The degree of commutativity of an infinite group

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Outline

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Motivation

(Joint work with Y. Antolín and A. Martino.)

Theorem (Gustafson, 1973)

Let G be a finite group. If the probability that two elements from G commute is bigger than $5/8$, then G is abelian.

Proof. *Suppose G is not abelian. Then,*

$$dc(G) = \frac{|\{(u,v) \in G^2 \mid uv = vu\}|}{|G|^2} = \frac{1}{|G|^2} \sum_{u \in G} |C_G(u)| =$$

$$= \frac{1}{|G|^2} \Big(|Z(G)||G| + \sum_{u \in G \setminus Z(G)} |C_G(u)| \Big) \leqslant$$

$$\leqslant \frac{1}{|G|^2} \Big(|Z(G)||G| + (|G| - |Z(G)|) \frac{|G|}{2} \Big) =$$

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because $G/Z(G)$ cannot be cyclic and so, $|Z(G)| \leqslant |G|/4$. □

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The quaternion group has $dc(Q) = 5/8$.

"There is no live between $5/8$ and 1"

(Goal)

Is there a version of dc for infinite groups ?

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$$dc_X(G) = \limsup_{n\to\infty} \frac{|\{(u,v) \in \mathbb{B}_X(n) \times \mathbb{B}_X(n) \mid uv = vu\}|}{|\mathbb{B}_X(n)|^2} \in [0,1],$$

where $\mathbb{B}_X(n) = \{g \in G \mid |g|_X \leqslant n\}$.

Question

Is this a real lim ? Does it depend on X ?

About limsup we have no idea:

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A f.g. group $G = \langle X \rangle$ is of

- *subexponential growth* if $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n+1)|}{|\mathbb{B}_X(n)|} = 1$;
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Let $G = \langle X \rangle$. A map $f\colon G \to \mathbb{N}$ is an estimation of the X-metric if $\exists$ $K > 0$ such that $\forall w \in G$

$$\frac{1}{K} f(w) \leqslant |w|_X \leqslant K\, f(w).$$

Example

It is well known that, for $G = \langle X \rangle = \langle Y \rangle$, $|\cdot|_X$ is an estimation of the Y-metric, and $|\cdot|_Y$ is an estimation of the X-metric.

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Define the f-ball and the f-dc:

$$\mathbb{B}_f(n) = \{w \in G \mid f(w) \leqslant n\},$$

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Proposition

Let $G = \langle X \rangle$ be of polynomial growth, and $f\colon G \to \mathbb{N}$ be an estimation of the X-metric. Then,

$$dc_X(G) > 0 \iff dc_f(G) > 0.$$

Proof. *Clearly, $\mathbb{B}_f(n) \subseteq \mathbb{B}_X(Kn) \subseteq \mathbb{B}_f(K^2 n)$ so,*

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$$\left(\frac{|\{(u,v)\in(\mathbb{B}_f(n))^2 \mid uv=vu\}|}{|\mathbb{B}_f(n)|^2}\right)\left(\frac{|\mathbb{B}_f(n)|}{|\mathbb{B}_X(Kn)|}\right)^2$$

So, $dc_X(G)=0 \quad \Rightarrow \quad dc_f(G)=0$, because

$$\frac{|\mathbb{B}_f(n)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{|\mathbb{B}_X(n/K)|}{|\mathbb{B}_X(Kn)|} \geqslant \frac{C(n/K)^d}{D(Kn)^d} = \frac{C}{DK^{2d}} > 0. \quad \square$$

Corollary

If $G=\langle X\rangle=\langle Y\rangle$ is of polynomial growth, then

$$dc_X(G)=0 \quad \Longleftrightarrow \quad dc_Y(G)=0.$$

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Let $\langle Y \rangle = H \leqslant G = \langle X \rangle$. The subgroup H is undistorted if $\exists\, K > 0$ s.t. $\forall\, h \in H,\ \ |h|_Y / K \leqslant |h|_X \leqslant K |h|_Y$.
In this case, $|\cdot|_X$ restricted to H is an estimation of the Y-metric for H.

Corollary

Let $G = \langle X \rangle$ be of polynomial growth, and $\langle Y \rangle = H \leqslant G$ be a non-distorted subgroup. Then,

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Finite index subgroups

Lemma (Burillo–Ventura, 2002)

If $H \leqslant_{f.i.} G = \langle X \rangle$ and G has subexponential growth then, for every $g \in G$, there exists $\lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap gH|}{|\mathbb{B}_X(n)|} = \lim_{n\to\infty} \frac{|\mathbb{B}_X(n) \cap Hg|}{|\mathbb{B}_X(n)|} = \frac{1}{[G:H]}$.

Remark

This is false in the free group: $H = \{\text{even words}\} \leqslant_2 F_r$.

Proposition

Let $\langle Y \rangle = H \leqslant_{f.i.} G = \langle X \rangle$ be of polynomial growth. Then,

$$dc_X(G) \geqslant \frac{1}{[G:H]^2} dc_X(H).$$

In particular, $dc_Y(H) > 0 \Rightarrow dc_X(H) > 0 \Rightarrow dc_X(G) > 0$.

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$|\{(u,v) \in (\mathbb{B}_X(n))^2 \mid uv = vu\}| \geq |\{(u,v) \in (H \cap \mathbb{B}_X(n))^2 \mid uv = vu\}|$.
Therefore, given $\varepsilon > 0$, we have for $n \gg 0$

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$$\frac{|\{(u,v) \in (H \cap \mathbb{B}_X(n))^2 \mid uv = vu\}|}{|H \cap \mathbb{B}_X(n)|^2} \left(\frac{1}{[G:H]} - \varepsilon \right)^2,$$

Taking limsups, $dc_X(G) \geqslant dc_X(H) \left(\frac{1}{[G:H]} - \varepsilon \right)^2$. And this is true $\forall \varepsilon > 0$ so, $dc_X(G) \geqslant \frac{1}{[G:H]^2} dc_X(H)$. $\square$

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Finite index subgroups

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Proposition (Gallagher, 1970)

Let G be a finite group and $H \trianglelefteq G$. Then, $dc(G) \leqslant dc(H) \cdot dc(G/H)$.

Proposition

Let $G = \langle X \rangle$ be subexponentially growing. Then, for any finite quotient G/N, we have $dc_X(G) \leqslant dc(G/N)$.

Proof. Let $N \trianglelefteq G$ with $[G : N] = d$.
By B–V, $\forall g \in G$ $\lim_{n\to\infty} |gN \cap \mathbb{B}_X(n)|/|\mathbb{B}_X(n)| = 1/d$, indep. X and g.
But $|G/N| < \infty$, so this lim is uniform on g, i.e.,
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Outline

The main result

Theorem

Let $G = \langle X \rangle$ be of subexponential growth and residually finite. Then,

(i) $dc_X(G) > 5/8 \Rightarrow G$ is abelian;

(ii) $dc_X(G) > 0 \Leftrightarrow G$ is virtually abelian.

In particular, (i) and (ii) is true for polynomially growing groups.

Corollary

Let $G = \langle X \rangle = \langle Y \rangle$ be of subexponential growth and residually finite. Then,

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Let $G = \langle X \rangle$ be of subexponential growth and residually finite. Then,
(i) $dc_X(G) > 5/8 \;\Rightarrow\; G$ is abelian;
(ii) $dc_X(G) > 0 \;\Leftrightarrow\; G$ is virtually abelian.

Proof. *(i). Suppose $dc_X(G) > 5/8$. Then, $dc(G/N) > 5/8$ for every $N \trianglelefteq_{f.i.} G$. Hence, by Gustafson's thm, every finite quotient of G is abelian. Residual finiteness implies G abelian.*

(ii, $\Leftarrow$). Suppose $G = \langle X \rangle$ is virtually abelian, $\langle Y \rangle = H \leqslant_{f.i.} G$ with H abelian. Then G is polynomially growing and $dc_Y(H) = 1 > 0$ so, $dc_X(G) > 0$.

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Then $\forall i, \quad K_i \trianglelefteq G, \quad (G/K_i)/(K_{i-1}/K_i) = G/K_{i-1}$ and, by Gallagher,

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By induction, $dc(G/K_i) \leqslant (5/8)^i$ and so,

$$dc_X(G) \leqslant dc(G/K_i) \leqslant (5/8)^i,$$

for every i. Therefore, $dc_X(G) = 0$. $\square$

Outline

Generalizations

- *We can replace $xy = yx$ by any system of equations.*
- *We can replace the uniform measures on balls to any sequence of measures (random walks, etc).*

Definition

Let $\{X_1, \dots, X_k\}$ be a set of abstract variables and $\mathcal{F}$ the free group on it. Think elements $w \in \mathcal{F}$ as equations, $w = 1$, and subsets $\mathcal{E} \subseteq \mathcal{F}$ as systems of equations. Define solutions on a group G in the obvious way.

Definition

Given $G = \langle X \rangle$ and a system of equations $\mathcal{E} \subseteq \mathcal{F}$, we define the degree of satisfiability of $\mathcal{E}$ in G as

$$ds_X(G, \mathcal{E}) = \limsup_{n \to \infty} \frac{|\{(g_1, \dots, g_k) \in (\mathbb{B}_X(n))^k \mid (g_1, \dots, g_k) \text{ sol. } \mathcal{E}\}|}{|\mathbb{B}_X(n)|^k} \in [0, 1].$$

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Let G and $\mathcal{E}$ be as before. Fix a collection of measures μ_n in G with finite support, $|\operatorname{Supp}\mu_n| < \infty$, and such that

$$\operatorname{Supp}\mu_1 \subseteq \operatorname{Supp}\mu_2 \subseteq \cdots$$

and $\cup_{n\in\mathbb{N}} \operatorname{Supp}\mu_n = G$. We define the degree of satisfiability of $\mathcal{E}$ in G w.r.t. μ_n as

$$ds_X(G, \mathcal{E}, \{\mu_n\}_n) =$$

$$\limsup_{n\to\infty} \mu_n^{\times k}(\{(g_1, \ldots, g_k) \in G^k \mid (g_1, \ldots, g_k) \text{ sol. } \mathcal{E}\}) \in [0, 1].$$

Generalizations

Conjecture

Let G, $\mathcal{E}$, and $\{\mu_n\}_n$ be as above, with μ_n "reasonable". Then,

$$ds(G, \mathcal{E}, \{\mu_n\}_n) > 0 \iff \mathcal{E} \text{ is a virtual law in } G.$$

Definition

$\mathcal{E}$ is a law in G if every $(g_1, \ldots, g_k) \in G^k$ is a solution of $\mathcal{E}$ in G.

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THANKS