact sequence described in Proposition 2.28 splits.*

One may also easily that if $\mathrm{Im}[\alpha] \cong C_r$ with $r = 2, 3, 4, 6$, then the short exact sequence for $\mathrm{Out}_{F_n}(M_\alpha)$ in Proposition 2.27 splits and $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}} = \mathrm{N}(\mathrm{Im}[\alpha])$.

3. The Case $n = 2$

From now on we will restrict our attention to the study of the case $n = 2$, and we will denote $F_2 = \langle a, b | - \rangle$. This case is a lot easier to treat than the general case $n \geq 3$, namely because the structure of $\mathrm{Out}(F_2)$ is a lot simpler than that of $\mathrm{Out}(F_n)$ for $n \geq 3$. More precisely, we have the following well-known result:

Lemma 3.1. *The kernel of the abelianization from* $\mathrm{Aut}(F_2)$ *to* $\mathrm{GL}_2(\mathbb{Z})$ *consists of precisely the inner automorphism of* F_2*. That is,* $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$.

Proof. See [10, Ch. I, Prop. 4.5]. □

For the remainder of this work, we will identify $\mathrm{Out}(F_2)$ with $\mathrm{GL}_2(\mathbb{Z})$, and so a class $[\psi] \in \mathrm{Out}(F_2)$ corresponds to the matrix $\psi^{\mathrm{ab}} \in \mathrm{GL}_2(\mathbb{Z})$ whose rows are the vectors $(a\psi)^{\mathrm{ab}} = (|a\psi|_a, |a\psi|_b)$ and $(b\psi)^{\mathrm{ab}} = (|b\psi|_a, |b\psi|_b)$ counting the total number of occurrences of a and b in the words $a\psi, b\psi \in F_2$, respectively. For convenience, we will simply write $[\psi] = \psi^{\mathrm{ab}}$. According to context, we will regard elements of $\mathrm{Out}(F_2) = \mathrm{GL}_2(\mathbb{Z})$ either as outer automorphism classes $[\psi] \in \mathrm{Out}(F_2)$ or as matrices $A \in \mathrm{GL}_2(\mathbb{Z})$. Moreover, given $\psi \in \mathrm{Aut}(F_2)$, we will write it as $\psi : (a \mapsto a\psi,\ b \mapsto b\psi)$, specifying the images of the generators.

3.1 Abelian Subgroups of $\mathrm{GL}_2(\mathbb{Z})$

Most of the results we have seen in Sections 1 and 2 for $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^m$ depend on the structure of $\mathrm{Im}[\alpha]$, which in this case is a finitely generated abelian subgroup of $\mathrm{GL}_2(\mathbb{Z})$. Therefore, we will dedicate this subsection to the study of the finitely generated abelian subgroups of $\mathrm{GL}_2(\mathbb{Z})$. More precisely, we will classify these subgroups up to conjugacy.

The matrices $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ generate a free subgroup F of rank 2 of finite index in $\mathrm{GL}_2(\mathbb{Z})$ (cf. [9, Section 4.4]), so that $\mathrm{GL}_2(\mathbb{Z})$ is virtually free. Therefore, for any subgroup $H \leq \mathrm{GL}_2(\mathbb{Z})$ we have that

$$[H : H \cap F] \leq [\mathrm{GL}_2(\mathbb{Z}) : F] < \infty.$$

We will use this result to study the abelian subgroups of $\mathrm{GL}_2(\mathbb{Z})$. Even though the following result is not required for our purposes, since we only care about finitely generated abelian subgroups, we include it for completeness:

Lemma 3.2. *Every abelian subgroup of* $\mathrm{GL}_2(\mathbb{Z})$ *is finitely generated.*

Proof. Let H be an abelian subgroup of $\mathrm{GL}_2(\mathbb{Z})$. The intersection $H \cap F$ is an abelian subgroup of F, hence either trivial or isomorphic to $\mathbb{Z}$; in particular, it is finitely generated. Then H is finitely generated too, since $H \cap F$ has finite index in H. □

The following result shows that $\mathrm{Im}[\alpha]$ is either finite or isomorphic to $\mathbb{Z} \oplus T$, where T is a finite abelian group. Consequently, the structure of $\mathrm{Im}[\alpha]$ is relatively simple when $n = 2$, which makes this case significantly easier to handle than the general one.

Lemma 3.3. *Suppose* $H \leq \mathrm{GL}_2(\mathbb{Z})$ *is isomorphic to* $\mathbb{Z}^r$ *for some integer* $r \geq 0$*. Then* $r \leq 1$.

Proof. We have that $H \cap F$ is a finite index subgroup of $H \cong \mathbb{Z}^r$, which implies that $H \cap F \cong \mathbb{Z}^r$ too. However, note that $H \cap F$ is also an abelian subgroup of F, and so we must have $r \leq 1$. □

Observation 3.4. More generally, one has that an abelian subgroup of virtually free group is either finite or isomorphic to $\mathbb{Z} \oplus T$, where T is a finite abelian group.

Therefore, our next objective will be to study the finite abelian groups of $\mathrm{GL}_2(\mathbb{Z})$. Recall there is an isomorphism $\mathrm{GL}_2(\mathbb{Z}) \cong D_4 *_{D_2} D_6$, where D_n denotes the dihedral group of order $2n$ (cf. [6, Example 5.2, Ch. I]). More precisely, consider the following matrices of $\mathrm{GL}_2(\mathbb{Z})$

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then D_4 and D_6 correspond, respectively, to the following subgroups of $\mathrm{GL}_2(\mathbb{Z})$

$$\langle S, R \mid R^2 = 1, \; S^4 = 1, \; RSR = S^{-1} \rangle \cong D_4,$$

$$\langle T, R \mid R^2 = 1, \; T^6 = 1, \; RTR = T^{-1} \rangle \cong D_6,$$

while the common subgroup isomorphic to D_2 is $\langle R, S^2 \rangle$ and $\langle R, T^3 \rangle$, where $S^2 = T^3$. In order to study the finite abelian subgroups of $\mathrm{GL}_2(\mathbb{Z})$, we can make use of the following result:

Lemma 3.5. *Let $G = H_1 *_K H_2$ be an amalgamated free product of two groups H_1 and H_2 over a common subgroup K. Then every finite subgroup of G is conjugate to a subgroup of H_1 or H_2.*

Proof. See [13, I.4, Cor. to Thm. 8]. □

From the prior discussion, we may deduce the following classification of the finite abelian subgroups of $\mathrm{GL}_2(\mathbb{Z})$ up to conjugacy:

Lemma 3.6. *Every finite abelian subgroup of $\mathrm{GL}_2(\mathbb{Z})$ is conjugate to one and only one of the subgroups in the following list:*

- $\{\mathrm{Id}_2\}$,
- $\langle -\mathrm{Id}_2 \rangle$, $\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$, $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \cong C_2$,
- $\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -\mathrm{Id}_2 \right\rangle$, $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -\mathrm{Id}_2 \right\rangle \cong C_2 \times C_2$,
- $\left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle \cong C_3$,
- $\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong C_4$,
- $\left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong C_6$.

Proof. By Lemma 3.5 we have that every finite abelian subgroup of $\mathrm{GL}_2(\mathbb{Z})$ is conjugate to an abelian subgroup of D_4 or D_6. Now, one may easily check that every non-trivial abelian subgroup of D_4 is conjugate to one of the following:

$$\langle S^2 \rangle = \langle -\mathrm{Id}_2 \rangle, \; \langle R \rangle = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \; \langle RS \rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle, \; \langle S \rangle = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

$$\langle R, S^2\rangle = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -\mathrm{Id}_2 \right\rangle,\ \langle RS, S^2\rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -\mathrm{Id}_2 \right\rangle.$$

Similarly, every non-trivial abelian subgroup of D_6 is conjugate to one of the following:

$$\langle T^3\rangle = \langle S^2\rangle,\ \langle R\rangle,\ \langle RT\rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right\rangle,\ \langle T^2\rangle = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle,\ \langle T\rangle = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

$$\langle R, T^3\rangle = \langle R, S^2\rangle,\ \langle RT, T^3\rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, -\mathrm{Id}_2 \right\rangle,$$

We also observe that $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and so $\langle RT\rangle$ and $\langle R\rangle$ are conjugate in $\mathrm{GL}_2(\mathbb{Z})$, and so are $\langle R, S^2\rangle$ and $\langle RT, T^3\rangle$. This proves that every finite abelian subgroup in $\mathrm{GL}_2(\mathbb{Z})$ is conjugate to one of the subgroups appearing in the statement of the lemma.

Finally, if two subgroups from the statement of the lemma were conjugate to each other in $\mathrm{GL}_2(\mathbb{Z})$, then they would fall under the same isomorphism class. Now, the conjugacy class of $-\mathrm{Id}_2$ in $\mathrm{GL}_2(\mathbb{Z})$ is just $\{-\mathrm{Id}_2\}$, while $\pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ cannot be conjugate in $\mathrm{GL}_2(\mathbb{Z})$ since the reduction of these matrices in $\mathrm{GL}_2(\mathbb{F}_2)$ have order 2 and 1, respectively. Hence, the three subgroups from the statement of the lemma isomorphic to C_2 cannot be conjugate to each other in $\mathrm{GL}_2(\mathbb{Z})$, and neither the subgroups isomorphic to $C_2 \times C_2$; this finishes the proof. □

Using the classification of finite abelian subgroups up to conjugacy obtained above, we determine the isomorphism type of any abelian subgroup of $\mathrm{GL}_2(\mathbb{Z})$ and describe such subgroups explicitly up to conjugacy:

Lemma 3.7. *Every abelian subgroup H of $\mathrm{GL}_2(\mathbb{Z})$ is either finite and conjugate to one of the subgroups from Lemma 3.6, or there exists a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ of infinite order such that $H = \langle A\rangle \cong \mathbb{Z}$ or $H = \langle -\mathrm{Id}_2\rangle \times \langle A\rangle \cong C_2 \times \mathbb{Z}$.*

Proof. Suppose that H is not finite. By Lemma 3.2 and the structure theorem of finitely generated abelian groups, we can write $H = T \times K$ with T a finite abelian group and K a finitely generated free abelian group. By Lemma 3.3 we must have $K \cong \mathbb{Z}$ or, equivalently, there exists a matrix of infinite order $A \in \mathrm{GL}_2(\mathbb{Z})$ such that $K = \langle A\rangle$. By conjugating by a certain matrix $B \in \mathrm{GL}_2(\mathbb{Z})$, we may suppose that BTB^{-1} is one of the subgroups appearing in Lemma 3.6. Now, some straightforward calculations show that if $T \neq \{\mathrm{Id}_2\}, \langle -\mathrm{Id}_2\rangle$, then the centralizer of BTB^{-1} in $\mathrm{GL}_2(\mathbb{Z})$ is finite. However, the matrix BAB^{-1} commutes with all the elements of BTB^{-1}, hence belongs to the centralizer of BTB^{-1}; a contradiction because BAB^{-1} has infinite order. Therefore, we must have that $T = \{\mathrm{Id}_2\}$ or $T = \langle -\mathrm{Id}_2\rangle$, and so $H = \langle A\rangle$ or $H = \langle -\mathrm{Id}_2\rangle \times \langle A\rangle$. □

Later on, we will show that the number $r_\alpha \in \{0, 1, 2\}$ can be determined from $\mathrm{Im}[\alpha]$ up to conjugacy in $\mathrm{GL}_2(\mathbb{Z})$. For this purpose, we will use the following result, which corresponds to cases in which $r_\alpha = 1$.

Lemma 3.8. *Let $A \in \mathrm{GL}_m(\mathbb{Z})$ be a matrix of infinite order such that $\mathrm{rank}_{\mathbb{Z}}(A - \mathrm{Id}_2) = 1$. Then there is a unique integer $k > 0$ such that $\langle A\rangle$ is conjugate to $\left\langle \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \right\rangle$.*

Proof. By hypothesis both rows of $A - \mathrm{Id}_2$ are proportional to a primitive vector $(z_1, z_2) \in \mathbb{Z}^2$, with at least one of the rows not being zero. Therefore, there are integers $a, b \in \mathbb{Z}$, at least one of them not zero, such that

$$A = \begin{pmatrix} 1 + az_1 & az_2 \\ bz_1 & 1 + bz_2 \end{pmatrix}.$$

Since (z_1, z_2) is primitive there exist $u, v \in \mathbb{Z}$ such that $uz_2 - vz_1 = \pm 1$ and, by replacing (u, v) for $(-u, -v)$ if necessary, we might suppose $k := au + bv \geq 0$. Now, the matrix

$$B := \begin{pmatrix} u & v \\ z_1 & z_2 \end{pmatrix} \begin{pmatrix} 1 + az_1 & az_2 \\ bz_1 & 1 + bz_2 \end{pmatrix} \begin{pmatrix} u & v \\ z_1 & z_2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & k \\ 0 & \det A \end{pmatrix}$$

is conjugate to A in $\mathrm{GL}_2(\mathbb{Z})$. If $\det A = -1$ or $k = 0$ then one can easily check that $B^2 = \mathrm{Id}$, contradicting the fact that A has infinite order. Therefore, $\det A = 1$, $k > 0$ and $\langle A \rangle$ is conjugate to $\langle B \rangle$ in $\mathrm{GL}_2(\mathbb{Z})$. Lastly, if $k_2 > k_1 > 0$ are integers, then $\left(\begin{smallmatrix} 1 & k_2 \\ 0 & 1 \end{smallmatrix}\right)$ belongs to the kernel of the reduction morphism $\mathrm{GL}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}/k_2\mathbb{Z})$, but $\left(\begin{smallmatrix} 1 & k_1 \\ 0 & 1 \end{smallmatrix}\right)$ doesn't. Hence the subgroups generated by them cannot be conjugate in $\mathrm{GL}_2(\mathbb{Z})$, and this proves the uniqueness of k. □

The preceding discussion yields the following classification of abelian subgroups of $\mathrm{GL}_2(\mathbb{Z})$ up to conjugacy, which will suffice for our analysis of $\mathrm{im}[\alpha]$ and, consequently, the group $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^m$.

Lemma 3.9. *Every abelian subgroup of* $\mathrm{GL}_2(\mathbb{Z})$ *is, up to conjugacy, exactly one of the following:*

(i) a finite subgroup from Lemma 3.6,

(ii) $\left\langle \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \right\rangle$ *for some integer* $k > 0$,

(iii) $\langle A \rangle$ *with* $A \in \mathrm{GL}_2(\mathbb{Z})$ *of infinite order and* $\mathrm{rank}_{\mathbb{Z}}(A - \mathrm{Id}_2) = 2$,

(iv) $\langle -\mathrm{Id}_2 \rangle \times \langle A \rangle$ *with* $A \in \mathrm{GL}_2(\mathbb{Z})$ *of infinite order.*

Moreover, the finite subgroups are pairwise non-conjugate, and the subgroups in case (ii) corresponding to distinct values of k are also non-conjugate.

Proof. Follows immediately from Lemmas 3.6, 3.7 and 3.8, the fact that $\mathrm{rank}_{\mathbb{Z}}(A - \mathrm{Id}_2)$ is a conjugacy invariant for $A \in \mathrm{GL}_2(\mathbb{Z})$ and that conjugate subgroups are isomorphic. □

Observation 3.10. In the setting of Lemma 3.9, we have:

- In case (iii), two subgroups $\langle A \rangle$ and $\langle B \rangle$ are conjugate if and only if A is conjugate to B or B^{-1} in $\mathrm{GL}_2(\mathbb{Z})$.
- In case (iv), two subgroups $\langle -\mathrm{Id}_2 \rangle \times \langle A \rangle$ and $\langle -\mathrm{Id}_2 \rangle \times \langle B \rangle$ are conjugate if and only if A is conjugate to B, B^{-1}, $-B$, or $-B^{-1}$ in $\mathrm{GL}_2(\mathbb{Z})$.

Proof. The statement concerning when two subgroups $\langle A \rangle$ and $\langle B \rangle$ in case (iii) are conjugate follows from the fact that if there exists $C \in \mathrm{GL}_n(\mathbb{Z})$ such that $\langle A \rangle = C^{-1}\langle B \rangle C$, then $C^{-1}BC$ must generate $\langle A \rangle \cong \mathbb{Z}$, hence $C^{-1}BC = A^{\pm 1}$; the other implication is straightforward. Finally, if $\langle -\mathrm{Id}_2 \rangle \times \langle A \rangle$ and $\langle -\mathrm{Id}_2 \rangle \times \langle B \rangle$ are two subgroups in case (iv), then they are conjugate if and only if there exists a matrix $C \in \mathrm{GL}_2(\mathbb{Z})$ such that $\langle -\mathrm{Id}_2 \rangle \times \langle A \rangle = \langle -\mathrm{Id}_2 \rangle \times \langle C^{-1}BC \rangle$, which implies that $-\mathrm{Id}_2$ and CBC^{-1} must generate $\langle -\mathrm{Id}_2 \rangle \times \langle A \rangle \cong C_2 \times \mathbb{Z}$, and this is only possible if $C^{-1}BC = \pm A^{-\pm 1}$; the other implication is straightforward. □

3.2 Isomorphism Problem For $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^m$

The objective of this subsection is to solve the isomorphism problem for the family of groups of the form $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^m$ with $m \geq 1$, which we refer to as the family of FBFA groups with $n = 2$. More precisely, we consider the following problem:

Problem 3.11. Let $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^m$ and $M_\beta = F_2 \rtimes_\beta \mathbb{Z}^{m_2}$ be two FBFA groups with $n = 2$, and suppose that they are given by the following "data": two integers $m_1, m_2 \geq 1$, automorphisms $\alpha_1, ..., \alpha_{m_1} \in \mathrm{Aut}(F_2)$ commuting pair-wise and $\beta_1, ..., \beta_{m_2} \in \mathrm{Aut}(F_2)$ also commuting pair-wise. The *isomorphism problem for the family of FBFA groups with* $n = 2$ is the algorithmic problem of deciding, given such data, whether $M_\alpha \cong M_\beta$.

We begin by classifying M_α according to which case of Lemma 3.9 the image $\mathrm{Im}[\alpha]$ falls under. This classification shows that, up to direct products with free abelian groups, the only FBFA groups with $n = 2$ are $F_2 \rtimes \mathbb{Z}$ and $F_2 \rtimes \mathbb{Z}^2$. More precisely, every group M_α is isomorphic to one of these groups, or to a direct product of one of them with $\mathbb{Z}^k$ for some $k \geq 0$, after a suitable change of basis of $\mathbb{Z}^m$ as provided by Lemma 1.20.

Moreover, this classification substantially simplifies the study of the automorphism group $\mathrm{Aut}(M_\alpha)$, which will be carried out in Section 3.3. Although the full classification is not strictly necessary to solve the isomorphism problem, it provides a convenient framework that helps with both the study of the automorphism group and the isomorphism problem of FBFA groups with $n = 2$.

We now present the classification. It will be organized according to the number $r_\alpha \in \{0, 1, 2\}$ defined in 1.6, and according to the minimal number of generators of $\mathrm{Im}[\alpha]$, which can take the values 0, 1, or 2. Moreover, we note that the case $r_\alpha = 2$ corresponds precisely to those cases in which M_α is full.

Proposition 3.12. *Let $\beta : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_2)$ be a group homomorphism. Then there exists a group homomorphism $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_2)$ such that $\mathrm{Im}[\beta]$ is conjugate to $\mathrm{Im}[\alpha]$ in $\mathrm{GL}_2(\mathbb{Z})$, $M_\beta \cong M_\alpha$, and M_α falls into one of the following cases:*

Case $r_\alpha = 0$:

(1) $M_\alpha = \mathbb{Z}^m \times F_2$, *with* $\alpha_i = \mathrm{Id}$ *for all* $i = 1, \dots, m$.

Cases $r_\alpha = 1$: *here* $M_\alpha = \mathbb{Z}^{m-1} \times (F_2 \rtimes_{\alpha_1} \mathbb{Z})$, *with* $\alpha_i = \mathrm{Id}$ *for* $i = 2, \dots, m$, *and*

(2) $\alpha_1 : (a \mapsto a,\ b \mapsto b^{-1})$,

(3) $\alpha_1 : (a \mapsto b,\ b \mapsto a)$,

(4.k) [4] $\alpha_1 : (a \mapsto ab^k,\ b \mapsto b)$.

Cases $r_\alpha = 2$, $M_\alpha = \mathbb{Z}^{m-1} \times (F_2 \rtimes_{\alpha_1} \mathbb{Z})$: *here* $\alpha_i = \mathrm{Id}$ *for* $i = 2, \dots, m$, *and*

(5) $\mathrm{ord}[\alpha_1] = +\infty$ *and* $\mathrm{rank}_{\mathbb{Z}}([\alpha_1] - \mathrm{Id}_2) = 2$,

(6) $\alpha_1 : (a \mapsto a^{-1},\ b \mapsto b^{-1})$,

(7) $\alpha_1 : (a \mapsto b^{-1},\ b \mapsto ab^{-1})$,

[4] Treated as separate cases for each integer $k > 0$. When referring collectively, we denote them as case (4).

(8) $\alpha_1 : (a \mapsto b,\ b \mapsto a^{-1})$,

(9) $\alpha_1 : (a \mapsto ab^{-1},\ b \mapsto a)$.

Cases [5] $r_\alpha = 2$**,** $M_\alpha = \mathbb{Z}^{m-2} \times (F_2 \rtimes_{\alpha_1,\alpha_2} \mathbb{Z}^2)$**:** *here* $\alpha_i = \mathrm{Id}$ *for* $i = 3, \dots, m$*, and*

(10) $\alpha_1 : (a \mapsto a,\ b \mapsto b^{-1})$, $\alpha_2 : (a \mapsto a^{-1},\ b \mapsto b)$,

(11) $\alpha_1 : (a \mapsto a,\ b \mapsto b^{-1})$, $\alpha_2 : (a \mapsto a^{-1},\ b \mapsto b)$,

(12) [6] $\mathrm{ord}[\alpha_1] = +\infty$, $[\alpha_2] = -\mathrm{Id}_2$.

Proof. We have shown that $\mathrm{Im}[\beta]$ is conjugate via a certain $[\chi] \in \mathrm{GL}_2(\mathbb{Z})$ to one of the subgroups appearing in Lemma 3.9. We can actually suppose that M_α is equal to one of those subgroups by applying Proposition 1.5 with $P = \mathrm{Id}_2$ and $\chi \in \mathrm{Aut}(F_2)$. Applying Lemma 1.20 we get that there exists a group homomorphism $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_n)$ such that $M_\beta \cong M_\alpha$, $\mathrm{Im}[\alpha] = \mathrm{Im}[\beta]$ and

$$\mathrm{Ker}[\alpha] = \langle d_{m-\ell+1}\mathbf{e_{m-\ell+1}}, \dots, d_m\mathbf{e_m}\rangle$$

with ℓ the rank of $\mathrm{Ker}[\beta]$ and $d_m \mid d_{m-1} \mid \cdots \mid d_{m-\ell}$ positive integers (here we reordered the α_i's by reordering the t_i's). Now, note that

$$\mathrm{Im}[\beta] = \mathrm{Im}[\alpha] \cong \mathbb{Z}^{m-\ell} \oplus \bigoplus_{i=1}^{\ell} \mathbb{Z}/d_i\mathbb{Z}.$$

We now have the following cases:

(1) If $\mathrm{Im}[\alpha] = \{\mathrm{Id}_2\}$ then $\ell = m$ and $d_i = 1$ for all $i = 1, \dots m$, hence $\mathrm{Ker}[\alpha] = \mathbb{Z}^m$ and so $[\alpha_i] = \mathrm{Id}_2$ for all $i = 1, \dots, m$. By Proposition 1.5 we may just suppose that $\alpha_i = \mathrm{Id}$ for all $i = 1, \dots, m$, and so $M_\alpha = \mathbb{Z}^m \times F_2$.

(2) If $\mathrm{Im}[\alpha] = \langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle \cong C_2$ then we must have that $\ell = m$, $d_1 = 2$ and $d_i = 1$ for all $i = 2, \dots, m$. We have that $[\alpha_1] = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $[\alpha_i] = \mathrm{Id}_2$ for all $i = 2, \dots, m$, hence by Proposition 1.5 we can suppose that $\alpha_1 : (a \mapsto b,\ b \mapsto a)$ and $\alpha_i = \mathrm{Id}$ for all $i = 2, \dots, m$, and so $M_\alpha = \mathbb{Z}^{m-1} \times (F_2 \rtimes_{\alpha_1} \mathbb{Z})$.

(3) If $\mathrm{Im}[\alpha] = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\rangle$ then analogous argument to the one used in the prior case (2) shows that we may suppose $\alpha_1 : (a \mapsto a,\ b \mapsto b^{-1})$ and $\alpha_i = \mathrm{Id}$ for all $i = 2, \dots, m$; hence $M_\alpha = \mathbb{Z}^{m-1} \times (F_2 \rtimes_{\alpha_1} \mathbb{Z})$.

(4.k) If $\mathrm{Im}[\alpha] = \left\langle \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right) \right\rangle \cong \mathbb{Z}$ with $k > 0$ an integer, then $\ell = m - 1$ and $d_i = 1$ for all $i = 1, \dots, m-1$. In particular, we have that $[\alpha_i] = \mathrm{Id}$ for all $i = 2, \dots, m$ and $[\alpha_1]$ is a generator for $\mathrm{Im}[\alpha]$, hence $[\alpha_1] = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)^{\pm 1}$. We might then suppose that $[\alpha_1] = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ and $\alpha_i = \mathrm{Id}$ for $i = 2, \dots, m$ by applying Proposition 1.5 with $P = \mathrm{diag}(\pm 1, 1, \dots, 1)$ and $\chi = \mathrm{Id}$, and we have that $M_\alpha = \mathbb{Z}^{m-1} \times (F_2 \rtimes_{\alpha_1} \mathbb{Z})$.

(5) If $\mathrm{Im}[\alpha] = \langle A \rangle \cong \mathbb{Z}$ with $A \in \mathrm{GL}_2(\mathbb{Z})$ of infinite order and $\mathrm{rank}_{\mathbb{Z}}(A - \mathrm{Id}_2) = 2$, then an analogous argument to the one used in (4) shows that we may suppose $[\alpha_1] = A$ and $\alpha_i = \mathrm{Id}$ for all $i = 2, \dots, m$, so that $M_\alpha = \mathbb{Z}^{m-1} \times (F_2 \rtimes_{\alpha_1} \mathbb{Z})$.

[5] These cases can only occur if $m \geq 2$.

[6] In this context, we have not been able to prove or disprove the existence of two commuting automorphisms $\alpha_1, \alpha_2 \in \mathrm{Aut}(F_2)$ such that $\mathrm{ord}([\alpha_1]) = +\infty$ and $[\alpha_2] = -\mathrm{Id}_2$ in $\mathrm{GL}_2(\mathbb{Z})$. If such a pair exists, it would be of interest to obtain a characterization of all such commuting pairs. On the other hand, it is clear that all other cases can be realized by choosing an appropriate α.

(6) - (9) If $\mathrm{Im}[\alpha] \cong C_r$ with $r = 2, 3, 4$ or 6, then $\ell = m$, $d_1 = r$ and $d_i = 1$ for $i = 2, ..., m$. Note that no other element apart from $[\alpha_1]^{\pm 1}$ generates C_r, then and analogous argument to the one used in (4) shows the result for these cases.

(10) If $\mathrm{Im}[\alpha] = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), -\mathrm{Id}_2 \right\rangle \cong C_2 \times C_2$, we must have $\ell = m$, $d_1 = d_2 = 2$ and $d_i = 1$ for all $i = 3, ..., m$. Therefore, we have that $[\alpha_i] = \mathrm{Id}$ for all $i = 3, ..., m$ and $[\alpha_1], [\alpha_2]$ generate $\mathrm{Im}[\alpha]$. Let us denote $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and note that either $\{[\alpha_1], [\alpha_2]\} = \{A, -A\}$ or $\{\pm A, -\mathrm{Id}_2\}$. In the first case, we can suppose that $[\alpha_1] = A$ and $[\alpha_2] = -A$ by permuting t_1 and t_2 if necessary. In the second case, we can suppose that $[\alpha_1] = \pm A$ and $[\alpha_2] = -\mathrm{Id}_2$, again by permuting t_1 and t_2 if necessary. Observe that $[\alpha_1][\alpha_2] = \mp A$, so that if we define $P' = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, then applying Proposition 1.5 with $P = \left(\begin{smallmatrix} P' & 0 \\ 0 & \mathrm{Id}_{m-2} \end{smallmatrix}\right)$ and $\chi = \mathrm{Id}$ we may suppose that $[\alpha_1] = \pm A$ and $[\alpha_2] = \mp A$, so that we are again in the first case. Hence, we may suppose either way that $[\alpha_1] = A$ and $[\alpha_2] = -A$. Then, applying again Proposition 1.5 we can suppose that $\alpha_1 : (a \mapsto a,\ b \mapsto b^{-1})$, $\alpha_2 : (a \mapsto a^{-1},\ b \mapsto b)$ and $\alpha_i = \mathrm{Id}$ for all $i = 3, ..., m$ (note that α_1 and α_2 do indeed commute); hence $M_\alpha = \mathbb{Z}^{m-2} \times (F_2 \rtimes_{\alpha_1, \alpha_2} \mathbb{Z}^2)$.

(11) The case in which $\mathrm{Im}[\alpha] = \left\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), -\mathrm{Id}_2 \right\rangle$ is dealt in exactly the same way as case (10).

(12) If $\mathrm{Im}[\alpha] = \langle -\mathrm{Id} \rangle \times \langle A \rangle \cong C_2 \times \mathbb{Z}$ with $A \in \mathrm{GL}_2(\mathbb{Z})$ of infinite order, then $\ell = m - 1$, $d_2 = 2$ and $d_i = 1$ for all $i = 3, ..., m$. In particular, we have that $\mathrm{ord}[\alpha_1] = +\infty$, $\mathrm{ord}[\alpha_2] = 2$, $[\alpha_i] = \mathrm{Id}$ for all $i = 3, ..., m$ and $[\alpha_1]$, $[\alpha_2]$ generate $\mathrm{Im}[\alpha]$. Now, the only element of $\mathrm{Im}[\alpha]$ of order 2 is $-\mathrm{Id}_2$, therefore, we must have $[\alpha_2] = -\mathrm{Id}_2$. Since $-\mathrm{Id}_2$ and $[\alpha_1]$ generate $\mathrm{Im}[\alpha]$, we must have that $[\alpha_1] = \pm A^{\pm 1}$ is of infinite order. Therefore, applying Proposition 1.5 we can suppose that $\mathrm{ord}[\alpha_1] = \infty$, $[\alpha_2] = -\mathrm{Id}_2$ and $\alpha_i = \mathrm{Id}$ for all $i = 3, ..., m$, and so $M_\alpha = \mathbb{Z}^{m-2} \times (F_2 \rtimes_{\alpha_1, \alpha_2} \mathbb{Z}^2)$.

□

We summarize the above classification of the group M_α in the table given in Appendix A, together with the computation of several invariants: the number r_α, the centre $Z(M_\alpha)$ (see Proposition 1.21), the inner automorphism group $\mathrm{Inn}(M_\alpha)$ (see Proposition 2.22), the abelianization M_α^{ab}, which can be computed by a straightforward calculation, and the image $\mathrm{Im}[\alpha]$. Using this classification, we now prove the following lemma, which is the last result needed to solve the isomorphism problem for the family of FBFA groups with $n = 2$.

Lemma 3.13. *Let $\alpha, \beta : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_2)$ be group homomorphisms. Then $M_\alpha \cong M_\beta$ if and only if $\mathrm{Im}[\alpha]$ and $\mathrm{Im}[\beta]$ are conjugate in $\mathrm{GL}_2(\mathbb{Z})$.*

Proof. It is enough to prove the result in the case where both M_α and M_β fall under one of the cases considered in Proposition 3.12. We will first deal with the only if implication; hence suppose that $M_\alpha \cong M_\beta$, and note that by Observation 1.7 we must have $r_\alpha = r_\beta =: r$. If $r = 0$ then both M_α and M_β fall under case (1), hence $\mathrm{Im}[\alpha] = \mathrm{Im}[\beta] = \{\mathrm{Id}_2\}$. If $r = 1$ then M_α and M_β both fall under cases (2) - (4). A quick calculation shows that a group falling under case (2) abelianizes to $\mathbb{Z}^{m-1} \times C_2$, in case (3) to $\mathbb{Z}^{m+1}$ and in case (4.k) to $\mathbb{Z}^{m+1} \oplus C_k$, hence these cases are non-isomorphic, except possibly cases (2) and (4.2). However, these last two cases are also non-isomorphic because, by Proposition 2.22, the inner automorphism group of M_α in case (2) is $F_2 \rtimes C_2$ and has torsion, while in (4.2) it is $F_2 \rtimes \mathbb{Z}$ and does not have torsion by Lemma 1.1. Therefore, M_α and M_β must fall under exactly the same case and $\mathrm{Im}[\alpha]$ is conjugate to $\mathrm{Im}[\beta]$ in $\mathrm{GL}_2(\mathbb{Z})$. Lastly, if $r = 2$ then $M_\alpha \cong M_\beta$ and by Corollary 1.16 we have that $\mathrm{Im}[\alpha]$ and $\mathrm{Im}[\beta]$ are conjugate in $\mathrm{GL}_2(\mathbb{Z})$.

For the if implication, suppose that $\mathrm{Im}[\alpha]$ is conjugate to $\mathrm{Im}[\beta]$ in $\mathrm{GL}_2(\mathbb{Z})$. Looking at Proposition 3.12 and Lemma 3.9 we can easily deduce that M_α and M_β both must fall under exactly the same case of those in Proposition 3.12. It is then clear that $M_\alpha \cong M_\beta$, unless maybe if they fall under case (5) or (12). If they fall under case (5), according to Lemma 3.9 we must have that $[\alpha_1]$ and $[\beta_1]^{\pm 1}$ are conjugate in $\mathrm{GL}_2(\mathbb{Z})$ via a certain $[\chi] \in \mathrm{GL}_2(\mathbb{Z})$, and Proposition 1.5 with $P = \mathrm{diag}(\pm 1, 1, ..., 1)$ and χ shows that $M_\alpha \cong M_\beta$. If they fall under case (12) then according to Lemma 3.9 we must have that $[\alpha_1]$ and $\pm[\beta_1]^{\pm 1}$ are conjugate in $\mathrm{GL}_2(\mathbb{Z})$ via a certain $[\chi] \in \mathrm{GL}_2(\mathbb{Z})$. Since $[\beta_2] = -\mathrm{Id}_2$, this means that $[\beta_1]^{\pm 1}[\beta_2]^{\epsilon} = [\chi]^{-1}[\alpha_1][\chi]$ with $\epsilon \in \{0, 1\}$. Hence, if we define $P' = \left(\begin{smallmatrix} \pm 1 & \epsilon \\ 0 & 1 \end{smallmatrix}\right)$, then applying Proposition 1.5 with $P = \left(\begin{smallmatrix} P' & 0 \\ 0 & \mathrm{Id}_{m-2} \end{smallmatrix}\right)$ and χ we get that $M_\alpha \cong M_\beta$. □

We are now ready to state and prove the main result of this section, which forms the basis for solving the isomorphism problem for the family of FBFA groups with $n = 2$. The statement is as follows:

Theorem 3.14. *Let $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^{m_1}$ and $M_\beta = F_2 \rtimes_\beta \mathbb{Z}^{m_2}$. Then $M_\alpha \cong M_\beta$ if and only if $m_1 = m_2$ and $\mathrm{Im}[\alpha]$ and $\mathrm{Im}[\beta]$ are conjugate in $\mathrm{GL}_2(\mathbb{Z})$.*

Proof. It suffices to show that if $M_\alpha \cong M_\beta$ then $m_1 = m_2$; the result will then easily follow from Lemma 3.13. Hence, suppose that M_α is isomorphic to M_β; we will show that $r_\alpha = r_\beta$, and then looking at the free rank of the abelianizations $M_\alpha^{\mathrm{ab}} \cong M_\beta^{\mathrm{ab}}$ we will get that $m_1 = m_2$. If $r_\alpha \neq r_\beta$, then both of this numbers must be different from 2 by Theorem 1.15 (i.e. neither M_α nor M_β are full), and we may suppose without loss of generality that $r_\alpha = 0$ and $r_\beta = 1$. In particular, we have that $M_\alpha \cong F_2 \times \mathbb{Z}^{m_1}$, while M_β falls under cases (2) - (4) of Proposition 3.12 (with $m = m_2$). Abelianizing we get that $M_\beta^{\mathrm{ab}} \cong M_\alpha^{\mathrm{ab}} \cong \mathbb{Z}^{m_1+2}$ has no torsion, hence M_β must fall under case (3), since case (2) abelianizes to $\mathbb{Z}^{m_2+1} \oplus C_2$ and (4.k) abelianizes to $\mathbb{Z}^{m_2+1} \oplus C_k$. But then, we have by Proposition 2.22 that $\mathrm{Inn}(M_\alpha) \cong F_2$ is isomorphic to $\mathrm{Inn}(M_\beta) \cong F_2 \rtimes C_2$, which is not possible because $F_2 \rtimes C_2$ has torsion but F_2 does not. Therefore, we must have that $r_\alpha = r_\beta$; this finishes the proof. □

We note that the above result generalizes a theorem of Bogopolski–Martino–Ventura [3]. In that work, the authors proved that for $\psi, \phi \in \mathrm{Aut}(F_2)$, the groups $F_2 \rtimes_\psi \mathbb{Z}$ and $F_2 \rtimes_\phi \mathbb{Z}$ are isomorphic if and only if $[\psi]$ is conjugate to $[\phi]^{\pm 1}$ in $\mathrm{GL}_2(\mathbb{Z})$. We remark that the approach adopted in the present work is different. Our proof relies primarily on the study of the isomorphism problem for the family of FBFA groups and, in particular, on the main result of Theorem 1.15. By contrast, the argument in [3] is based mainly on an explicit computation of the outer automorphism group $\mathrm{Out}(M_\psi)$, although some of the techniques developed there have also proved to be very useful in our study.

In particular, Theorem 3.14 implies that the different cases appearing in the classification of Proposition 3.12 are non-isomorphic, since the corresponding subgroups $\mathrm{Im}[\alpha]$ are non-conjugate in $\mathrm{GL}_2(\mathbb{Z})$ by Lemma 3.9. Moreover, using the decomposition $\mathrm{GL}_2(\mathbb{Z}) \cong D_4 *_{D_2} D_6$ as an amalgamated product of finite groups, and the fact that the conjugacy problem for finitely generated subgroups in such amalgams is algorithmically decidable (cf. [11, Corollary 5.17]), Theorem 3.14 then immediately implies the following result, confirming that the answer to Problem 3.11 is affirmative:

Corollary 3.15. *The isomorphism problem for the family of FBFA groups with $n = 2$, as stated in 3.11, is algorithmically decidable.*

The algorithm developed in [11] for amalgams of finite groups is rather involved, relying on a generalized Stallings folding method, and lies beyond the scope of this work. Since we are only concerned with the conjugacy problem for abelian subgroups of $\mathrm{GL}_2(\mathbb{Z})$, whose structure is quite simple as seen in

Subsection 3.1, it would be of interest to develop a simpler algorithm that, given finite generating sets, determines whether two such abelian subgroups are conjugate.

3.3 The Automorphism Group of $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^m$

In this section, we compute the automorphism group of M_α using the results established in Section 2. To simplify the computations, we assume that M_α falls under one of the cases (1) to (12) from the classification in Proposition 3.12. We first consider cases (5) - (12), which corresponds to those where M_α is full. In these cases, the computation of the automorphism group becomes significantly simpler, since we have that $\mathrm{Aut}_{F_2}(M_\alpha) = \mathrm{Aut}(M_\alpha)$ and, as we will see, the short exact sequence from Theorem 2.13 splits. We then treat cases (1) - (4), corresponding to those where M_α is not full. Case (1) is straightforward, while cases (2) - (4) are more involved: the structure of $\mathrm{Aut}(M_\alpha)$ in these instances is considerably more complex and requires additional work. Nonetheless, the analysis remains feasible thanks to our study of $\mathrm{Aut}_{F_2}(M_\alpha)$ in Section 2. We give the following lemma for later reference:

Lemma 3.16. *If $A \in \mathrm{GL}_2(\mathbb{Z})$ is a matrix of infinite order, then A cannot be conjugate to $-A$ in $\mathrm{GL}_2(\mathbb{Z})$.*

Proof. If A is conjugate to $-A$ in $\mathrm{GL}_2(\mathbb{Z})$, then $\mathrm{tr}(A) = \mathrm{tr}(-A) = -\mathrm{tr}(A)$, hence $\mathrm{tr}(A) = 0$. The characteristic polynomial of A is then equal to $x^2 - \mathrm{tr}(A)x + \det(A) = x^2 \pm 1$, which has two different roots equal to either ± 1 or $\pm i$. In particular, A is conjugate over the complex numbers to either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ and the order of A is either equal to 2 or 4, a contradiction. □

In the following result, we compute the automorphism group of M_α for cases (5) - (12). The analysis shows that, when M_α is full, there is a decomposition $\mathrm{Aut}(M_\alpha) \cong S_\alpha \rtimes \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$ as a semidirect product. More precisely, the automorphism group is either a direct product $S_\alpha \times \pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha])$ or an extension of C_2 by this group. Moreover, in the cases where α is known explicitly, we will compute both $\pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha])$ and $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$. The precise statement is as follows:

Proposition 3.17. *Let $\alpha : \mathbb{Z}^m \longrightarrow \mathrm{Aut}(F_2)$ be a group homomorphism, and suppose that M_α falls under cases (5) - (12) of Proposition 3.12. Then*

(i) either $\mathrm{N}(\mathrm{Im}[\alpha]) = \mathrm{C}(\mathrm{Im}[\alpha])$ and

$$\mathrm{Aut}(M_\alpha) \cong S_\alpha \times \pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha]),$$

(ii) or $[\mathrm{N}(\mathrm{Im}[\alpha]) : \mathrm{C}(\mathrm{Im}[\alpha])] = 2$ and there exists a group homomorphism $s : \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha]) \longrightarrow \mathrm{GL}_m(\mathbb{Z})$ with kernel $\pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha])$ such that

$$\mathrm{Aut}(M_\alpha) \cong S_\alpha \rtimes \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$$

with $\psi \in \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$ acting on $A \in S_\alpha$ via $\psi^{-1}A\psi = s_\psi^{-1} A s_\psi$.

More precisely, we have the following cases:

(5) If M_α falls under case (5), then $S_\alpha = \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid A\mathbf{e_1}^\mathsf{T} = \mathbf{e_1}^\mathsf{T}\}$ and

(5.1) either $[\alpha_1]$ is not conjugate to $[\alpha_1]^{-1}$ in $\mathrm{GL}_2(\mathbb{Z})$ and condition (i) is satisfied,

(5.2) or $[\alpha_1]$ is conjugate to $[\alpha_1]^{-1}$ in $\mathrm{GL}_2(\mathbb{Z})$ and condition (ii) is satisfied with $s_\psi = \mathrm{diag}(-1, 1, ..., 1)$ if $[\psi] \notin \mathrm{C}(\mathrm{Im}[\alpha])$.

(6) *If M_α falls under case (6), then $S_\alpha = \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid A\mathbf{e_1}^\mathsf{T} \equiv \mathbf{e_1}^\mathsf{T} \pmod 2\}$ and $\mathrm{Aut}(M_\alpha) \cong S_\alpha \times \mathrm{Aut}(F_2)$.*

(7) *If M_α falls under case (7), then $S_\alpha = \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid A\mathbf{e_1}^\mathsf{T} \equiv \mathbf{e_1}^\mathsf{T} \pmod 3\}$ and condition (ii) is satisfied. In this case*

$$\pi^{-1}\,\mathsf{N}(\mathrm{Im}[\alpha]) = \{\delta^i\beta^j\gamma_u \mid i = 0,1,\ j = 0,\dots,5,\ u \in F_2\},$$

where $\delta : (a \mapsto b,\ b \mapsto a)$ and $\beta : (a \mapsto ab^{-1},\ b \mapsto a)$. The map s is defined exactly as in (5.2) and $\mathsf{C}(\mathrm{Im}[\alpha]) = \langle[\beta]\rangle$.

(8) *If M_α falls under case (8), then $S_\alpha = \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid A\mathbf{e_1}^\mathsf{T} \equiv \mathbf{e_1}^\mathsf{T} \pmod 4\}$ and condition (ii) is satisfied. In this case*

$$\pi^{-1}\,\mathsf{N}(\mathrm{Im}[\alpha]) = \{\delta^i\alpha_1^j\gamma_u \mid i = 0,1,\ j = 0,\dots,3,\ u \in F_2\} \cong F_2 \rtimes D_4,$$

where $\delta : (a \mapsto a,\ b \mapsto b^{-1})$. The map s is defined exactly as in (5.2) and $\mathsf{C}(\mathrm{Im}[\alpha]) = \langle[\alpha_1]\rangle$.

(9) *If M_α falls under case (9), then $S_\alpha = \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid A\mathbf{e_1}^\mathsf{T} \equiv \mathbf{e_1}^\mathsf{T} \pmod 6\}$ and condition (ii) is satisfied. In this case*

$$\pi^{-1}\,\mathsf{N}(\mathrm{Im}[\alpha]) = \{\delta^i\alpha_1^j\gamma_u \mid i = 0,1,\ j = 0,\dots,5,\ u \in F_2\},$$

where $\delta : (a \mapsto b,\ b \mapsto a)$. The map s is defined exactly as in (5.2) and $\mathsf{C}(\mathrm{Im}[\alpha]) = \langle[\alpha_1]\rangle$.

(10) - (11) *If M_α falls under case (10) or (11), then $S_\alpha = \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid A\mathbf{e_i}^\mathsf{T} \equiv \mathbf{e_i}^\mathsf{T} \pmod 2,\ i = 1,2\}$ and condition (ii) is satisfied. In this case*

$$\pi^{-1}\,\mathsf{N}(\mathrm{Im}[\alpha]) = \{\delta^i\alpha_1^j\gamma_u \mid i = 0,\dots,3,\ j = 0,1,\ u \in F_2\} \cong F_2 \rtimes D_4,$$

where $\delta : (a \mapsto b,\ b \mapsto a^{-1})$, and $s_\psi = \begin{pmatrix} Q & 0 \\ 0 & \mathrm{Id}_{m-2} \end{pmatrix}$ with $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ if $[\psi] \notin \langle[\alpha_1]\rangle$.

(12) *If M_α falls under case (12) then $S_\alpha = \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid A\mathbf{e_1}^\mathsf{T} = \mathbf{e_1}^\mathsf{T}$ and $A\mathbf{e_2}^\mathsf{T} \equiv \mathbf{e_2}^\mathsf{T}$ (mod 2)$\}$ and*

(12.1) *either $[\alpha_1]$ is not conjugate to neither $[\alpha_1]^{-1}$ nor $-[\alpha_1]^{-1}$ in $\mathrm{GL}_2(\mathbb{Z})$ and condition (i) is satisfied,*

(12.2) *or $[\alpha_1]$ is conjugate to $[\alpha_1]^{-1}$ in $\mathrm{GL}_2(\mathbb{Z})$ and condition (ii) is satisfied with s defined as in (5.2),*

(12.3) *or $[\alpha_1]$ is conjugate to $-[\alpha_1]^{-1}$ in $\mathrm{GL}_2(\mathbb{Z})$ and condition (ii) is satisfied with $s_\psi = \begin{pmatrix} Q & 0 \\ 0 & \mathrm{Id}_{m-2} \end{pmatrix}$, $Q = \left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$, if $[\psi] \notin \mathsf{C}(\mathrm{Im}[\alpha])$.*

Proof. Note that in all of these cases M_α is full, hence $\mathrm{Aut}_{F_n}(M_\alpha) = \mathrm{Aut}(M_\alpha)$ by Observation 2.2. Moreover, the calculation of S_α is straightforward in each case by Observation 2.12. We will now go through each case:

(5.1) If $[\psi] \in \mathsf{N}(\mathrm{Im}[\alpha])$ then $[\alpha_1]\Lambda_{[\psi]}$ must generate $\mathrm{Im}[\alpha] = \langle[\alpha_1]\rangle \cong \mathbb{Z}$, hence must be equal to $[\alpha_1]^{\pm 1}$. But $[\alpha_1]$ is not conjugate to $[\alpha_1]^{-1}$ in $\mathrm{GL}_2(\mathbb{Z})$, hence $[\alpha_1]\Lambda_{[\psi]} = [\alpha_1]$. In particular, we have that $\mathsf{N}(\mathrm{Im}[\alpha]) = \mathsf{C}(\mathrm{Im}[\alpha])$ and the result in this case follows from Corollary 2.16.

(5.2) Similarly, if $\psi \in \pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha])$ then $[\alpha_1]\Lambda_{[\psi]} = [\alpha_1]^{\pm 1}$. In this case, there exists $[\phi] \in \mathsf{GL}_2(\mathbb{Z})$ such that $[\phi]^{-1}[\alpha_1][\phi] = [\alpha_1]^{-1}$, hence $[\phi] \in \mathsf{N}(\mathsf{Im}[\alpha])$ and $[\mathsf{N}(\mathsf{Im}[\alpha]) : \mathsf{C}(\mathsf{Im}[\alpha])] = 2$. We may define $s_\psi = \mathsf{Id}_m$ if $[\psi] \in \mathsf{C}(\mathsf{Im}[\alpha])$ and $s_\psi = \mathsf{diag}(-1, 1, ..., 1)$ otherwise, and in any case we have that s_ψ lifts $\Lambda_{[\psi]}$. In particular, $\mathsf{N}(\mathsf{Im}[\alpha])_{\mathsf{lift}} = \mathsf{N}(\mathsf{Im}[\alpha])$ and we have a group homomorphism $s : \pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha]) \longrightarrow \mathsf{GL}_m(\mathbb{Z})$ satisfying the conditions of Proposition 2.14; which proves the result in this case.

(6) In this case, we have that $[\alpha_1] = -\mathsf{Id}_2$ and $\mathsf{Im}[\alpha] = \langle -\mathsf{Id}_2 \rangle$, hence $\mathsf{N}(\mathsf{Im}[\alpha]) = \mathsf{C}(\mathsf{Im}[\alpha]) = \mathsf{GL}_2(\mathbb{Z})$ and $\pi^{-1}\,\mathsf{C}(\mathsf{Im}[\alpha]) = \mathsf{Aut}(F_2)$. The result in this case then follows from Corollary 2.16.

(7) In this case, we have that $\mathsf{Im}[\alpha] = \langle [\alpha_1] \rangle \cong C_3$, hence $\mathsf{Im}[\alpha]$ can only be generated by $[\alpha_1]^{\pm 1}$. Moreover, if $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ then $[\alpha_1]^{-1} = P^{-1}[\alpha_1]P$, which shows $[\mathsf{N}(\mathsf{Im}[\alpha]) : \mathsf{C}(\mathsf{Im}[\alpha])] = 2$. We then define $s : \pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha]) \longrightarrow \mathsf{GL}_m(\mathbb{Z})$ as in (5.2), which shows that $\mathsf{N}(\mathsf{Im}[\alpha])_{\mathsf{lift}} = \mathsf{N}(\mathsf{Im}[\alpha])$ and provides a group homomorphism satisfying the conditions from Proposition 2.14; hence it only remains to calculate $\pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha])$. Some elementary calculations with matrices show that $\mathsf{C}(\mathsf{Im}[\alpha])$ is the set of matrices in $\mathsf{GL}_2(\mathbb{Z})$ of the form $\left(\begin{smallmatrix} p & -q \\ q & p-q \end{smallmatrix}\right)$ with $p, q \in \mathbb{Z}$ integers. The determinant of this matrix must be equal to ± 1, hence $p^2 + q^2 - pq = (p - \frac{q}{2})^2 + \frac{3}{4}q^2 = \pm 1$, and the solutions to this Diophantine equation are $(p, q) = (0, \pm 1), (\pm 1, 0), \pm(1, 1)$, and one can then check that $\mathsf{C}(\mathsf{Im}[\alpha])$ is of order 6 generated by the matrix $B = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. In particular, $\mathsf{N}(\mathsf{Im}[\alpha])$ is the set of matrices of the form $P^i B^j$ with $i = 0, 1$ and $j = 0, ..., 5$. We can choose $\delta : (a \mapsto b,\ b \mapsto a)$ and $\beta : (a \mapsto ab^{-1},\ b \mapsto a)$ as representatives of $P, B \in \mathsf{GL}_2(\mathbb{Z})$ in $\mathsf{Aut}(F_2)$, respectively. This shows that $\pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha])$ is the set of automorphisms of the form $\delta^i \beta^j \gamma_u$ with $i = 0, 1$, $j = 0, ..., 5$ and $u \in F_2$.

(8) In this case, we have that $\mathsf{Im}[\alpha] = \langle [\alpha_1] \rangle \cong C_4$ can only be generated by $[\alpha_1]^{\pm 1}$. Moreover, if $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ then $P^{-1}[\alpha_1]P = [\alpha_1]^{-1}$. Again, one then proceeds as we did in (5.2), and it only remains to calculate $\pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha])$. An elementary calculation shows that $\mathsf{C}(\mathsf{Im}[\alpha]) = \langle [\alpha_1] \rangle$, hence $\mathsf{N}(\mathsf{Im}[\alpha])$ consists precisely of the matrices $P^i[\alpha_1]^j$ for $i = 0, 1$ and $j = 0, ..., 3$. We may choose $\delta : (a \mapsto a,\ b \mapsto b^{-1})$ as a representative for $P \in \mathsf{GL}_2(\mathbb{Z})$ in $\mathsf{Aut}(F_2)$, and we have that $\pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha])$ is the set of automorphisms of the form $\delta^i \alpha_1^j \gamma_u$ with $i = 0, 1$, $j = 0, ..., 3$ and $u \in F_2$.

(9) In this case, we have that $\mathsf{Im}[\alpha] = \langle [\alpha_1] \rangle \cong C_6$ can only be generated by $[\alpha_1]^{\pm 1}$. Moreover, if $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ then $P^{-1}[\alpha_1]P = [\alpha_1]^{-1}$. Again, one then proceeds as we did in (5.2), and it only remains to calculate $\pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha])$. An elementary calculation shows that $\mathsf{C}(\mathsf{Im}[\alpha]) = \langle [\alpha_1] \rangle$, hence $\mathsf{N}(\mathsf{Im}[\alpha])$ consists precisely of the matrices $P^i[\alpha_1]^j$ for $i = 0, 1$ and $j = 0, ..., 5$. We may choose $\delta : (a \mapsto b,\ b \mapsto a)$ as a representative for $P \in \mathsf{GL}_2(\mathbb{Z})$ in $\mathsf{Aut}(F_2)$, and we have that $\pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha])$ is the set of automorphisms of the form $\delta^i \alpha_1^j \gamma_u$ with $i = 0, 1$, $j = 0, ..., 5$ and $u \in F_2$.

(10) - (11) In these cases, we have that $[\alpha_2] = -[\alpha_1]$ and $\mathsf{Im}[\alpha] = \{\mathsf{Id}_2, -\mathsf{Id}_2, [\alpha_1], -[\alpha_1]\}$. If $\psi \in \pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha])$, then $\pm\mathsf{Id}_2\,\Lambda_{[\psi]} = \pm\mathsf{Id}_2$ and, since $\Lambda_{[\psi]}$ is an automorphism of $\mathsf{Im}[\alpha]$, either $\Lambda_{[\psi]} = \mathsf{Id}$, or $[\alpha_1]\Lambda_{[\psi]} = [\alpha_2]$ and $[\alpha_2]\Lambda_{[\psi]} = [\alpha_1]$. If we let $P = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, then one might check that in both cases $P^{-1}[\alpha_1]P = [\alpha_2]$, hence $[\mathsf{N}(\mathsf{Im}[\alpha]) : \mathsf{C}(\mathsf{Im}[\alpha])] = 2$. We may then define $s_\psi = \mathsf{Id}_m$ if $[\psi] \in \mathsf{C}(\mathsf{Im}[\alpha])$, and $s_\psi = \left(\begin{smallmatrix} Q & 0 \\ 0 & \mathsf{Id}_{m-2} \end{smallmatrix}\right)$ with $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ otherwise. In any case, one has that s_ψ always lifts $\Lambda_{[\psi]}$, hence $\mathsf{N}(\mathsf{Im}[\alpha])_{\mathsf{lift}} = \mathsf{N}(\mathsf{Im}[\alpha])$ and we have a group homomorphism $s : \pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha]) \longrightarrow \mathsf{GL}_m(\mathbb{Z})$ which satisfies the hypothesis of Proposition 2.14; hence it only remains to calculate $\pi^{-1}\,\mathsf{N}(\mathsf{Im}[\alpha])$. By means of an elementary computation, one can show in both cases that $\mathsf{C}(\mathsf{Im}[\alpha]) = \mathsf{Im}[\alpha]$, and we have that $\mathsf{N}(\mathsf{Im}[\alpha]) = \mathsf{C}(\mathsf{Im}[\alpha]) \sqcup P \cdot \mathsf{C}(\mathsf{Im}[\alpha])$. Taking into account that $P^2 = -\mathsf{Id}_2$, the elements of $\mathsf{N}(\mathsf{Im}[\alpha])$ are precisely the matrices of the form $P^i[\alpha_1]^j$ with $i = 0, ..., 3$ and $j = 0, 1$. We may

choose $\delta : (a \mapsto b,\ b \mapsto a^{-1})$ as a representative for $P \in \mathrm{GL}_2(\mathbb{Z})$ in $\mathrm{Aut}(F_2)$, and we have that $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$ is the set of automorphisms of the form $\delta^i\alpha_1^j\gamma_u$ with $i = 0, ..., 3$, $j = 0, 1$ and $u \in F_2$.

(12) In this case, we have that $\mathrm{ord}[\alpha_1] = +\infty$, $[\alpha_2] = -\mathrm{Id}_2$ and $\mathrm{Im}[\alpha] = \langle[\alpha_1]\rangle \times \langle -\mathrm{Id}_2\rangle \cong \mathbb{Z} \times C_2$. If $\psi \in \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$ then $[\alpha_2]\Lambda_{[\psi]} = [\alpha_2]$ and, since $\Lambda_{[\psi]}$ is an automorphism of $\mathrm{Im}[\alpha]$, we must have that $[\alpha_1]\Lambda_{[\psi]} = \pm[\alpha_1]^{\pm 1}$ (where the two signs are independent). However, note that $[\alpha_1]\Lambda_{[\psi]} \neq -[\alpha_1]$ by Lemma 3.16, therefore we either have $\Lambda_{[\psi]} = \mathrm{Id}$, $[\alpha_1]\Lambda_{[\psi]} = [\alpha_1]^{-1}$ or $[\alpha_1]\Lambda_{[\psi]} = -[\alpha_1]^{-1}$. Moreover, there cannot exist $\psi_1, \psi_2 \in \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$ such that $[\alpha_1]\Lambda_{[\psi_1]} = [\alpha_1]^{-1}$ and $[\alpha_1]\Lambda_{[\psi_2]} = -[\alpha_1]^{-1}$, because in such case we would have that $[\alpha_1]^{-1}$ is conjugate to $-[\alpha_1]^{-1}$ in $\mathrm{GL}_2(\mathbb{Z})$, contradicting Lemma 3.16. This shows that M_α falls under one of the following subcases:

(12.1) If $[\alpha_1]$ is not conjugate to neither $[\alpha_1]^{-1}$ nor $-[\alpha_1]^{-1}$ in $\mathrm{GL}_2(\mathbb{Z})$, then $\Lambda_{[\psi]} = \mathrm{Id}$ for all $\psi \in \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$, hence $\mathrm{N}(\mathrm{Im}[\alpha]) = \mathrm{C}(\mathrm{Im}[\alpha])$, and the result in this case follows from Corollary 2.16.

(12.1) If $[\alpha_1]$ is conjugate to $[\alpha_1]^{-1}$, then there exists $[\phi] \in \mathrm{GL}_2(\mathbb{Z})$ such that $[\psi]^{-1}[\alpha_1][\psi] = [\alpha]^{-1}$. Moreover, if $\psi \in \mathrm{N}(\mathrm{Im}[\alpha])$ then we must either have $\Lambda_{[\psi]} = \mathrm{Id}$ or $[\alpha_1]\Lambda_{[\psi]} = [\alpha_1]^{-1}$, hence $[\mathrm{N}(\mathrm{Im}[\alpha]) : \mathrm{C}(\mathrm{Im}[\alpha])] = 2$. We now define $s_\psi = \mathrm{Id}_m$ if $\psi \in \mathrm{C}(\mathrm{Im}[\alpha])$ and $s_\psi = \mathrm{diag}(-1, 1, ..., 1)$ otherwise, and note that s_ψ always lifts $\Lambda_{[\psi]}$. In particular $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}} = \mathrm{N}(\mathrm{Im}[\alpha])$ and we have a group homomorphism $s : \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha]) \longrightarrow \mathrm{GL}_m(\mathbb{Z})$ which satisfies the hypothesis of Proposition 2.14; this proves the result in this case.

(12.2) If $[\alpha_1]$ is conjugate to $-[\alpha_1]^{-1}$, then there exists $[\phi] \in \mathrm{GL}_2(\mathbb{Z})$ such that $[\psi]^{-1}[\alpha_1][\psi] = -[\alpha]^{-1}$. Moreover, if $\psi \in \mathrm{N}(\mathrm{Im}[\alpha])$ then we must either have $\Lambda_{[\psi]} = \mathrm{Id}$ or $[\alpha_1]\Lambda_{[\psi]} = -[\alpha_1]^{-1} = [\alpha_1]^{-1}[\alpha_2]$, hence $[\mathrm{N}(\mathrm{Im}[\alpha]) : \mathrm{C}(\mathrm{Im}[\alpha])] = 2$. We now define $s_\psi = \mathrm{Id}_m$ if $\psi \in \mathrm{C}(\mathrm{Im}[\alpha])$ and $s_\psi = \begin{pmatrix} Q & 0 \\ 0 & \mathrm{Id}_{m-2} \end{pmatrix}$ with $Q = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ otherwise, and note that s_ψ always lifts $\Lambda_{[\psi]}$. In particular $\mathrm{N}(\mathrm{Im}[\alpha])_{\mathrm{lift}} = \mathrm{N}(\mathrm{Im}[\alpha])$ and we have a group homomorphism $s : \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha]) \longrightarrow \mathrm{GL}_m(\mathbb{Z})$ which satisfies the hypothesis of Proposition 2.14; this proves the result in this case.

□

We will now compute $\mathrm{Aut}(M_\alpha)$ for the cases where M_α is not full, that is, M_α falls under cases (1) - (4). In case (1), we have $M_\alpha = F_2 \times \mathbb{Z}^m$, and in this case we will actually compute the automorphism group of $F_n \times \mathbb{Z}^m$ for $n \geq 2$. The strategy will be first determining $\mathrm{Aut}_{F_n}(F_n \times \mathbb{Z}^m)$ using Proposition 2.14, and then the structure of $\mathrm{Aut}(F_n \times \mathbb{Z}^m)$ will be easily determined from here. The result is the following:

Proposition 3.18. *There is an isomorphism*

$$\mathrm{Aut}(F_n \times \mathbb{Z}^m) \cong (\mathbb{Z}^m)^n \rtimes (\mathrm{GL}_m(\mathbb{Z}) \times \mathrm{Aut}(F_n))\,,$$

where $A \in \mathrm{GL}_m(\mathbb{Z})$ *and* $\psi \in \mathrm{Aut}(F_n)$ *act on* $v = (\mathbf{v_1}, ..., \mathbf{v_n}) \in (\mathbb{Z}^m)^n$ *via* $A^{-1}vA = (\mathbf{v_1}A, ..., \mathbf{v_n}A)$ *and* $\psi v \psi^{-1} = \psi^{\mathrm{ab}}(\mathbf{v_1}, ..., \mathbf{v_n})^T$.[7]

Proof. We have that $M_\alpha = \mathbb{Z}^m \times F_n$ with $\alpha_i = \mathrm{Id}$ for $i = 1, ..., m$. In particular, $\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha]) = \mathrm{Aut}(F_n)$, $S_\alpha = \mathrm{GL}_m(\mathbb{Z})$ and Id_m lifts any element of $\mathrm{Aut}(F_n)^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha])$, hence

$$\mathrm{Aut}_{F_n}(M_\alpha) \cong \mathrm{GL}_m(\mathbb{Z}) \times \mathrm{Aut}(F_n)$$

[7] This product is computed as usual by treating ψ^{ab} as an $n \times n$ matrix and $v = (\mathbf{v_1}, ..., \mathbf{v_n})$ as a vector with n coordinates. The result is then another vector of $(\mathbb{Z}^m)^n$.

is immediate by Proposition 2.14. It is easy to see that $\mathrm{GL}_m(\mathbb{Z})$ is identified with the subgroup of $\mathrm{Aut}(M_\alpha)$ formed by those automorphisms of M_α of the form $t^{\mathbf{a}} \mapsto t^{\mathbf{a}A}$ and $x \mapsto x$, with $A \in \mathrm{GL}_m(\mathbb{Z})$, $\mathbf{a} \in \mathbb{Z}^m$ and $x \in F_n$; abusing the notation we will denote these by A. Meanwhile, $\mathrm{Aut}(F_n)$ corresponds to the subgroup formed by those automorphisms of the form $t^{\mathbf{a}} \mapsto t^{\mathbf{a}}$ and $x \mapsto x\psi$ with $\psi \in \mathrm{Aut}(F_n)$; again abusing the notation we will denote these by ψ. Now, let $v = (\mathbf{v_1}, ..., \mathbf{v_n}) \in (\mathbb{Z}^m)^n$ and consider the automorphism of M_α, which we shall also denote by v, given by $t^{\mathbf{a}} \mapsto t^{\mathbf{a}}$ and $x_i \mapsto t^{\mathbf{v_i}} x_i$ with $i = 1, ..., n$; it is immediate to check that the set of these automorphisms forms a subgroup of $\mathrm{Aut}(M_\alpha)$ isomorphic to $(\mathbb{Z}^m)^n$ which, by abusing the notation, we shall also denote by $(\mathbb{Z}^m)^n$. Note that $(\mathbb{Z}^m)^n \cap \mathrm{Aut}_{F_n}(M_\alpha) = \{\mathrm{Id}\}$.

Now, a $\Psi \in \mathrm{Aut}(M_\alpha)$ restricts to an automorphism of $Z(M_\alpha) = \mathbb{Z}^m$, hence $t^{\mathbf{a}}\Psi = t^{\mathbf{a}A}$ for a certain $A \in \mathrm{GL}_m(\mathbb{Z})$. We also write $x_i\Psi = t^{\mathbf{u_i}}\omega_i$ for certain $\mathbf{u_i} \in \mathbb{Z}^m$ and $\omega_i \in F_n$, $i = 1, ..., n$. Letting $v = (-\mathbf{u_1}A^{-1}, ..., -\mathbf{u_n}A^{-1}) \in (\mathbb{Z}^m)^n$, a quick calculation shows that $x_i v\Psi = \omega_i$ for $i = 1, ..., n$, hence $v\Psi \in \mathrm{Aut}_{F_n}(M_\alpha)$; this shows that $\mathrm{Aut}(M_\alpha) = (\mathbb{Z}^m)^n \cdot \mathrm{Aut}_{F_n}(M_\alpha)$.

Finally, a straightforward calculation shows that given $v \in (\mathbb{Z}^m)^n$, $A \in \mathrm{GL}_m(\mathbb{Z})$ and $\psi \in \mathrm{Aut}(F_n)$, then $A^{-1}vA = (\mathbf{v_1}A, ..., \mathbf{v_m}A)$ and $\psi v \psi^{-1} = \psi^{\mathrm{ab}}(\mathbf{v_1}, ..., \mathbf{v_n})^T$. In particular, $(\mathbb{Z}^m)^n$ is a normal subgroup of $\mathrm{Aut}(M_\alpha)$, and this proves that $\mathrm{Aut}(M_\alpha) = (\mathbb{Z}^m)^n \rtimes \mathrm{Aut}_{F_n}(M_\alpha)$ with the given action; this finishes the proof. □

We will now compute $\mathrm{Aut}(M_\alpha)$ for cases (2) and (3). The structure of the automorphism group in these cases is very similar; hence, we shall prove the result in full detail only for case (3), and merely outline how the argument can be adapted to obtain the result in case (2). As will become evident shortly, the structure of $\mathrm{Aut}(M_\alpha)$ in these cases is considerably more complicated and requires a more delicate analysis. Nonetheless, the general strategy is the same as in case (1). Namely, we first compute $\mathrm{Aut}_{F_2}(M_\alpha)$ using Proposition 2.14, and then exhibit an explicit subgroup of $\mathrm{Aut}(M_\alpha)$ with the property that, for any automorphism $\Psi \in \mathrm{Aut}(M_\alpha)$, there exists Φ in this subgroup such that $\Phi^{-1}\Psi \in \mathrm{Aut}_{F_2}(M_\alpha)$. With this description in hand, a further analysis allows us to make the structure of the automorphism group transparent. More precisely, the structure of $\mathrm{Aut}(M_\alpha)$ is as follows:

Proposition 3.19. *Let $M_\alpha = \mathbb{Z}^{m-1} \times (F_2 \rtimes_{\alpha_1} \mathbb{Z})$ with $\alpha_1 : (a \mapsto b,\ b \mapsto a)$, that is M_α falls under case (3) of Proposition 3.12 (resp. $\alpha_1 : (a \mapsto a,\ b \mapsto b^{-1})$ and M_α falls under case (2) of Proposition 3.12). Consider the following:*

- *For each $\mathbf{v} \in 2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$, we also denote by $\mathbf{v}$ the automorphism of M_α mapping $t^{\mathbf{a}} \mapsto t^{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}^m$, $a \mapsto t^{\mathbf{v}}a$ and $b \mapsto t^{\mathbf{v}}b$ (resp. $b \mapsto b$). The set of these automorphisms forms a subgroup of $\mathrm{Aut}(M_\alpha)$ which can be identified with $\mathbb{Z}^m$.*
- *For each $A \in S_\alpha = \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid A\mathbf{e_1}^{\mathsf{T}} \equiv \mathbf{e_1}^{\mathsf{T}} \pmod 2\}$, we also denote by A the automorphism of M_α mapping $t^{\mathbf{a}} \mapsto t^{\mathbf{a}A}$, $\mathbf{a} \in \mathbb{Z}^m$, and $x \mapsto x$, $x \in F_n$. The set of these automorphisms forms a subgroup of $\mathrm{Aut}(M_\alpha)$ which can be identified with S_α.*
- *For each $u \in F_2$, we denote by γ_u the automorphism of M_α which maps $t_1 \mapsto t_1(u\alpha_1)^{-1}u$, $t_j \mapsto t_j$, $j = 2, ..., m$, and $x \mapsto x\gamma_u$, $x \in F_2$. The set of these automorphisms forms a subgroup of $\mathrm{Aut}(M_\alpha)$ which can be identified with F_2.*
- *We define $\beta : (a \mapsto a^{-1},\ b \mapsto b^{-1})$, and denote by α_1 and β the automorphisms of M_α which map $t^{\mathbf{a}} \mapsto t^{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}^m$, $x \mapsto x\alpha_1$ and $x \mapsto x\beta$, $x \in F_2$, respectively. The subgroup generated by these forms a subgroup of $\mathrm{Aut}(M_\alpha)$ which can be identified with $V = C_2 \times C_2$.*

With this notation, we have that

$$K := \mathbb{Z}^m \rtimes (S_\alpha \times (F_2 \rtimes V)),$$

with V acting naturally on F_2, S_α acting naturally on $\mathbb{Z}^m$, F_2 acting trivially on $\mathbb{Z}^m$ and V acting on $\mathbb{Z}^m$ via $\alpha_1^{-1}\mathbf{v}\alpha_1 = \mathbf{v}$ and $\beta^{-1}\mathbf{v}\beta = -\mathbf{v}$ is a normal subgroup of $\mathrm{Aut}(M_\alpha)$ of index 2. Moreover, if we consider $\sigma \in \mathrm{Aut}(M_\alpha)$ mapping $t^{\mathbf{a}} \mapsto t^{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}^m$, $a \mapsto t_1 a$ and $b \mapsto t_1 b$ (resp. $b \mapsto b$), then $\mathrm{Aut}(M_\alpha) = K \sqcup \sigma K$, and σ satisfies that

$$\sigma^2 = 2\mathbf{e_1}, \quad \sigma^{-1}\mathbf{v}\sigma = \mathbf{v}, \quad \sigma^{-1}A\sigma = A \circ \mathbf{e_1}(\mathrm{Id}_m - A),$$

$$\sigma^{-1}\alpha_1\sigma = \alpha_1, \quad \sigma^{-1}\beta\sigma = -2\mathbf{e_1} \circ \beta \quad (\textit{resp. } \sigma^{-1}\beta\sigma = -2\mathbf{e_1} \circ \alpha_1\beta),$$

$$\sigma^{-1}\gamma_a\sigma = \alpha_1\gamma_a, \quad \sigma^{-1}\gamma_b\sigma = \alpha_1\gamma_b \quad (\textit{resp. } \sigma^{-1}\gamma_b\sigma = \gamma_b),$$

this characterizes completely $\mathrm{Aut}(M_\alpha)$.

Proof. We first prove the result for when M_α falls under case (3) of Proposition 3.12. Let $\Psi \in \mathrm{Aut}(M_\alpha)$ and write $a\Psi = t^{\mathbf{c_a}}\omega_a$ and $b\Psi = t^{\mathbf{c_b}}\omega_b$, with $\mathbf{c_a}, \mathbf{c_b} \in \mathbb{Z}^m$ and $\omega_a, \omega_b \in F_2$. We note that Ψ induces an automorphism on the abelianization $M_\alpha^{\mathrm{ab}} \cong \mathbb{Z}^{m+1}$, which has basis $t_1, ..., t_m$, and $a = b$, hence we must have $\mathbf{c_a} = \mathbf{c_b} := \mathbf{c}$. We also have that Ψ restricts to an automorphism of $Z(M_\alpha) = 2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$ (which has basis $t_1^2, t_2, ..., t_m$), hence we may write $t_1\Psi = t^{\mathbf{a_1}}y$, with $\mathbf{a_1} \in \mathbb{Z}^m$ and $y \in F_2$, and $t_j\Psi = t^{\mathbf{a_j}}$ with $\mathbf{a_j} \in \mathbb{Z}^m$ for $j = 2, ..., m$. Now, we must have that $t_1^2\Psi = t^{2\mathbf{a_1}}$, hence $2\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_m}$ must be a basis for $2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$. In particular, if we write $\mathbf{a_i} = (a_{i,1}, ..., a_{i,m})$ for $i = 1, ..., m$ then the matrix

$$\begin{pmatrix} a_{1,1} & 2a_{1,2} & \cdots & 2a_{1,m} \\ a_{2,1}/2 & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}/2 & a_{m,2} & \cdots & a_{m,m} \end{pmatrix}$$

has determinant ± 1, and it is straightforward to check using the properties of the determinant that the matrix $A = (a_{i,j})_{1 \le i,j \le m}$ also has determinant ± 1, and so $A \in \mathrm{GL}_m(\mathbb{Z})$. Given any $\mathbf{u} \in \mathbb{Z}^m$, it is immediate to check that the map $t^{\mathbf{a}} \mapsto t^{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}^m$, $a \mapsto t^{\mathbf{u}}a$ and $b \mapsto t^{\mathbf{u}}b$, which we will denote by $\mathbf{u}$ too, is a group homomorphism since it preserves the relations of M_α, and has inverse $\mathbf{u}^{-1} = -\mathbf{u}$, hence $\mathbf{u} \in \mathrm{Aut}(M_\alpha)$. Now, if we let $\mathbf{u} = \mathbf{c}A^{-1}$ then a quick calculation shows that the projection of both $a\mathbf{u}^{-1}\Psi$ and $b\mathbf{u}^{-1}\Psi$ to the quotient $M_\alpha/F_2 = \mathbb{Z}^m$ is equal to $t^{\mathbf{c}-\mathbf{u}A} = 1$, hence both $a\mathbf{u}^{-1}\Psi$ and $b\mathbf{u}^{-1}\Psi$ belong to F_2 and $\mathbf{u}^{-1}\Psi$ leaves F_2 invariant i.e. $\mathbf{u}^{-1}\Psi \in \mathrm{Aut}_{F_n}(M_\alpha)$. Therefore, we have shown that every $\Psi \in \mathrm{Aut}(M_\alpha)$ can be written as $\Psi = \mathbf{u}\Phi$ for certain $\mathbf{u} \in \mathbb{Z}^m$ and $\Phi \in \mathrm{Aut}_{F_2}(M_\alpha)$. Moreover, if we let $\sigma = \mathbf{e_1} \in \mathrm{Aut}(M_\alpha)$, then $\mathbf{u}$ can be written in a unique way as $\mathbf{u} = \sigma^\varepsilon \circ \mathbf{v}$ for a certain $\varepsilon = 0, 1$ and $\mathbf{v} \in 2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$.

We will now determine the structure of $\mathrm{Aut}_{F_2}(M_\alpha)$. Observe that $\mathrm{Im}[\alpha] \cong C_2$, hence $\mathrm{Aut}(\mathrm{Im}[\alpha]) \cong \mathrm{Aut}(C_2) = \{\mathrm{Id}\}$, which implies that $\mathrm{N}(\mathrm{Im}[\alpha]) = \mathrm{C}(\mathrm{Im}[\alpha])$. Using Proposition 2.14 and the fact that Id_m lifts any $\psi \in \pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha])$, we deduce that

$$\mathrm{Aut}_{F_2}(M_\alpha) \cong S_\alpha \times \pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha]),$$

with $S_\alpha = \{A \in \mathrm{GL}_m(\mathbb{Z}) \mid A\mathbf{e_1}^{\mathsf{T}} \equiv \mathbf{e_1}^{\mathsf{T}} \pmod 2\}$. Moreover, we have that $\mathrm{C}(\mathrm{Im}[\alpha])$ is the centralizer of $[\alpha_1] = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbb{Z})$, and an elementary calculation shows that the matrices in this centralizer are precisely $\pm\,\mathrm{Id}_2$ and $\pm[\alpha_1]$. If we consider the automorphism $\beta : (a \mapsto a^{-1},\ b \mapsto b^{-1})$ of F_2, then $[\beta] = -\,\mathrm{Id}_2$ and

$$\pi^{-1}\,\mathrm{C}(\mathrm{Im}[\alpha]) = \{\alpha_1^i\beta^j\gamma_u \mid i, j = 0, 1,\ u \in F_2\} \cong F_2 \rtimes V$$

where we identify F_2 with $\{\gamma_u \mid u \in F_2\}$ and $V = C_2 \times C_2$ with the subgroup generated by α_1 and β, and V acts naturally on F_2. Applying Observation 2.4 we see that $A \in \mathrm{GL}_m(\mathbb{Z})$, $\gamma_u \in F_2$, α_1 and β each correspond to the automorphisms of M_α given in the statement of the lemma.

Now, we identify $\mathbb{Z}^m$ with the subgroup of $\mathrm{Aut}(M_\alpha)$ consisting of the automorphisms $\mathbf{v} \in 2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$. Since the only of these automorphisms leaving F_2 invariant is $\mathbf{0} = \mathrm{Id}$, we have that $\mathbb{Z}^m \cap \mathrm{Aut}_{F_2}(M_\alpha) = \{\mathrm{Id}\}$. Now, it is immediate to check that $A^{-1} \circ \mathbf{v} \circ A = \mathbf{v}A$, $\alpha_1^{-1}\mathbf{v}\alpha_1 = \mathbf{v}$ and $\beta^{-1}\mathbf{v}\beta = -\mathbf{v}$ by applying these to the generators of M_α. Similarly, a quick calculation also shows that $\gamma_a^{-1}\mathbf{v}\gamma_a = \gamma_b^{-1}\mathbf{v}\gamma_b = \mathbf{v}$, so that $\gamma_u^{-1}\mathbf{v}\gamma_u = \mathbf{v}$ for all $u \in F_2$. We have then shown that

$$K = \mathbb{Z}^m \rtimes \mathrm{Aut}_{F_2}(M_\alpha) = \mathbb{Z}^m \rtimes (S_\alpha \times (F_2 \rtimes V)),$$

with the actions described as in the statement of the lemma, is a subgroup of $\mathrm{Aut}(M_\alpha)$. Furthermore, if we had $\sigma \in K$ then $\sigma = \mathbf{v}\Phi$ for certain $\mathbf{v} \in 2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$ and $\Phi \in \mathrm{Aut}_{F_2}(M_\alpha)$, which implies that $\mathbf{v}^{-1}\sigma = \Phi$ leaves F_2 invariant, and this is only possible if $\sigma = \mathbf{v}$; a contradiction. Therefore, we have that $\sigma \notin K$, and by a prior observation we can deduce that $\mathrm{Aut}(M_\alpha) = K \sqcup \sigma K$, so that K is a normal subgroup of $\mathrm{Aut}(M_\alpha)$ of index 2. Finally, the identities involving σ in the statement of the lemma are easily checked by looking at the images of the generators of M_α; this finishes the proof for when M_α falls under case (3).

Suppose now that M_α falls instead under case (2) of Proposition 3.12. If $\Psi \in \mathrm{Aut}(M_\alpha)$, we may also write $b\Psi = t^{\mathbf{c_b}}\omega_b$, with $\mathbf{c_b} \in \mathbb{Z}^m$ and $\omega_b \in F_2$. Since Ψ induces an automorphism on the abelianization $M_\alpha^{\mathrm{ab}} = \langle t_1, ..., t_m, a\rangle \oplus \langle b\rangle \cong \mathbb{Z}^{m+1} \oplus C_2$, and $b^2 = 1$ in M_α^{ab}, we deduce that $2\mathbf{c_b} = \mathbf{0}$ i.e. $\mathbf{c_b} = \mathbf{0}$. Given any $\mathbf{u} \in \mathbb{Z}^m$, it is immediate to check that the map $t^{\mathbf{a}} \mapsto t^{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}^m$, $a \mapsto t^{\mathbf{u}}a$ and $b \mapsto b$, which we will denote by $\mathbf{u}$ too, is a group homomorphism since it preserves the relations of M_α, and has inverse $\mathbf{u}^{-1} = -\mathbf{u}$, hence $\mathbf{u} \in \mathrm{Aut}(M_\alpha)$. The result in this case then follows from an analogous argument to the one used in the previous case; the proof being essentially the same. □

Lastly, we will compute $\mathrm{Aut}(M_\alpha)$ in case (4). This is by far the most difficult case to treat and requires a substantial amount of work. The strategy in this case differs slightly from the one used in the other non-full cases. Namely, we will reduce the study of $\mathrm{Aut}(M_\alpha)$ to that of $\mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$, that is, to the case $m = 1$. We note that it will be convenient to use another notation for the generators of $\mathbb{Z}^m$. More exactly, we have that $M_\alpha = \mathbb{Z}^{m-1} \times (F_2 \rtimes_{\alpha_1} \mathbb{Z})$, and we will write s for the generator of $\mathbb{Z}$, $t_1, ..., t_{m-1}$ for the generators of $\mathbb{Z}^{m-1}$ and $t^{\mathbf{a}} = t_1^{a_1} \cdots t_{m-1}^{a_{m-1}}$ if $\mathbf{a} = (a_1, ..., a_{m-1}) \in \mathbb{Z}^{m-1}$. With this notation, we then have that $Z(M_\alpha) = \{t^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}^{m-1}\}$ and $s^{-1}xs = x\alpha_1$ for $x \in F_2$.

Therefore, we first study the automorphism group for $m = 1$. The techniques used to analyze this group are very similar to those employed in the previous non-full cases: we begin by computing $\mathrm{Aut}_{F_2}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$ using Proposition 2.14, and then deduce the general form of an automorphism of $F_2 \rtimes_{\alpha_1} \mathbb{Z}$ from the following lemma:

Lemma 3.20. *Let $\alpha_1 : (a \mapsto ab^k,\ b \mapsto b)$ with $k > 0$ an integer, $M_\alpha = F_2 \rtimes_{\alpha_1} \mathbb{Z}$ and write s for the generator of $\mathbb{Z}$. Consider the automorphism $\Theta \in \mathrm{Aut}(M_\alpha)$ mapping $a \mapsto sa$, $b \mapsto b$ and $s \mapsto s$. Then for every $\Psi \in \mathrm{Aut}(M_\alpha)$ there exists an integer m such that $\Psi\Theta^m$ leaves F_2 invariant.*

Proof. See [3, Proposition 4.4]. □

We remark that the proof of the above result is significantly more involved than one might initially expect, and for this reason we have chosen to omit it. The proof essentially reduces to showing that for an automorphism $\Psi \in \mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z}^m)$, if we write $a\Psi = s^p\omega$ with $p \in \mathbb{Z}$ and $\omega \in F_2$, then the a-exponent sum of ω is equal to ± 1; then one may easily check that $\Psi\Theta^{\pm p}$ leaves F_2 invariant. Now we can compute $\mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$ using the above lemma:

Lemma 3.21. *Let $\alpha_1 : (a \mapsto ab^k,\ b \mapsto b)$ with $k > 0$ an integer, $M_\alpha = F_2 \rtimes_{\alpha_1} \mathbb{Z}$ and write s for the generator of $\mathbb{Z}$. Consider the following:*

- *For each $u \in F_2$, we denote by γ_u the automorphism of M_α which maps $s \mapsto s(u\alpha_1)^{-1}u$ and $x \mapsto x\gamma_u$, $x \in F_2$. We have that $\langle \gamma_a, \gamma_b \rangle \cong F_2$.*
- *Let δ be the automorphism of M_α mapping $a \mapsto ab$, $b \mapsto b$ and $s \mapsto s$. We have that $\langle \delta \rangle \cong \mathbb{Z}$.*
- *Let Θ be the automorphism of M_α mapping $a \mapsto sa$, $b \mapsto b$ and $s \mapsto s$. We have that $\langle \Theta \rangle \cong \mathbb{Z}$.*
- *Let β_1 mapping $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ and $s \mapsto sb^{-k}$ and β_2 mapping $a \mapsto a$, $b \mapsto b^{-1}$ and $s \mapsto s^{-1}$, which are automorphisms of M_α. We have that $\langle \beta_1, \beta_2 \rangle \cong C_2 \times C_2 =: V$.*

With this notation, we have that

$$\mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z}) = ((\langle \gamma_a, \gamma_b \rangle \rtimes \langle \delta \rangle) \rtimes \langle \Theta \rangle) \rtimes \langle \beta_1, \beta_2 \rangle \cong ((F_2 \rtimes_\delta \mathbb{Z}) \rtimes_{\Theta^k} \mathbb{Z}) \rtimes V,$$

where the actions are given by

$$\delta^{-1}\gamma_a\delta = \gamma_a\gamma_b, \quad \delta^{-1}\gamma_b\delta = \gamma_b,$$
$$\Theta^{-1}\gamma_a\Theta = \delta^k\gamma_a, \quad \Theta^{-1}\gamma_b\Theta = \gamma_b, \quad \Theta^{-1}\delta\Theta = \delta,$$
$$\beta_1^{-1}\gamma_a\beta_1 = \gamma_a^{-1}, \quad \beta_1^{-1}\gamma_b\beta_1 = \gamma_b^{-1}, \quad \beta_1^{-1}\delta\beta_1 = \delta\gamma_b^{-1}, \quad \beta_1^{-1}\Theta\beta_1 = \Theta^{-1},$$
$$\beta_2^{-1}\gamma_a\beta_2 = \gamma_a, \quad \beta_2^{-1}\gamma_b\beta_2 = \gamma_b^{-1}, \quad \beta_2^{-1}\delta\beta_2 = \delta^{-1}, \quad \beta_2^{-1}\Theta\beta_2 = \Theta^{-1}.$$

Proof. If $\Psi \in \mathrm{Aut}(M_\alpha)$ then by Lemma 3.20 there exists an integer m such that $\Psi\Theta^{-m}$ leaves F_2 invariant, so that we can write $\Psi = \Phi\Theta^m$ for some $\Phi \in \mathrm{Aut}_{F_2}(M_\alpha)$. In this case we have that S_α is the trivial group, hence

$$\mathrm{Aut}_{F_2}(M_\alpha) \cong \pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha]).$$

We have $\mathrm{Im}[\alpha] = \langle [\alpha_1] \rangle \cong \mathbb{Z}$ with $[\alpha_1] = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$. Note that $[\alpha_1]$ is conjugate to $[\alpha_1]^{-1}$ via $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, hence $[\mathrm{N}(\mathrm{Im}[\alpha]) : \mathrm{C}(\mathrm{Im}[\alpha])] = 2$. A quick calculation shows that $\mathrm{C}(\mathrm{Im}[\alpha])$ consists precisely of the matrices of the form $\pm A^q$ with $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $q \in \mathbb{Z}$, hence $\mathrm{N}(\mathrm{Im}[\alpha])$ consists precisely of the matrices of the form $\pm P^\varepsilon A^q$ with $\varepsilon \in \{0, 1\}$ and $q \in \mathbb{Z}$. If we consider the automorphisms of F_2 given by $\beta_1 : (a \mapsto a^{-1},\ b \mapsto b^{-1})$, $\beta_2 : (a \mapsto a,\ b \mapsto b^{-1})$ and $\delta : (a \mapsto ab,\ b \mapsto b)$, then we have that

$$\pi^{-1}\,\mathrm{N}(\mathrm{Im}[\alpha]) = \{\beta_1^{\varepsilon_1}\beta_2^{\varepsilon_2}\delta^q\gamma_u \mid \varepsilon_1, \varepsilon_2 \in \{0, 1\},\ q \in \mathbb{Z},\ u \in F_2\}.$$

Note that, by Observation 2.4, the automorphisms $\beta_1, \beta_2, \delta, \gamma_u$ correspond to the automorphisms of M_α given in the statement in the lemma. By our first remark, we have that

$$\mathrm{Aut}(M_\alpha) = \{\beta_1^{\varepsilon_1}\beta_2^{\varepsilon_2}\delta^q\gamma_u\Theta^m \mid \varepsilon_1, \varepsilon_2 \in \{0, 1\},\ q \in \mathbb{Z},\ u \in F_2,\ m \in \mathbb{Z}\},$$

so that $\beta_1, \beta_2, \delta, \gamma_a, \gamma_b, \Theta$ generate $\mathrm{Aut}(M_\alpha)$. Now, consider the subgroups $\langle \gamma_a, \gamma_b \rangle \cong F_2$ and $\langle \delta \rangle \cong \mathbb{Z}$, which have trivial intersection since $[\delta] = A$ has infinite order in $\mathrm{GL}_2(\mathbb{Z})$ i.e. no power of δ can be an inner automorphism of F_2. This, together with the identities $\delta^{-1}\gamma_a\delta = \gamma_a\gamma_b$ and $\delta^{-1}\gamma_b\delta = \gamma_b$, show that

$$\langle \gamma_a, \gamma_b, \delta \rangle = \langle \gamma_a, \gamma_b \rangle \rtimes \langle \delta \rangle \cong F_2 \rtimes_\delta \mathbb{Z}$$

with the given action. Now, we have that $a\Theta^m = s^m a$, hence no non-trivial power of Θ leaves F_2 invariant, which implies that $\langle \gamma_a, \gamma_b, \delta \rangle \leq \mathrm{Aut}_{F_2}(M_\alpha)$ and $\langle \Theta \rangle$ intersect trivially. Moreover, it is straightforward to

check by looking at the images of the generators that $\Theta^{-1}\gamma_a\Theta = \delta^k\gamma_a, \Theta^{-1}\gamma_b\Theta = \gamma_b, \Theta^{-1}\delta\Theta = \delta$, hence we have that

$$\langle \gamma_a, \gamma_b, \delta, \Theta\rangle = (\langle \gamma_a, \gamma_b\rangle \rtimes \langle\delta\rangle) \rtimes \langle\Theta\rangle \cong (F_2 \rtimes_\delta \mathbb{Z}) \rtimes_{\Theta^k} \mathbb{Z}$$

with the given action. The above group is torsion-free by Observation 1.2, because $F_2 \rtimes_\delta \mathbb{Z}$ has no torsion by Lemma 1.1. In particular, the subgroups $\langle\beta_1, \beta_2\rangle \cong V$ and $\langle \gamma_a, \gamma_b, \delta, \Theta\rangle$ must intersect trivially, and again one can easily check by looking at the images of the generators that $\beta_1^{-1}\gamma_a\beta_1 = \gamma_a^{-1}, \beta_1^{-1}\gamma_b\beta_1 = \gamma_b^{-1}, \beta_1^{-1}\delta\beta_1 = \delta\gamma_b^{-1}, \beta_1^{-1}\Theta\beta_1 = \Theta^{-1}, \beta_2^{-1}\gamma_a\beta_2 = \gamma_a, \beta_2^{-1}\gamma_b\beta_2 = \gamma_b^{-1}, \beta_2^{-1}\delta\beta_2 = \delta^{-1}, \beta_2^{-1}\Theta\beta_2 = \Theta^{-1}$, and this proves that

$$\mathrm{Aut}(M_\alpha) = \langle \gamma_a, \gamma_b, \delta, \Theta, \beta_1, \beta_2\rangle = ((\langle \gamma_a, \gamma_b\rangle \rtimes \langle\delta\rangle) \rtimes \langle\Theta\rangle) \rtimes \langle\beta_1, \beta_2\rangle \cong ((F_2 \rtimes_\delta \mathbb{Z}) \rtimes_{\Theta^k} \mathbb{Z}) \rtimes V,$$

with the given action; this finishes the proof. □

The above result together with the following proposition completely determines the structure of $\mathrm{Aut}(M_\alpha)$ in case (4):

Proposition 3.22. *Let $\alpha_1 : (a \mapsto ab^k,\ b \mapsto b)$ with $k > 0$ an integer and let $M_\alpha = \mathbb{Z}^{m-1} \times (F_2 \rtimes_{\alpha_1} \mathbb{Z})$, that is M_α falls under case (4.k) of Proposition 3.12. Write s for the generator of $\mathbb{Z}$, $t_1, ..., t_{m-1}$ for the generators of $\mathbb{Z}^{m-1}$ and $t^{\mathbf{a}} = t_1^{a_1} \cdots t_{m-1}^{a_{m-1}}$ if $\mathbf{a} = (a_1, ..., a_{m-1}) \in \mathbb{Z}^{m-1}$. Consider the following:*

- *For each $A \in \mathrm{GL}_{m-1}(\mathbb{Z})$, we also denote by A the automorphism of M_α mapping $t^{\mathbf{a}} \mapsto t^{\mathbf{a}A}$, $\mathbf{a} \in \mathbb{Z}^{m-1}$, $z \mapsto z$, $z \in F_2 \rtimes_{\alpha_1} \mathbb{Z}$. The set of these automorphisms forms a subgroup of $\mathrm{Aut}(M_\alpha)$ which can be identified with $\mathrm{GL}_{m-1}(\mathbb{Z})$.*
- *For each $v = (\mathbf{v_1}, \mathbf{v_2}) \in (\mathbb{Z}^{m-1})^2$, we also denote by v the automorphism of M_α mapping $t^{\mathbf{a}} \mapsto t^{\mathbf{a}}, \mathbf{a} \in \mathbb{Z}^{m-1}$, $s \mapsto t^{\mathbf{v_1}}s$, $a \mapsto t^{\mathbf{v_2}}a$ and $b \mapsto b$. The set of these automorphisms forms a subgroup of $\mathrm{Aut}(M_\alpha)$ which can be identified with $(\mathbb{Z}^{m-1})^2$.*
- *For each $\Phi \in \mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$, we also denote by Φ the automorphism of M_α mapping $t^{\mathbf{a}} \mapsto t^{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}^{m-1}$, $z \mapsto z\Phi$, $z \in F_2 \rtimes_{\alpha_1} \mathbb{Z}$. The set of these automorphisms forms a subgroup of $\mathrm{Aut}(M_\alpha)$ which can be identified with $\mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$.*

With this notation, we have that

$$\mathrm{Aut}(M_\alpha) = ((\mathbb{Z}^{m-1})^2 \rtimes \mathrm{GL}_{m-1}(\mathbb{Z})) \rtimes \mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z}),$$

with $A \in \mathrm{GL}_{m-1}(\mathbb{Z})$ acting on $v = (\mathbf{v_1}, \mathbf{v_2}) \in (\mathbb{Z}^{m-1})^2$ via $A^{-1}vA = (\mathbf{v_1}A, \mathbf{v_2}A)$, $\mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$ acting trivially on $\mathrm{GL}_{m-1}(\mathbb{Z})$ and $\Phi \in \mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$ acting on $(\mathbb{Z}^{m-1})^2$ via (see Lemma 3.21):

$$\gamma_a v\gamma_a^{-1} = v \quad , \gamma_b v\gamma_b^{-1} = v, \quad \delta v\delta^{-1} = v,$$

$$\Theta v\Theta^{-1} = (\mathbf{v_1}, \mathbf{v_1} + \mathbf{v_2}), \quad \beta_1 v\beta_1^{-1} = (\mathbf{v_1}, -\mathbf{v_2}), \quad \beta_2 v\beta_2^{-1} = (-\mathbf{v_1}, \mathbf{v_2}).$$

Proof. First of all, if $A \in \mathrm{GL}_{m-1}(\mathbb{Z})$, $\Phi \in \mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$ and $v = (\mathbf{v_1}, \mathbf{v_2}) \in (\mathbb{Z}^{m-1})^2$, then these define automorphisms of M_α as described in the statement of the lemma. Indeed, one may easily check that the relations of M_α are preserved and the inverses are given by A^{-1}, Φ^{-1} and $-v = (-\mathbf{v_1}, -\mathbf{v_2})$, respectively. Moreover, the subgroups of $\mathrm{Aut}(M_\alpha)$ generated by these can be identified with $\mathrm{GL}_{m-1}(\mathbb{Z})$, $\mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$ and $(\mathbb{Z}^{m-1})^2$, respectively.

Let $\Psi \in \mathrm{Aut}(M_\alpha)$, and note that Ψ restricts to an automorphism of $Z(M_\alpha) = \langle t_1, ..., t_{m-1} \rangle \cong \mathbb{Z}^{m-1}$, which is represented by a certain matrix $A \in \mathrm{GL}_{m-1}(\mathbb{Z})$ in the basis $t_1, ..., t_{m-1}$, i.e. $t^{\mathbf{a}}\Psi = t^{\mathbf{a}A}$ for all $\mathbf{a} \in \mathbb{Z}^m$. We now write $s\Psi = t^{\mathbf{v_s}}z_s$, $a\Psi = t^{\mathbf{v_a}}z_a$ and $b\Psi = t^{\mathbf{v_b}}z_b$ for certain $\mathbf{v_s}, \mathbf{v_a}, \mathbf{v_b} \in \mathbb{Z}^{m-1}$ and $z_a, z_b, z_s \in F_2 \rtimes_{\alpha_1} \mathbb{Z}$. Note that Ψ restricts to an automorphism of $M_\alpha^{\mathrm{ab}} = \langle t_1, ..., t_{m-1}, s, a \rangle \oplus \langle b \rangle \cong \mathbb{Z}^{m+1} \oplus C_k$, and $b^k = 1$ in M_α^{ab}, hence we must have $k\mathbf{v_b} = \mathbf{0}$ i.e. $v_b = \mathbf{0}$. Moreover, Ψ induces an automorphism Φ of $\mathrm{Inn}(M_\alpha) \cong M_\alpha/Z(M_\alpha) = M_\alpha/\mathbb{Z}^{m-1} = F_2 \rtimes_{\alpha_1} \mathbb{Z}$ mapping $s \mapsto z_s$, $a \mapsto z_a$ and $b \mapsto z_b$. It is then straightforward to check that we can decompose $\Psi = A \circ v \circ \Phi$ in $\mathrm{Aut}(M_\alpha)$.

Hence, we have shown that $\mathrm{Aut}(M_\alpha) = \mathrm{GL}_{m-1}(\mathbb{Z}) \cdot (\mathbb{Z}^{m-1})^2 \cdot \mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$. Now, observe that $\mathrm{GL}_{m-1}(\mathbb{Z})$ and $(\mathbb{Z}^{m-1})^2$ have trivial intersection, and we have that $A^{-1} \circ v \circ A = (\mathbf{v_1}A, \mathbf{v_2}A)$, hence $\mathrm{GL}_{m-1}(\mathbb{Z})$ and $(\mathbb{Z}^{m-1})^2$ generate a subgroup of $\mathrm{Aut}(M_\alpha)$ isomorphic to $(\mathbb{Z}^{m-1})^2 \rtimes \mathrm{GL}_{m-1}(\mathbb{Z})$ with the given action. We now have that $(\mathbb{Z}^{m-1})^2 \rtimes \mathrm{GL}_{m-1}(\mathbb{Z})$ and $\mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$ intersect trivially, and $\Psi^{-1}A\Psi = A$. Moreover, we have that $t^{\mathbf{a}}\Phi^{-1}v\Phi = t^{\mathbf{a}}$ for $a \in \mathbb{Z}^m$. Let $B_\Phi \in \mathrm{GL}_2(\mathbb{Z})$ be the matrix whose first (resp. second) row is the s-exponent and $a-$exponent of $s\Phi$ (resp. $a\Phi$), which is well defined since $(F_2 \rtimes_{\alpha_1} \mathbb{Z})^{\mathrm{ab}} = \langle s, a \rangle \oplus \langle b \rangle \cong \mathbb{Z}^2 \oplus C_k$. Now, if we write $B_\Phi = (b_{i,j})_{i,j}$ then

$$s\Phi v\Phi^{-1} = (t^{b_{1,1}\mathbf{v_1}+b_{1,2}\mathbf{v_2}}s\Phi)\Phi^{-1} = t^{b_{1,1}\mathbf{v_1}+b_{1,2}\mathbf{v_2}}s,$$

$$a\Phi v\Phi^{-1} = (t^{b_{1,1}\mathbf{v_1}+b_{1,2}\mathbf{v_2}}a\Phi)\Phi^{-1} = t^{b_{2,1}\mathbf{v_1}+b_{2,2}\mathbf{v_2}}a,$$

$$b\Phi v\Phi^{-1} = (b\Phi)\Phi^{-1} = b,$$

which shows that $\Phi v\Phi^{-1} = B_\Phi v^T = (b_{1,1}\mathbf{v_1} + b_{1,2}\mathbf{v_2}, b_{2,1}\mathbf{v_1} + b_{2,2}\mathbf{v_2})$. In particular, these calculations show that $\mathrm{Aut}(M_\alpha) = \left((\mathbb{Z}^{m-1})^2 \rtimes \mathrm{GL}_{m-1}(\mathbb{Z})\right) \rtimes \mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$ with $\mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$ acting on $\mathrm{GL}_{m-1}(\mathbb{Z})$ trivially and on $(\mathbb{Z}^{m-1})^2$ via the given action. Using the notation from Lemma 3.21, we have that $B_{\gamma_a} = B_{\gamma_b} = B_\delta = \mathrm{Id}_2$, $B_\Theta = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $B_{\beta_1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $B_{\beta_2} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and hence the generators of $\mathrm{Aut}(F_2 \rtimes_{\alpha_1} \mathbb{Z})$ act on $(\mathbb{Z}^{m-1})^2$ as described in the statement of the lemma. □

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A. Classification of $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^m$

We summarize the classification of $M_\alpha = F_2 \rtimes_\alpha \mathbb{Z}^m$ obtained in Proposition 3.12 in the following table, together with the computation of certain invariants of the group M_α. We also note that distinct cases correspond to non-isomorphic groups by Theorem 3.14.

M_α	α	r_α	$Z(M_\alpha)$	$\mathrm{Inn}(M_\alpha)$	M_α^{ab}	$\mathrm{Im}[\alpha]$
$F_2 \times \mathbb{Z}^m$	Id	0	$\mathbb{Z}^m$	F_2	$\mathbb{Z}^{m+2}$	$\{\mathrm{Id}_2\}$
$(F_2 \rtimes_{\alpha_1} \mathbb{Z}) \times \mathbb{Z}^{m-1}$	$\alpha_1 : (a \mapsto a,\ b \mapsto b^{-1})$	1	$2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$	$F_2 \rtimes_{\alpha_1} C_2$	$\mathbb{Z}^{m+1} \oplus C_2$	$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$
	$\alpha_1 : (a \mapsto b,\ b \mapsto a)$	1	$2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$	$F_2 \rtimes_{\alpha_1} C_2$	$\mathbb{Z}^{m+1}$	$\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$
	$\alpha_1 : (a \mapsto ab^k,\ b \mapsto b)$ $k \in \mathbb{Z}_{>0}$	1	$\mathbb{Z}^{m-1}$	$F_2 \rtimes_{\alpha_1} \mathbb{Z}$	$\mathbb{Z}^{m+1} \oplus C_k$	$\left\langle \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \right\rangle$
	$\mathrm{ord}\,[\alpha_1] = +\infty$ $\mathrm{rank}([\alpha_1] - \mathrm{Id}_2) = 2$	2	$\mathbb{Z}^{m-1}$	$F_2 \rtimes_{\alpha_1} \mathbb{Z}$	$\mathbb{Z}^m \oplus T_\alpha$	$\langle [\alpha_1] \rangle$
	$\alpha_1 : (a \mapsto a^{-1},\ b \mapsto b^{-1})$	2	$2\mathbb{Z} \oplus \mathbb{Z}^{m-1}$	$F_2 \rtimes_{\alpha_1} C_2$	$\mathbb{Z}^m \oplus C_2 \oplus C_2$	$\langle -\mathrm{Id}_2 \rangle$
	$\alpha_1 : (a \mapsto b^{-1},\ b \mapsto ab^{-1})$	2	$\mathbb{Z}^{m-1}$	$F_2 \rtimes_{\alpha_1} C_3$	$\mathbb{Z}^m \oplus C_3$	$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle$
	$\alpha_1 : (a \mapsto b,\ b \mapsto a^{-1})$	2	$4\mathbb{Z} \oplus \mathbb{Z}^{m-1}$	$F_2 \rtimes_{\alpha_1} C_4$	$\mathbb{Z}^m \oplus C_2$	$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$
	$\alpha_1 : (a \mapsto ab^{-1},\ b \mapsto a)$	2	$\cong 6\mathbb{Z} \oplus \mathbb{Z}^{m-1}$	†	$\mathbb{Z}^m$	$\left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$
$(F_2 \rtimes_{\alpha_1,\alpha_2} \mathbb{Z}^2) \times \mathbb{Z}^{m-2}$	$\alpha_1 : (a \mapsto a,\ b \mapsto b^{-1})$ $\alpha_2 : (a \mapsto a^{-1},\ b \mapsto b)$	2	$2\mathbb{Z} \oplus 2\mathbb{Z} \oplus \mathbb{Z}^{m-2}$	$F_2 \rtimes_{\alpha_1,\alpha_2} (C_2 \times C_2)$	$\mathbb{Z}^m \oplus C_2 \oplus C_2$	$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -\mathrm{Id}_2 \right\rangle$
	$\alpha_1 : (a \mapsto b,\ b \mapsto a)$ $\alpha_2 : (a \mapsto b^{-1},\ b \mapsto a^{-1})$	2	$2\mathbb{Z} \oplus 2\mathbb{Z} \oplus \mathbb{Z}^{m-2}$	$F_2 \rtimes_{\alpha_1,\alpha_2} (C_2 \times C_2)$	$\mathbb{Z}^m \oplus C_2$	$\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -\mathrm{Id}_2 \right\rangle$
	$\mathrm{ord}\,[\alpha_1] = +\infty$ $[\alpha_2] = -\mathrm{Id}_2$	2	$\cong 2\mathbb{Z} \oplus \mathbb{Z}^{m-2}$	‡	$\mathbb{Z}^m \oplus C_2 \oplus C_2$ or $\mathbb{Z}^m \oplus C_2$ or $\mathbb{Z}^m$.	$\langle -\mathrm{Id}_2 \rangle \times \langle [\alpha_2] \rangle$

† In this case we have that $\alpha_1^6 = \gamma_\omega$ with $\omega = b^{-1}aba^{-1}$. Therefore, $Z(M_\alpha) = \langle t_1^6\omega^{-1}, t_2, ..., t_m \rangle$ and $\mathrm{Inn}(M_\alpha) \cong (F_2 \rtimes_{\alpha_1} \mathbb{Z})/\langle s^6\omega^{-1} \rangle$, where s denotes the corresponding generator of $\mathbb{Z}$.

‡ If $\alpha_2^2 = \mathrm{Id}$, then $Z(M_\alpha) = \langle t_2^2, t_3, ..., t_m \rangle$ and $\mathrm{Inn}(M_\alpha) \cong F_2 \rtimes_{\alpha_1,\alpha_2} (\mathbb{Z} \times C_2)$.

Otherwise, there exists a non-trivial word $\omega \in F_2$ such that $\alpha_2^2 = \gamma_\omega$. Then $Z(M_\alpha) = \langle t_2^2\omega^{-1}, t_3, ..., t_m \rangle$ and $\mathrm{Inn}(M_\alpha) \cong (F_2 \rtimes_{\alpha_1,\alpha_2} \mathbb{Z}^2)/\langle s_2^2\omega^{-1} \rangle$, where s_1, s_2 denote the corresponding generators of $\mathbb{Z}^2$.