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Degree in Mathematics

Title: *Pro-*$\mathcal{V}$ topologies in groups: the closure problem on the free group and $\mathcal{V}_p$

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Abstract

The main objective of this bachelor thesis is to develop and understand the known properties of $Pro{-}\mathcal{V}$ topologies in groups, to then apply these concepts in the notorious problem of computing generators of the closure of finitely generated subgroups of the free group. After the first part of this manuscript, where we will study the general attributes of the $Pro{-}\mathcal{V}$ topology on an arbitrary group, and characterize the case of topologies on subgroups, finite direct product groups and quotient groups, we will focus on the case of $G = \mathbb{F}_A$: here, and with the nice tool of automata theory, we will be able to prove that, under the family of extension–closed pseudovarieties, the closure of a finitely generated subgroup of $\mathbb{F}_A$ is also finitely generated. This will lead us to, finally, and in the case $\mathcal{V} = \mathcal{V}_p$, reproduce a known algorithm to effectively compute generators of the closure of a finitely generated subgroup $H \leqslant \mathbb{F}_A$.

Contents

Chapter 1

Introduction

Since modern times, groups, as a mathematical structure, have been an object of study for several scientific disciplines, not only mathematics. Indeed, they appear in a large list of subjects that vary from pure algebra and geometry into more applied areas such as physics (the *Poincaré group* consists of the symmetries of spacetime in special relativity), or even chemistry (*point groups* describe symmetry in molecular chemistry).

A special type of these mathematical structures are the so–called *free groups.* Since its formal definition and the study of its algebraic properties as we know them today in the early 1900s, they have always been a subject of interest to mathematicians because of its simple presentations and interesting lattice of subgroups (it is known that every group can be expressed as a quotient of a free group).

It was Hall in the 1950s [5], [6] who, with this goal of studying the subgroups of the free group, introduced the notion of $Pro{-}\mathcal{V}$ topology on a group (see Section 3.1 for the formal definition), where $\mathcal{V}$ is a pseudovariety of finite groups (a class of finite groups closed under taking subgroups, finite direct products and quotients). Hall mainly considered $\mathcal{V}_f$, the pseudovariety of all finite groups (which yield the so–called *profinite topology*) and was able to prove that every finitely generated subgroup of the free group is closed under the profinite topology [6, Theorem 5.1].

With the introduction of *Stallings automata* by J.R. Stallings in [16], relating free groups with graph theory, important advances in the field were made, such as giving an algorithm to compute generators of subgroups of the free group. Of course, this new area of study was also applied in $Pro{-}\mathcal{V}$ topologies on the free group.
For instance, in this thesis we will return to a Ribes and Zalesskii article [14] that combines known facts about $Pro{-}\mathcal{V}$ topologies with *Stallings* automata theory to first prove the following result:

Theorem. *Let $\mathbb{F}_A$ be the free group over the alphabet A, and consider $\mathcal{V}_p$, the pseudovariety of all finite $p-$groups. Then, if H is a finitely generated subgroup of $\mathbb{F}_A$, the closure of H under the $Pro{-}\mathcal{V}_p$ topology on $\mathbb{F}_A$, $Cl(H)$, is also finitely generated.*

Then, they use this theorem to develop an algorithm to effectively compute generators of the closure of finitely generated subgroups of the free group, (under the $Pro{-}\mathcal{V}_p$ topology, of course). This algorithm was then clarified and sped up by Margolis, Sapir and Weil in [9]. The correct understandability of this new version of the algorithm will be one of the main goals of this thesis, on Chapter 4.

But, why would we want to compute generators of $Cl(H)$? One of the main reasons to study closed subgroups under the $Pro{-}\mathcal{V}$ topology on a group G (those with $Cl(H) = H$) lies in the tight relation they have with the algebraic properties of the $Pro{-}\mathcal{V}$ completion of G, $\hat{G}$ (the completion of G thought as a metric space). Even though, this escapes the content of this thesis and will not be covered here.

This manuscript is at the confluence of general topology and group theory, and is organized as follows: in Chapter 2, we first present some basic definitions and results about general topology and group theory, that will be used throughout all the thesis. Then, two concise sections introducing free groups and *Stallings* automata follow for completion, as most of the results presented there will be used later, in Chapter 4.

In Chapter 3 we first recover the definition of a pseudovariety of finite groups $\mathcal{V}$, and then formally define the *Pro*$-\mathcal{V}$ topology on an arbitrary group G, namely $\mathcal{T}_{\mathcal{V}}^{G}$. The rest of the chapter is dedicated to study topological and algebraic properties of $\mathcal{T}_{\mathcal{V}}^{G}$: for instance, it is known [9, Section 1.1] that the *Pro*$-\mathcal{V}$ topology on an arbitrary group G can also be defined through a pseudodistance $d_{\mathcal{V}}^{G}$:

Theorem. *Let $\mathcal{V}$ be a pseudovariety of finite groups and G an arbitrary group. Then, there exists a pseudodistance $d_{\mathcal{V}}^{G}\colon G \times G \longrightarrow \mathbb{R}$ that induces the Pro$-\mathcal{V}$ topology on G, that is*

$$\mathcal{T}_{\mathcal{V}}^{G} = \mathcal{T}_{d_{\mathcal{V}}^{G}}.$$

This theorem, that gives G the structure of a (pseudo) metric space, will then alow us to characterize the *Pro*$-\mathcal{V}$ topology on a finite direct product of groups via the product topology, among other results.
Then, we turn to study the topological properties of subgroups $H \leqslant G$: under which algebraic conditions is H an open set? Or, what conditions do we have to ask H to ensure that the restriction to H of $\mathcal{T}_{\mathcal{V}}^{G}$ is equivalent to $\mathcal{T}_{\mathcal{V}}^{H}$?
Finally, the last part of the chapter is dedicated to study *Pro*$-\mathcal{V}$ topologies on quotient groups and its relation with the quotient topology. A surprising result arises when we prove that, if H is a normal subgroup of G, then $\mathcal{T}_{\mathcal{V}}^{G/H} = \mathcal{T}_{\mathcal{V}}^{G}/H$ holds in complete generality.

In Chapter 4, we return to the initial discussion and focus on the case $G = \mathbb{F}_A$ since, as previously said, our goal is to understand and reproduce Margolis, Sapir and Weil [9] algorithm to compute generators of the closure of a finitely generated subgroup of the free group (when $\mathcal{V} = \mathcal{V}_p$). Following the work of Ribes and Zalesskii in [14], we will first prove that, under a more general family of pseudovarieties (that, of course, includes $\mathcal{V}_p$), the closure of a finitely generated subgroup of the free group is also finitely generated.
Finally, in the last chapter we present the conclusions we have reached, and outline potential avenues for further research.

Chapter 2

Preliminaries

In this chapter, we review some basic notions that will be used throughout this thesis. Basically, we will go through general concepts of topology and algebra, focusing on group theory on this last subject. Also, an additional section introducing free groups and Stalling automata is presented, as for its fundamental relevance on this thesis (Chapter 4).
Even though many basic statements are left unproven here, we provide a reference with a detailed proof for the more specific results presented in the chapter.

2.1 On general topology

Because of the pivotal role topology plays in this thesis, and for the sake of completion, in this section we introduce the most basic notions of the subject. The general references used here are [7] and [12].

Definition 2.1.1. Let X be a set. A *topology* on X is a colection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying the following properties:

1. $\emptyset, X \in \mathcal{T}$;
2. If $U_i \in \mathcal{T}$, for $i \in I$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$;
3. If $U_1, \ldots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

$\mathcal{T}_0 = \{\emptyset, X\}$ is the trivial topology, and $\mathcal{T}_{dis} = \mathcal{P}(X)$ is the so–called *discrete* topology. The elements of a topology are called open sets. We will also call the pair $(X, \mathcal{T})$ a *topological space*.

When working with topological spaces we can characterize the notion of continuity of functions.

Definition 2.1.2. Let $(X, \mathcal{T})$, $(X', \mathcal{T}')$ be two topological spaces, and let $f\colon X \to X'$ be a function between them. We say that f is *continuous* if $\forall U \in \mathcal{T}'$, $f^{-1}(U) := \{x \in X : f(x) \in U\} \in \mathcal{T}$. As one could expect, the composition of continuous functions is also a continuous function. A continuous bijective function with continuous inverse is called an *homeomorphism*.

Metric spaces are tightly related with topological spaces. That is, we can give a topology on a metric space based on its distance function.

Definition 2.1.3. Let (X, d) be a metric space. Then, the collection of subsets

$$\mathcal{T}_d = \{U \subseteq X : \forall x \in U,\ \exists \varepsilon > 0 \text{ such that } x \in B_d(x, \varepsilon) \subseteq U\},$$

where $B_d(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}$ is the open ball of center $x \in X$ and radius $\varepsilon > 0$, is a topology on X, the so–called *metric topology.* That way, a topological space is a much more general notion than a metric space! It is also easy to see that, if d and d' are equivalent metrics (i.e. they yield the same continuous functions), then $\mathcal{T}_d = \mathcal{T}_{d'}$.

We also say that a topological space $(X, \mathcal{T})$ is *metrizable* if there exists a distance function d over X such that $\mathcal{T} = \mathcal{T}_d$. Let us now define the meaning behind a *Hausdorff* topological space:

Definition 2.1.4. We say that a topological space $(X, \mathcal{T})$ is *Hausdorff* if $\forall x, y \in X$, there exist $U_x, V_y \in \mathcal{T}$ such that $x \in U_x,\ y \in V_y$ and $U_x \cap V_y = \emptyset$. It is known that all metrizable spaces are Hausdorff, even though the converse is not true.

Definition 2.1.5. If we have two topologies $\mathcal{T}, \mathcal{T}'$ defined on the same set X satisfying $\mathcal{T} \subseteq \mathcal{T}'$, we then say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$.

Lots of times it will be sufficient to work with a small part of the elements of a topology, which will act as a representative of the whole collection of subsets. These sets of representatives are called *bases* of the topology.

Definition 2.1.6. Let $(X, \mathcal{T})$ be a topological space. A subset $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis* of $\mathcal{T}$ if $\forall U \in \mathcal{T}$, there exist $B_i^U \in \mathcal{B},\ i \in I$, such that $U = \bigcup_{i \in I} B_i^U$. The elements of $\mathcal{B}$ are often called *basic sets.* If $\mathcal{B} \subseteq \mathcal{B}'$ are both bases of $\mathcal{T}$, we will also say that $\mathcal{B}'$ is finer than $\mathcal{B}$.

An interesting concept that will appear in the definition of the *Pro*$-\mathcal{V}$ topology on a group, on Chapter 3, is the notion of *subbasis*:

Definition 2.1.7. Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$. We define $\langle \mathcal{S} \rangle$ as the least fine topology that contains $\mathcal{S}$, and we say that $\mathcal{S}$ is a subbasis of $\langle \mathcal{S} \rangle$.

We can also define now the notion of *neighborhood* of a point $x \in X$:

Definition 2.1.8. Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. An (open) *neighborhood* of x is a (open) subset $N \subseteq X$ such that there exist $U \in \mathcal{T}$, with $x \in U \subseteq N$.

Another important definition involves the concept of closed set, which will turn out to be key on this thesis:

Definition 2.1.9. Let $(X, \mathcal{T})$ be a topological space. We say that $D \subseteq X$ is *closed* if $D^c = X \backslash D \in \mathcal{T}$. It is easy to see that a finite union of closed sets is again a closed set, and that an arbitrary intersection of closed sets is also a closed set. If D is both an open and a closed set, we will refer to it as a *clopen* set.

We can now define some special types of points of a given subset $A \subseteq X$, with good topological properties.

Definition 2.1.10. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be any subset. Then,

- $x \in X$ is an **interior** point if $\exists U \in \mathcal{T}$ such that $x \in U \subseteq A$;

- $x \in X$ is an **adherent** point if $\forall U \in \mathcal{T}$ such that $x \in U$, then $U \cap A \neq \emptyset$;
- $x \in X$ is a **border** point if $\forall U \in \mathcal{T}$ such that $x \in U$, then $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$;
- $x \in X$ is an **accumulation** point if $\forall U \in \mathcal{T}$ such that $x \in U$, then $U \cap A \neq \emptyset$ and $U \cap A \neq \{x\}$.

We define the interior of A, A°, as the set of all interior points. In the same way we define the adherence[1], $\overline{A}$ or $Cl(A)$, the border, ∂A, and the set of accumulation points, A', of A.

The following result characterizes the aspect of the interior and the adherence of a given subset, and relates the four new concepts introduced in the previous definition.

Proposition 2.1.11. *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then, $A^\circ \subseteq A \subseteq Cl(A)$, and*

- *A° is the biggest open set contained in A, $A^\circ = \bigcup_{\substack{U \in \mathcal{T} \\ U \subseteq A}} U$;*
- *$Cl(A)$ is the smallest closed set that contains A, $Cl(A) = \bigcap_{\substack{D^c \in \mathcal{T} \\ A \subseteq D}} D$. Moreover, if $(X, \mathcal{T})$ is metrizable, with $\mathcal{T} = \mathcal{T}_d$ for some distance function d on X, we have that, for $x \in X$,*

$$x \in Cl(A) \iff \forall \varepsilon > 0,\ \exists a = a(\varepsilon) \in A \text{ such that } d(x, a) < \varepsilon.$$

Furthermore, ∂A is also a closed set, and one has that $Cl(A) = A^\circ \cup \partial A = A \cup A'$.

To conclude, let us define one last concept that will be key on the last chapter of this thesis.

Definition 2.1.12. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. We say that A is *dense* if $Cl(A) = X$.

Given a subset $Y \subseteq X$ of a topological space $(X, \mathcal{T})$, we want now to define a new topology on Y based on $\mathcal{T}$. This is what is introduced in the following definition:

Definition 2.1.13. Let $(X, \mathcal{T})$ be a topological space, and $Y \subseteq X$. Then, the collection of subsets $\mathcal{T}|_Y := \{U \cap Y : U \in \mathcal{T}\}$ is a topology on Y, and so $(Y,\ \mathcal{T}|_Y)$ is a topological space. We will refer to $\mathcal{T}|_Y$ as the *induced* topology on Y by X, or as the *restriction* to Y of $\mathcal{T}$.

Lemma 2.1.14. *If $Z \subseteq Y$ are both subsets of a topological space $(X, \mathcal{T})$, then $\mathcal{T}|_Z = (\mathcal{T}|_Y)|_Z$.*

Given a basis $\mathcal{B}$ of a topology $\mathcal{T}$ on X, we can easily obtain a basis for the induced topology on $Y \subseteq X$ with $\mathcal{B}|_Y := \{B \cap Y : B \in \mathcal{B}\}$.
Now, if $Y \subseteq X$ is an open set on the topology on X, we then have the following inclusion of topologies:

Proposition 2.1.15. *Let $(X, \mathcal{T})$ be a topological space, and let $Y \subseteq X$ such that $Y \in \mathcal{T}$. Then, $\mathcal{T}|_Y \subseteq \mathcal{T}$.*

Let us define a final concept involving the notion of induced topology:

[1] Or closure.

Definition 2.1.16. Let $(X, \mathcal{T})$ be a topological space, and $Y \subseteq X$. We denote by $\iota\colon Y \to X$ the natural inclusion, which is indeed continuous (using, of course, the induced topology on Y by X).

Moving on, given a collection of topological spaces $(X_i, \mathcal{T}_i),\ i \in I$, we can also define a topology for its product. The next definition formalizes this idea:

Definition 2.1.17. Let $(X_i, \mathcal{T}_i),\ i \in I$, be topological spaces. The *product topology* defined on $X := \prod_{i\in I} X_i,\ \mathcal{T}$, is the initial (least fine) topology that makes all projections $\pi_j\colon X \to X_j,\ (x_i)_{i\in I} \mapsto x_j$, continuous.

As one can see in the next proposition, the aspect of a basis of the product topology when I is a finite set is very simple.

Proposition 2.1.18. *Let $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$ be topological spaces with respective bases $\mathcal{B}_1, \dots, \mathcal{B}_n$. Consider $X := X_1 \times \cdots \times X_n$ and $\mathcal{T}$, the product topology on X. Then, a basis of $\mathcal{T}$ is given by the collection $\mathcal{B} := \{B_1 \times \cdots \times B_n : B_i \in \mathcal{B}_i,\ \forall i\}$*

Finally, and as we did in the case of subsets and products, we can also define a "natural" topology on a quotient.

Definition 2.1.19. Let $(X, \mathcal{T})$ be a topological space, and $\sim$ an equivalence relation on X. On the quotient $X/\sim$, we define the *quotient topology*, $\mathcal{T}/\sim$, as the final[2] topology that makes the canonical projection $\pi\colon X \twoheadrightarrow X/\sim,\ x \mapsto [x]$ continuous. That is, $\mathcal{T}/\sim = \{U \subseteq X/\sim\ : \pi^{-1}(U) \in \mathcal{T}\}$.

One should also have in mind that, to denote quotients of the form X/A, where A is a subset of X, we will use the convention $X/A := X/\sim_A$, where $x \sim_A y \iff x = y$ or $x, y \in A,\ \forall x, y \in X$, is a well–defined equivalence relation on X.

2.2 On group theory

This section is dedicated to introduce all the concepts from group theory that will be assumed throughout this manuscript. Even though the notions presented here are enough to continue, for further reading the reader is refered to [8] and [15].

Definition 2.2.1. A *group* is a pair[3] $(G, *)$, where G is a set and $*\colon G \times G \to G$ is a *binary operation* satisfying:

1. **Associativity:** $\forall g_1, g_2, g_3 \in G,\ (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$;
2. **Identity:** there exists $e \in G$ (the identity) such that $e * g = g * e = g,\ \forall g \in G$;
3. **Inverses:** $\forall g \in G$, there exists $g' \in G$ (the inverse of g) such that $g * g' = g' * g = e$.

Moreover, if $\forall g_1, g_2 \in G,\ g_1 * g_2 = g_2 * g_1$, we will say that the group is *abelian*.

Example 2.2.2. There are lots of natural examples of groups in all of mathematics: for instance, the set of integers with the usual addition is an (abelian) group. The sets of rational and real numbers are also abelian groups under addition, but under the usual multiplication too (excluding, of course, 0, which has no inverse). A simple example of non–abelian group is the set of invertible real $n \times n$ matrices under multiplication (excluding again the zero matrix for a correct definition).

[2] Most fine.

[3] To lighten notation, we will usually refer to simply G as the group.

As a convention, for non–abelian groups we will use the product notation: we will denote the operation by $\cdot$, even though many times we will omit it and denote by g_1g_2 the product $g_1 \cdot g_2 \in G$. Moreover, the identity element will be denoted by 1 and the inverse of $g \in G$ will be $g^{-1} \in G$. Similarly, for abelian groups we will use the additive notation: the operation will be denoted by $+$, the identity element by 0 and the inverse element of g by $-g$.

Definition 2.2.3. We define the *order* of a group $(G, \cdot)$ as the cardinal of G, and we denote it by $|G|$. We say that a group is finite if it has finite order, and infinite otherwise.

In the product notation presented before, we also define the n–th power of an element $g \in G$ as $g^n := \overbrace{g \cdots g}^{n)}$ (and $g^{-n} := (g^{-1})^n = (g^n)^{-1}$). Note that we can also extend this definition to the additive notation by $ng := \underbrace{g + \cdots + g}_{n)}$ (and $-ng := n(-g) = -(ng)$). This leads us to the following new concept:

Definition 2.2.4. Let $(G, \cdot)$ be a group. We define the *order* of $g \in G$, $|g|$, as the smallest positive integer n such that $g^n = 1$ (if such n does not exist, we then say that g has infinite order).

Let us now present a fundamental definition for constructing new groups from previous ones.

Definition 2.2.5. Let $(G, \cdot)$ be a group and $H \subseteq G$. We say that H is a *subgroup* of G, and denote it by $H \leqslant G$, if $(H, \cdot_H)$ is also a group, where $\cdot_H$ is the operation on G restricted to H. Even though the union of subgroups is not necessarily again a subgroup, it is easy to see that the arbitrary intersection of subgroups is also a subgroup.

Definition 2.2.6. Let $(G, \cdot)$ be a group, and $g \in G$. Then, we denote by $\langle g \rangle := \{g^n : n \in \mathbb{Z}\}$ the *subgroup generated by g*. Moreover, if $G = \langle g \rangle$, for some $g \in G$, we say that the group is *cyclic*. A known result states that, if a group is cyclic, then it must also be abelian.
If $S \subseteq G$, we also define the subgroup generated by S as $\langle S \rangle := \{s_1^{r_1} \cdots s_n^{r_n} : s_i \in S,\ r_i \in \mathbb{Z},\ \forall i,\ n \in \mathbb{N}\}$. If there exists $S \subseteq G$, with $|S| < +\infty$, such that $\langle S \rangle = G$, we then say that G is *finitely generated*. Otherwise, G is infinitely generated.

The first important result we present here is the so–called *Lagrange's* Theorem:

Theorem 2.2.7 (*Lagrange's* Theorem)**.** *Let G be a finite group and $H \leqslant G$ a subgroup. Then, $|H|$ divides $|G|$.*

We can now introduce the concept of *coset*.

Definition 2.2.8. Let G be a group, $H \leqslant G$ and $g \in G$. Then, the *left coset* of g with respect to H is the set $gH := \{gh : h \in H\}$ (in additive notation, $g+H := \{g+h : h \in H\}$). In a similar way, $Hg := \{hg : h \in H\}$ is the *right coset* of g with respect to H. As it can be easily seen that there is the same number of left and right cosets, we define the *index* of H in G, $[G : H]$, as the number of cosets of H in G.
A direct corollary of *Lagrange's* Theorem yields that, if G is a finite group, then $[G : H] = |G|/|H|$.

Our goal is now to define quotient groups. To be able to do so, we first have to consider a special type of subgroups, which are defined next.

Definition 2.2.9. Let G be a group and $N \leqslant G$ a subgroup. We say that N is a *normal* subgroup of G, and denote it by $N \trianglelefteqslant G$, if $gN = Ng$, $\forall g \in G$. Moreover, a group with no normal subgroups except for the trivial ones (itself and $\{1\}$) is called *simple*.

Definition 2.2.10. Let G be a group and $N \trianglelefteqslant G$ a normal subgroup. We define the *quotient group* of G by N as the set $G/N := \{gN : g \in G\}$ under the operation $(gN)(g'N) := gg'N$. We will normally use the notation $[g] := gN$, and refer to $[g]$ as the *class* of g (modulo N). From Definition 2.2.8 one gets that $|G/N| = |G|/|N|$.

It can be seen that to assure that the quotient group is well–defined, the normality of the subgroup N is required.

Now, we want to talk about functions between groups. If we want them to have useful properties, we have to ask them to respect groups operations. This is why we introduce the so–called *group homomorphisms*.

Definition 2.2.11. Let $(G,\ \cdot)$, $(G',\ *)$ be two arbitrary groups. Then a *group homomorphism* between G and G' is a function $\varphi\colon G \to G'$ such that

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) * \varphi(g_2),\ \forall g_1, g_2 \in G.$$

If the homomorphism is a bijective function, we also say that it is an *isomorphism*. If there exists an ismomorphism between two groups G and G', we say that they are isomorphic, and we denote it by $G \cong G'$.

One can think of isomorphic groups as essentially the same, but with different "labels" of their elements. The isomorphism that exists between them tells us the relation between the "labels" they have. Moving on, the following definition will arise commonly in the whole thesis:

Definition 2.2.12. We say that an homomorphism $\varphi\colon G \to G'$ *separates* two elements g and g' of G if $\varphi(g) \neq \varphi(g')$. We also say that G' separates $g, g' \in G$ if there exists an homomorphism $\varphi\colon G \to G'$ that separates g and g'.

The *kernel* of an homomorphism $\varphi\colon G \to G'$ is defined as one could expect, $ker(\varphi) := \{g \in G : \varphi(g) = 1_{G'}\}$, and it is easy to check that it is a normal subgroup of G. Similarly, we define the *image* of φ as $Im(\varphi) := \{\varphi(g) : g \in G\} \leqslant G'$. We are now ready to establish the famous Isomorphism Theorems:

Theorem 2.2.13 (First Theorem of Isomorphism)**.** *Let $\varphi\colon G \to G'$ be a group homomorphism. Then,*

$$G/ker(\varphi) \cong Im(\varphi).$$

Theorem 2.2.14 (Second Theorem of Isomorphism)**.** *Let G be a group, $H \leqslant G$ and $N \trianglelefteqslant G$. Consider $HN := \{hn : h \in H,\ n \in N\} \leqslant G$. Then,*

$$H/(H \cap N) \cong HN/N.$$

Theorem 2.2.15 (Third Theorem of Isomorphism)**.** *Let G be a group, with $H \trianglelefteqslant G$, $N \trianglelefteqslant G$ and $H \subseteq N$. Then,*

$$(G/H) \Big/ (N/H) \cong G/N.$$

Moving on, given some groups $G_1, \ldots, G_n$, we can construct a new group using the tool of the direct product.

Definition 2.2.16. Let $(G_1,\ \cdot_1), \dots, (G_n,\ \cdot_n)$ be groups. Then, $(G_1 \times \cdots \times G_n,\ \cdot)$ is also a well–defined group, where

$$(g_1, \dots, g_n) \cdot (g_1', \dots, g_n') \coloneqq (g_1 \cdot_1 g_1', \dots, g_n \cdot_n g_n'),$$

$\forall (g_1, \dots g_n), (g_1', \dots, g_n') \in G_1 \times \cdots \times G_n$.

A powerful result that we will be using is the known *Cauchy's* Theorem, which is presented below:

Theorem 2.2.17 (*Cauchy's* Theorem)**.** *Let G be a finite group, and $p \in \mathbb{Z}^+$ a prime number such that $p \mid |G|$. Then, G has an element of order p.*

Finally, let us take a look at the main results presented by *Sylow* in the late 1800s. Firstly, an easy definition:

Definition 2.2.18. Let G be a finite group, with $|G| = p^n r$, where $p \in \mathbb{Z}^+$ is a prime number, $r \in \mathbb{Z}^+$ is a positive integer such that $p \nmid r$ and $n \in \mathbb{Z}^+$. Then, a *Sylow* $p-$subgroup of G is any subgroup $H \leqslant G$ of order p^n.

We can now state the famous and powerful *Sylow* Theorem:

Theorem 2.2.19 (*Sylow* Theorem)**.** *Let G be a finite group, with $|G| = p^n r$, and $p \nmid r$ as defined above. Then,*

1. *G has at least one Sylow $p-$subgroup;*
2. *If H and K are two Sylow $p-$subgroups of G, then there exists $g \in G$ such that $g^{-1}Hg = K$. That is, H is conjugate to K;*
3. *If n_p is the number of Sylow $p-$subgroups of G, then $n_p \mid r$ and $n_p \equiv 1 \mod p$.*

2.3 The particular case of free groups

We present here an introduction to the theory of free groups and its algebraic construction, as well as its relation with *Stallings* automata, that will be used in Chapter 4 of this thesis. Even though all concepts presented in the previous section also apply here, due to the importance that free groups and *Stallings* automata have on this thesis we reserved a section for its presentation and correct understandability. The main references used here are [2] and [3].

Definition 2.3.1. Let $(G,\ *)$ be an arbitrary group. We say that $A \subseteq G$ is *free* on G if different reduced products on $A^{\pm} \coloneqq A \cup A^{-1}$ yield different elements of G.[4] That is, $a_1 * \cdots * a_n = a_1' * \cdots * a_m' \Rightarrow n = m,\ (a_1, \dots, a_n) = (a_1', \dots a_m'),\ \forall a_1, \dots, a_n, a_1', \dots, a_m' \in A^{\pm}$. We also say that A *generates* G if every element $g \in G$ can be written as a product of elements of $A^{\pm}$.
Finally, we say that A is a *basis* of G if A is free on G and it generates G. In this case, we also say that G is a *free group* over A.

Example 2.3.2. It is easy to see that $\mathbb{Z}$ is a free group, with basis $\{1\}$ or $\{-1\}$. The trivial group is also free with basis $\emptyset$, and these are the only abelian free groups.
One can also see that, apart from the trivial group, there are no finite free groups.

[4]Reduced products are those that do not contain consecutive inverse elements.

We now present the main parts of the construction of a free group with a basis of arbitrary cardinal.

Definition 2.3.3. We say that a set of elementary symbols A is an *alphabet*. A *word* over A is a finite, ordered sequence of symbols in A, $w = a_1 \cdots a_n$, with length $|w| = n,\ n \geq 0$. We also denote by 1 the (only) empty word, with length $|1| = 0$, and we will refer to A^* as the set of all words over the alphabet A.

Note that there is a natural operation we can introduce in A^*: given two words $w = a_1 \cdots a_n,\ v = b_1 \cdots b_m \in A^*$, we define the *concatenation* of w and v as

$$w \cdot v = wv = a_1 \cdots a_n b_1 \cdots b_m.$$

Clearly, concatenation is associative but it is not a commutative operation. Moreover, $1w = w1 = w,\ \forall w \in A^*$, and $|wv| = |w| + |v|,\ \forall w, v \in A^*$. Despite this, no word except for the trivial one has an inverse. So, unfortunately, we can not say that $(A^*,\ \cdot)$ is a group.
We can try to fix this: given an alphabet A we consider $A^{\pm} \coloneqq A \cup A^{-1}$, where $A^{-1} \coloneqq \{a^{-1} : a \in A\}$ is just a set of formal inverses of the symbols in A (observe that, in that way, if A is finite, then $A^{\pm}$ has always even cardinality). As before, we denote by $(A^{\pm})^*$ the set of all words over $A^{\pm}$, and we say that reduced words over $A^{\pm}$ are those that do not contain consecutive inverse elements. Consider now the following equivalence relation in $(A^{\pm})^*$

$$w \sim v \iff \begin{cases} w = v \text{ or} \\ \exists w_1, \ldots, w_n \text{ such that } w \leftrightsquigarrow w_1 \leftrightsquigarrow \cdots \leftrightsquigarrow w_n \leftrightsquigarrow v, \end{cases}$$

where $w \leftrightsquigarrow v$ means that w and v can be elementally transformed: that is, if $w = w_1 a a^{-1} w_2$ and $v = w_1 w_2$ (or vice-versa), or if $w = w_1 a^{-1} a w_2$ and $v = w_1 w_2$ (or vice-versa), with $w_1, w_2 \in (A^{\pm})^*, a, a^{-1} \in A^{\pm}$.
Now, consider the quotient of $(A^{\pm})^*$ by $\sim$ alongside with the natural well–defined operation

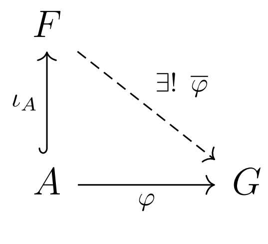
$$[w] \cdot_{\sim} [v] \coloneqq [w \cdot v].$$

The elements of $(A^{\pm})^*/\sim$ can be thought as reduced words over $(A^{\pm})^*$, as it can be seen that each equivalence class $[w]$ only contains a reduced word, which we will denote by $\overline{w}$. Now, $((A^{\pm})^*/\sim,\ \cdot_{\sim})$ forms a group, which is reflected on the following theorem (a detailed proof can be found in [2]).

Theorem 2.3.4. *Let A be an arbitrary alphabet. Then $\mathbb{F}_A \coloneqq (A^{\pm})^*/\sim$, alongside with the operation $\cdot_{\sim}$ is a group. Moreover, $\mathbb{F}_A$ is a free group over A.*

In this section we have introduced free groups within an algebraic approach. Despite this, one of the more known and useful characterizations of free groups is based on its categorical definition. This is reflected on the following proposition:

Proposition 2.3.5. *Let F be a group and $A \subseteq F$. We denote by $\iota_A \colon A \to F$ the natural inclusion. Then, F is a free group over A if and only if for every group G and every map $\varphi \colon A \to G$, there exists an unique group homomorphism $\overline{\varphi} \colon F \to G$ such that $\overline{\varphi} \circ \iota_A = \varphi$.*



With this characterization, that sometimes directly works as the definition of a free group, one can prove the following and intuitive result about when two free groups are isomorphic.

Theorem 2.3.6. *Let $\mathbb{F}$ be a free group over A, and let $\mathbb{F}'$ be another free group over A'. Then, $\mathbb{F} \cong \mathbb{F}'$ if and only if $|A| = |A'|$, where $|A|$ denotes the cardinal of A.*

Another important concept to take into account here is the *rank* of a free group $\mathbb{F}$, denoted $rk(\mathbb{F})$, which is defined as the minimum cardinality of a set of generators of $\mathbb{F}$. It is easy to see that, if A is a basis of $\mathbb{F}$, then $rk(\mathbb{F}) = |A|$. Also, and from now on, we will write $\mathbb{F}_k$ to refer the free group of rank k, without specifying any basis of it.

One of the main reasons to study free groups lies on the following result, which can be interpreted as this family of groups contain all the "information" about every possible group.

Theorem 2.3.7. *For every group G, there exist a cardinal k and a normal subgroup $N \trianglelefteq \mathbb{F}_k$ such that $G \cong \mathbb{F}_k/N$.*

One directly observes from this last theorem that if we were capable to study and fully understand the family of (normal) subgroups of the free group, then we would be able to understand the essence of every single group. However, this is not an easy problem because subgroups of $\mathbb{F}_A$ do not behave as one could generally expect (for instance, there are infinitely generated subgroups of a finitely generated free group!). Despite that, one tool that helps in many cases with this issue is *Stallings automata* theory, which is presented below.

2.3.1 An introduction to Stallings automata

The aim of this section is to construct an algorithmic approach to compute generators of subgroups of the free group, based on *Stallings* automata theory. We begin the section with some basic definitions that merge graph theory and free groups.

Definition 2.3.8. Let A be an alphabet. An A–*automaton* Γ (or simply automaton when A is understood) is a tuple $\Gamma = (V,\ E,\ o,\ f,\ l,\ \odot)$, where V and E are the sets of vertices and (directed) edges, respectively, $o, f \colon E \to V$ are maps designating the origin and end of each $e \in E$, $l \colon E \to A$ is the "label" of each edge and $\odot \in V$ is the so-called base vertex. Basically, we can think of an automaton as a directed graph with A-labeled edges and a distinguished vertex $\odot$ (see Figure 2.1).

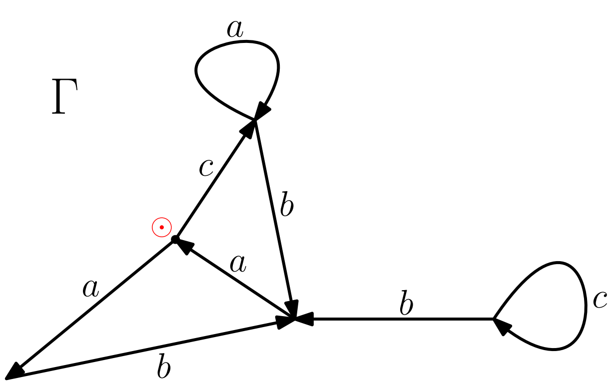


Figure 2.1: Example of an $\{a, b, c\}$–automaton Γ.

Definition 2.3.9. A (directed) path of an automaton Γ is a finite sequence $\gamma = p_0 e_1 p_1 \cdots p_{k-1} e_k p_k$ such that $p_i \in V,\ e_i \in E,\ o(e_i) = p_{i-1},\ f(e_i) = p_i,\ \forall i$. We call p_0 and p_k the origin and end of γ, respectively, and we write $\gamma\colon p_0 \rightsquigarrow p_k$. If $p_0 = p_k$, we say that the path is closed and, finally, we denote by $|\gamma| = k$ the length of the path (note that trivial paths, of length 0, correspond to just vertices).

Even though we are studying directed graphs here, consider now the following process: for each edge $e \in E$, with $o(e) = p,\ f(e) = p',\ l(e) = a$, we are going to duplicate it and construct an "inverse" edge e' with $o(e') = p',\ f(e') = p,\ l(e') = a^{-1} \in A^{\pm}$. We will call the resulting graph the *involutive* automaton of Γ (see Figure 2.2).

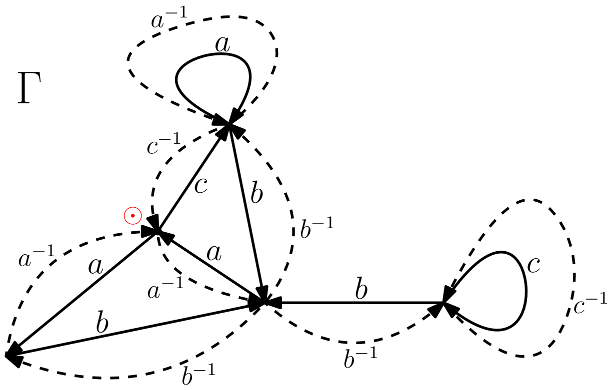


Figure 2.2: Involutive automaton of the example in Figure 2.1.

Note also that if we identify each pair $e, e' \in E$ (on a undirected edge), we recover a standard undirected graph, the so–called *subjacent graph* of Γ. Even though from now on we will always work with the involutive automaton of Γ and assign properties of undirected graphs to Γ based on its subjacent version (notion of connectedness, degree of a vertex...), we will always represent Γ as in Definition 2.3.8 and Figure 2.1 with the convention that we can always go through an edge $e = p \xrightarrow{a} q$ in the inverse way (then, with label a^{-1}).

Definition 2.3.10. We define the label of a path $\gamma = p_0 e_1 p_1 \cdots p_{k-1} e_k p_k$ as the word $l(\gamma) := l(e_1) \cdots l(e_k) \in (A^{\pm})^*$ (remember we consider the involutive automaton). Note also that $\overline{l(\gamma)} \in \mathbb{F}_A$. We also say that the path is reduced if it does not contain two consecutive inverse edges, $e_i, e_{i+1} \in \gamma$ having $l(e_i) = l(e_{i+1})^{-1}$.

We can now start to link this new concept of automaton with free groups. The first step is the next proposition.

Proposition 2.3.11. *Let Γ be an (involutive) connected A–automaton, and $p \in V$. Then,*

$$\langle \Gamma \rangle_p := \{\overline{l(\gamma)} : \gamma\colon p \rightsquigarrow p\},$$

is a subgroup of $\mathbb{F}_A$. We will also denote $\langle \Gamma \rangle_\odot$ as just $\langle \Gamma \rangle$.

We note now that every subgroup $H \leqslant \mathbb{F}_A$ can be represented by an A–automaton. Indeed, consider a family of generators $S = \{w_i\}_{i \in I} \subseteq \mathbb{F}_A$ for H (we can always take $S = H$), with $w_i = a_{i,1} \cdots a_{i,l_i},\ a_{i,j} \in A^{\pm},\ \forall i, j$, and construct Γ as follows (see Figure 2.3):

1. Create the base vertex $\odot \in V$.

2. For every $i \in I$, create $|w_i| - 1 = l_i - 1$ new vertices, $\{v^i_1, \dots, v^i_{l_i-1}\}$, and $|w_i| = l_i$ new edges, $\{e^i_1, \dots, e^i_{l_i}\}$, as follows:
Set $o(e^i_1) = f(e^i_{l_i}) = \odot$, $o(e^i_j) = v^i_{j-1}$, for $j = 2, \dots, l_i$, $f(e^i_k) = v^i_k$, for $k = 1, \dots, l_i - 1$. Also set $l(e^i_j) = a_{i,j}$, for $j = 1, \dots, l_i$. We will refer to Γ as $Fl((w_i)_{i\in I})$ (name comes from flower, see again Figure 2.3).

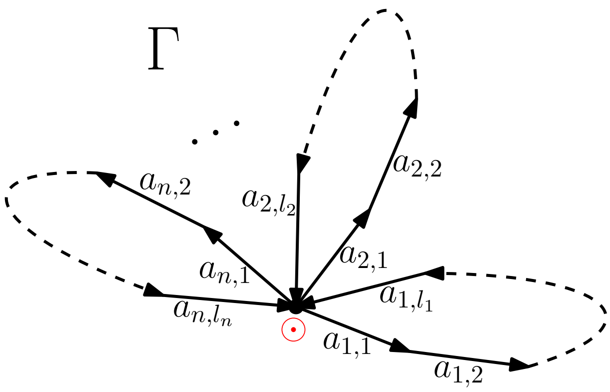


Figure 2.3: Flower automaton $\Gamma = Fl(w_1, \dots, w_n)$ that represent $H = \langle w_i \rangle_{i=1}^n$.

Note that, if $H = \langle w_i \rangle_{i\in I} \leqslant \mathbb{F}_A$ is finitely generated, i.e. $|I| < +\infty$, the computation of $Fl((w)_{i\in I})$ can be done in finite time.
It is clear that with this construction $\langle \Gamma \rangle = H$, as every reduced label of a closed path starting and ending at $\odot$ must be a combination of the generators $\{w_i\}_{i\in I}$ of H (or inverses of them). Even though, different generators of the same subgroup of $\mathbb{F}_A$ yield different automata, and hence we do not have a bijection between the family of subgroups of $\mathbb{F}_A$ and the set of A–automata (modulo isomorphism). Despite this, we can try to fix this problem restricting the form of the automaton we assign to each subgroup of $\mathbb{F}_A$.

Definition 2.3.12. An automaton Γ is called *deterministic* if it does not have two edges with the same label, origin. Note that, in the case of involutive automata, the definition is equivalent to not having two same-labeled edges with same origin or endpoint.

Definition 2.3.13. We say that an automaton Γ is *core* with respect to $p \in V$ (or simply $p - core$) if every vertex of Γ appears on some closed and reduced path starting (and ending) in p. Note that being $p-$core directly implies being connected.

Definition 2.3.14. An automaton Γ is said to be *reduced* if it is deterministic and $\odot - core$.

With all these new concepts, the following fundamental proposition can be proved, which is key to establish the desired bijection between subgroups of $\mathbb{F}_A$ and (reduced) A–automata.

Proposition 2.3.15. *Let Γ, Γ' be two reduced A–automata. Then,*

$$\langle \Gamma \rangle = \langle \Gamma' \rangle \iff \Gamma \cong \Gamma'.$$

Given a subgroup $H \leqslant \mathbb{F}_A$, we will call the only reduced automaton Γ satisfying $H = \langle \Gamma \rangle$ (modulo isomorphism) the Stallings automaton of H, $\Gamma = St(H)$.

With a careful construction that can be found again in [2], it can be seen that, given a subgroup $H \leqslant \mathbb{F}_A$, the *Stallings* automaton of H always exists. Then, the next theorem follows:

Theorem 2.3.16 (J.R. Stallings [16]). *There exists a bijection between the sets*

$$\{\textit{Stallings } A\text{–}\textit{automata (modulo isomorphism)}\} \longleftrightarrow \{\textit{subgroups of } \mathbb{F}_A\}$$

But, now, why would we use this bijection? For example, one of the main points of using the corresponding *Stallings* automaton of a subgroup $H \leqslant \mathbb{F}_A$ is to effectively compute a basis for H, as presented in the following proposition.

Proposition 2.3.17. *Let Γ be an involutive and connected A–automaton, and let $T = (V_T, E_T)$ be a spanning tree of Γ. Consider*

$$S_T \coloneqq \{w_e \in \mathbb{F}_A : e \in E \backslash E_T\},$$

where $w_e = \overline{l(\odot e_1 p_1 \cdots e \cdots p_{k-1} e_k \odot)}$, with $e_i \in T,\ \forall i$. Then,

1. *S_T is a set of generators of $\langle\Gamma\rangle$;*
2. *If Γ is deterministic, then $\langle\Gamma\rangle$ is a free group with basis S_T;*
3. *If Γ is reduced, then $\langle\Gamma\rangle$ is finitely generated if and only if Γ is finite.*

Note that, in addition, from this last proposition and Theorem 2.3.16 we can easily get the classical result stating that every subgroup of a free group is also free.

Note also that, in the case of a finite automaton Γ, this result gives us a way to effectively compute generators of the subgroup $\langle\Gamma\rangle \leqslant \mathbb{F}_A$, as there are known algorithms to compute a spanning tree of a connected graph and the computation of S_T is also finite (since $|E\backslash E_T| \leq |E| < +\infty$).
In a reverse way, we can also give the steps to construct the *Stallings* automaton Γ from a subgroup $H = \langle w_i \rangle_{i=1}^n \leqslant \mathbb{F}_A$. Indeed, consider the following process (called folding):

1. Construct $Fl(w_1, \ldots, w_n)$.
2. Iteratively identify edges that are equally-labeled and have same origin or endpoint (see Figure 2.4).

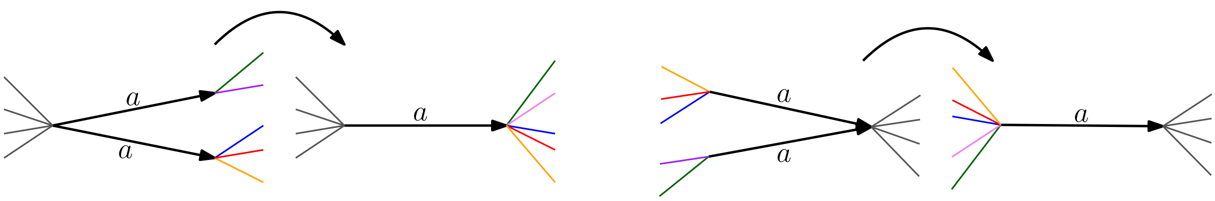


Figure 2.4: Identification of equally-labeled edges with same origin/endpoint.

One can easily check that this process can be done in finite time and that it indeed returns the *Stallings* automaton of H (proof in [2]). The figure below shows the process with a simple example, where $H = \langle aba,\ ba,\ b^{-1}ab\rangle$.

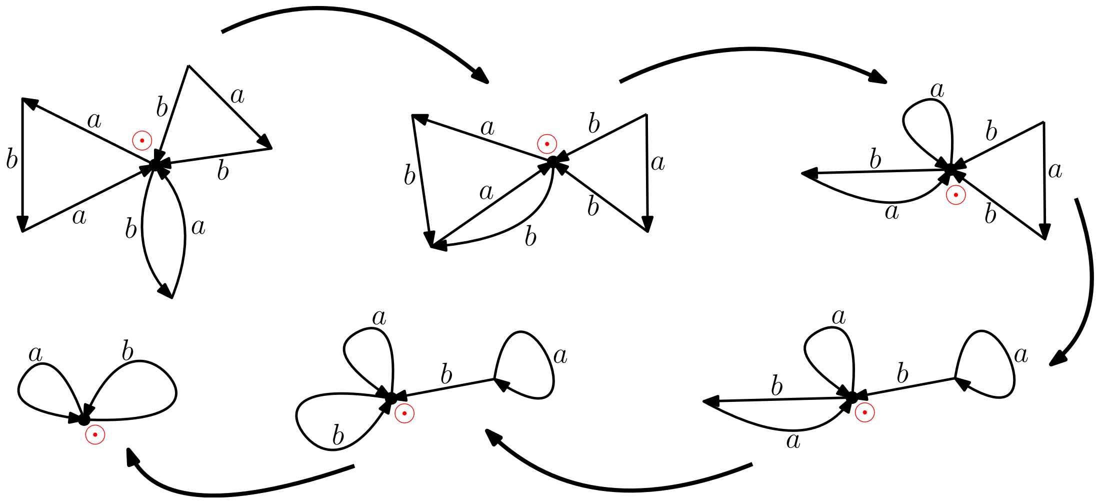


Figure 2.5: Folding of $Fl(aba,\ ba,\ b^{-1}ab)$. Note that $\langle aba,\ ba,\ b^{-1}ab\rangle = \mathbb{F}_{\{a,b\}}$. Trivially, a set of generators here is given by $\{a, b\}$.

As happened with groups, not all kind of functions between *Stallings* automata are useful. That is why we define the concept of automaton morphism:

Definition 2.3.18. Let Γ, Γ' be two *Stallings* A–automata. An *automaton morphism* between Γ and Γ' is a function $\varphi\colon V(\Gamma) \to V(\Gamma')$ such that

$$\begin{cases} \varphi(\odot) = \odot'; \\ \text{If } p \xrightarrow{a} q, \text{ then } \varphi(p) \xrightarrow{a} \varphi(q) \ (\varphi \text{ preserves labelled edges}). \end{cases}$$

An interesting property of automaton morphisms we will be using on Chapter 4 is presented in the following result.

Proposition 2.3.19. *Let H, H' be subgroups of the free group $\mathbb{F}_A$. Then, H is contained in H' if and only if there exists an automaton morphism from $St(H)$ to $St(H')$. That morphism, if it exists, is unique.*

As usual, a complete proof of the statement can be found in [2]. Note only that the uniqueness is easily derived from the fact that automaton morphisms map the base point to the base point and that *Stallings* automata are deterministic.

2.4 General notions

For the sake of completion, we present here some useful and known mixed results that we will be using throughout this thesis, as well as some conventions for the notation.

Fact 2.4.1. *Let $f\colon X \to Y$ be a function, $A \subseteq X$ and $B \subseteq Y$. Then,*

- $f^{-1}(f(A)) \supseteq A$*, and* $f^{-1}(f(A)) = A$ *if* f *is injective;*
- $f(f^{-1}(B)) \subseteq B$*, and* $f(f^{-1}(B)) = B$ *if* f *is exhaustive.*

Also, and to lighten the notation, in the case of having a function $f\colon X \to Y$ and $y \in Y$, we will denote by $f^{-1}(y)$ the preimage of the set $\{y\}$ under f. That is, $f^{-1}(y) := f^{-1}(\{y\}) = \{x \in X : f(x) = y\}$.

Definition 2.4.2. Let (X,d), (Y,d') be metric spaces and $k \in \mathbb{R}^+$. A function $f\colon X \to Y$ is called $k-$*Lipschitz* if, $\forall x, x' \in X$,

$$d'(f(x), f(x')) \leq k \cdot d(x, x').$$

In this thesis, the case $k = 1$ corresponds to contractive functions. Moreover, it is widely known that *Lipschitz* functions (regardless of the constant k) are uniformly continuous and, hence, continuous.

Definition 2.4.3. Let d, d' be two distance functions over a set X. We say that d and d' are *strongly equivalent* if, $\forall x, y \in X$, there exist positive constants $c_1, c_2 \in \mathbb{R}^+$ such that

$$c_1 \cdot d(x,y) \leq d'(x,y) \leq c_2 \cdot d(x,y).$$

Note that the relation is symmetric as $\frac{1}{c_1}, \frac{1}{c_2} \in \mathbb{R}^+$, and $\frac{1}{c_2} \cdot d'(x,y) \leq d(x,y) \leq \frac{1}{c_1} \cdot d'(x,y)$, $\forall x, y \in X$. It is easy to see that, if two metrics are strongly equivalent, they are also equivalent (see Definition 2.1.3) and, hence, they yield the same topology over X, $\mathcal{T}_d = \mathcal{T}_{d'}$.

Chapter 3

Properties of the *Pro*–$\mathcal{V}$ topology

This chapter is dedicated to define and study the general properties of the *Pro*–$\mathcal{V}$ topology on an arbitrary group G. After proving some easy results and characterizing the topology using a pseudodistance, we will analyze the *Pro*–$\mathcal{V}$ topology on finite direct product groups, study the topological properties of subgroups of G and, finally, we will focus on the case of topologies of subgroups and quotient groups, where we will relate the previously presented concepts of induced and quotient topology.

3.1 Definition of the *Pro*–$\mathcal{V}$ topology on a group

We begin the section by giving some fundamental definitions that will appear throughout all the thesis..

Definition 3.1.1. Let $\mathcal{V} = \{V_i\}_{i\in I}$ be a family of finite groups. We say that $\mathcal{V}$ is a *pseudovariety of finite groups* if the following properties are satisfied:

1. If $V \in \mathcal{V}$ and V' is a subgroup of V, then $V' \in \mathcal{V}$;
2. If $V_1, V_2 \in \mathcal{V}$, then $V_1 \times V_2 \in \mathcal{V}$;
3. If $V \in \mathcal{V}$ and V' is a normal subgroup of V, then $V/V' \in \mathcal{V}$.

Note that an alternative way of defining pseudovariety is by replacing property 3 by requiring $\mathcal{V}$ to be closed under homomorphic images, but it is easy to see that both definitions are equivalent, via the first Isomorphism Theorem.

We will always consider here pseudovarieties of finite groups, calling them just pseudovarieties by simplicity. Important examples are $\mathcal{V}_f$, the pseudovariety of all finite groups, $\mathcal{V}_{Ab}$, the pseudovariety of all finite abelian groups or $\mathcal{V}_p$, the pseudovariety of all finite p-groups (where p is a prime number). More specific examples we will also go through in this thesis include $\mathcal{V}_{Nil}$, the pseudovariety of all finite nilpotent groups and $\mathcal{V}_{Sol}$, the pseudovariety of all finite solvable groups, among many others.

With this new concept, and with usual terms from group theory and general topology presented in the preliminaries, the main definition of this section is presented next:

Definition 3.1.2. Let G be an arbitrary group. The *Pro*–$\mathcal{V}$ topology on G, $\mathcal{T}_{\mathcal{V}}^{G}$, is defined as the initial topology that makes all homomorphisms $\varphi\colon G \to V$ continuous, with $V \in \mathcal{V}$, and considering all elements of $\mathcal{V}$ equipped with the discrete topology. That is, $\mathcal{T}_{\mathcal{V}}^{G} = \langle \varphi^{-1}(v) : v \in V \in \mathcal{V},\ \varphi\colon G \to V \text{ homomorphism}\rangle$[1]. As G will usually be understood, we will normally denote the *Pro*–$\mathcal{V}$ topology on G as just $\mathcal{T}_{\mathcal{V}}$.

[1]Definition 2.1.7.

The first result we can establish directly from the definition of *Pro–$\mathcal{V}$* topology is the characterization of a basis[2] of $\mathcal{T}_{\mathcal{V}}$, which is reflected in the following proposition:

Proposition 3.1.3. $\mathcal{B}_{\mathcal{V}} = \{\varphi^{-1}(v) : v \in V \in \mathcal{V},\ \varphi\colon G \to V \text{ homomorphism}\}$ *is a basis of* $\mathcal{T}_{\mathcal{V}}$.

Proof. Given $V, W \in \mathcal{V}$, $\varphi\colon G \to V,\ \gamma\colon G \to W$ homomorphisms and $v \in V,\ w \in W$, it is sufficient to see that the intersection $\varphi^{-1}(v) \cap \gamma^{-1}(w)$ is in $\mathcal{B}_{\mathcal{V}}$. Indeed,

$$\varphi^{-1}(v) \cap \gamma^{-1}(w) = \delta^{-1}(v, w) \in \mathcal{B}_{\mathcal{V}},$$

where $\delta\colon G \to V \times W \in \mathcal{V},\ g \mapsto (\varphi(g), \gamma(g))$ is clearly an homomorphism to a group in $\mathcal{V}$. □

Also, and as one could expect, if $\mathcal{V}_1, \mathcal{V}_2$ are pseudovarieties of finite groups, with $\mathcal{V}_1 \subseteq \mathcal{V}_2$, then the *Pro–$\mathcal{V}_1$* topology on a group G is contained on the *Pro–$\mathcal{V}_2$* topology on the same group G. More formally, that is:

Proposition 3.1.4. *Let G be an arbitrary group, and let $\mathcal{V}_1, \mathcal{V}_2$ be pseudovarieties of finite groups such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$. Then, $\mathcal{T}_{\mathcal{V}_2}$ is finer[3] than $\mathcal{T}_{\mathcal{V}_1}$.*

Proof. Note that it is sufficient to see that $\mathcal{B}_{\mathcal{V}_1} \subseteq \mathcal{B}_{\mathcal{V}_2}$, and this is direct since every homomorphism $\varphi\colon G \to V \in \mathcal{V}_1$ is also an homomorphism to a group in $\mathcal{V}_2$. □

A simple example where the inclusion of topologies is strict can easily be found using the trivial pseudovariety $\mathcal{V}_1 = \mathcal{V}_{\{1\}} = \{1\}$ (which always yields the trivial topology) and a non–trivial pseudovariety $\mathcal{V}_2$, along with any group G that admits a non trivial homomorphism $\varphi\colon G \to V \in \mathcal{V}_2$. Then, $\mathcal{T}^G_{\mathcal{V}_2}$ is non–trivial and $\mathcal{T}^G_{\mathcal{V}_{\{1\}}} \subsetneq \mathcal{T}^G_{\mathcal{V}_2}$.

A question that can now arise is to ask oneself whether the *Pro–$\mathcal{V}$* topology on an arbitrary group G is Hausdorff or not (in the sense of Definition 2.1.4). The answer lies on the following definition and on the succeeding result.

Definition 3.1.5. Let $\mathcal{V}$ be a pseudovariety of finite groups, and let G be a group. We say that G is *residually–$\mathcal{V}$* if, for every $g \in G,\ g \neq 1$, there exists $V \in \mathcal{V}$ and an homomorphism $\varphi\colon G \to V$ such that $\varphi(g) \neq 1$.
Note that, directly from the definition, we have that if a group G is residually–$\mathcal{V}$, then every different pair of elements $g, g' \in G$ can be separated (Definition 2.2.12): indeed, $gg'^{-1} \neq 1$, so there exists $V \in \mathcal{V}$ and an homomorphism $\varphi\colon G \to V$ such that $\varphi(gg'^{-1}) \neq 1$, which directly implies that $\varphi(g) \neq \varphi(g')$, which is what we wanted to see. We will use this in the next proposition.

Proposition 3.1.6. *Let $\mathcal{V}$ be a pseudovariety of finite groups, and let G be a group. Then, G is residually–$\mathcal{V}$ if and only if the Pro–$\mathcal{V}$ topology on G is Hausdorff.*

Proof. Firstly, suppose that G is residually–$\mathcal{V}$, and let $g, g' \in G$ be any to elements of the group. Consider a homomorphism $\overline{\varphi}$ that separates them, and now define $v := \overline{\varphi}(g),\ v' := \overline{\varphi}(g')$. Then, we directly have that $\overline{\varphi}^{-1}(v),\ \overline{\varphi}^{-1}(v')$ are open neighborhoods that respectively contain g, g' by definition, and it holds too that $\overline{\varphi}^{-1}(v) \cap \overline{\varphi}^{-1}(v') = \emptyset$, also directly from the fact that $v \neq v'$, so $\mathcal{T}_{\mathcal{V}}$ is Hausdorff.
Reversely, suppose now that $\mathcal{T}_{\mathcal{V}}$ is Hausdorff, and let $g \in G,\ g \neq e$. Then, there exist open neighborhoods $U, V \in \mathcal{T}_{\mathcal{V}}$ of $g, e \in G$, respectively, such that $U \cap V = \emptyset$. Also,

[2]Definition 2.1.6.
[3]Definition 2.1.5

as $\mathcal{B}_{\mathcal{V}}$ is a basis of $\mathcal{T}_{\mathcal{V}}$, there exist homomorphisms $\varphi_1\colon G \to V_1 \in \mathcal{V}$, $\varphi_2\colon G \to V_2 \in \mathcal{V}$ and elements $v_1 \in V_1$, $v_2 \in V_2$ such that $g \in \varphi_1^{-1}(v_1) \subseteq U$, $e \in \varphi_2^{-1}(v_2) \subseteq V$. Finally, as $\varphi_1^{-1}(v_1) \cap \varphi_2^{-1}(v_2) = \emptyset$, we have that $g \notin \varphi_2^{-1}(v_2) = \varphi_2^{-1}(e)$, so $\varphi_2(g) \neq e$, and G is residually$-\mathcal{V}$. □

Now, let $\mathcal{V}$ be a pseudovariety, and consider the *pseudodistance*

$$\begin{aligned} d_{\mathcal{V}}^{G}\colon G \times G &\longrightarrow \mathbb{R} \\ (g, g') &\longmapsto 2^{-\rho(g,g')}, \end{aligned}$$

where $\rho(g, g') \coloneqq \min\{|V| : V \in \mathcal{V},\ V \text{ separates } g \text{ and } g'\}$ (if such homomorphism does not exist, we let $\rho(g, g') = +\infty$ and $d_{\mathcal{V}}^{G}(g, g') = 0$). It is clear that $d_{\mathcal{V}}^{G} \geq 0$, $d_{\mathcal{V}}^{G}(g, g') = d_{\mathcal{V}}^{G}(g', g)$, $\forall g, g'$, even though $d_{\mathcal{V}}^{G}(g, g') = 0 \not\Rightarrow g = g'$ in general (see Example 3.1.7). If G is residually$-\mathcal{V}$, then this last implication is true and $d_{\mathcal{V}}^{G}$ is in fact a distance (the verification of the triangle inequality is checked on Lemma 3.1.8 and below). From now on, when G and $\mathcal{V}$ are understood, we will denote $d_{\mathcal{V}}^{G}$ as just d and usually refer to it as a normal metric or distance. This pseudometric will be crucial in the following sections, as we will see that it induces the *Pro*$-\mathcal{V}$ topology on G, i.e. $\mathcal{T}_{\mathcal{V}} = \mathcal{T}_d$.

Example 3.1.7. Consider the trivial pseudovariety, $\mathcal{V}_{\{1\}}$. That is, the pseudovariety formed by the group of just one element. It is clear that for every group G and every pair of elements $g, g' \in G$, one has $d(g, g') = 0$, as the only homomorphisms from G to the only element of $\mathcal{V}_{\{1\}}$ are the trivial ones. Here, $\mathcal{T}_{\mathcal{V}_{\{1\}}} = \{\emptyset,\ G\}$ is the trivial topology.
Another interesting example is constructed as follows: consider $\mathcal{V}_{Ab}$, the pseudovariety of all finite abelian groups, and let G be any group with two non–trivial elements, $g, g' \in G$, that do not commute. Then, for every homomorphism $\varphi\colon G \to V \in \mathcal{V}_{Ab}$ one has that

$$\varphi(g \cdot g') = \varphi(g) + \varphi(g') = \varphi(g') + \varphi(g) = \varphi(g' \cdot g),$$

and so $d(g \cdot g', g' \cdot g) = 0$, while $g \cdot g' \neq g' \cdot g$ because g and g' do not commute in G.

In the following lemma we see that d verifies a stronger property than the triangle inequality.

Lemma 3.1.8. *d is an ultrametric distance on G. That is, $d(x, y) \leq \max\{d(x, z), d(z, y)\}$, $\forall x, y, z \in G$.*

Proof. First suppose that $d(x, z) = d(z, y)$. If $d(x, y)$ had a strictly greater value (and thus $\rho(x, y) < \rho(x, z) = \rho(z, y)$), then it would exist an homomorphism φ from G to a group in $\mathcal{V}$ of cardinal less than $\rho(x, z) = \rho(z, y)$, with $\varphi(x) \neq \varphi(y)$. But then $\varphi(z) \neq \varphi(x)$ and $\varphi(z) \neq \varphi(y)$, giving us a contradiction.
Now suppose, without loss of generality, that $\max\{d(x, z),\ d(z, y)\} = d(x, z) > d(z, y)$. Then, let $\varphi\colon G \to V \in \mathcal{V}$ be an homomorphism from the definition of $\rho(x, z)$, with $\varphi(x) \neq \varphi(z)$. We observe that φ also separates x and y and, by the definition of d and ρ, $d(x, y) \leq d(x, z)$ and we are done. Indeed, if we had $\varphi(x) = \varphi(y)$, then $\varphi(z) \neq \varphi(y)$ and φ would separate z and y, which can not happen since V has strictly smaller order than $\rho(z, y)$. □

A known and basic result that directly follows from the ultrametric inequality is that it implies the triangle inequality: that is, if (X, d) is an ultrametric space, then it is also a metric space. Indeed, $\forall x, y, z \in X$,

$$\begin{aligned} d(x, y) \leq \max\{d(x, z),\ d(z, y)\} &\leq \max\{d(x, z),\ d(z, y)\} + \min\{d(x, z),\ d(z, y)\} = \\ &= d(x, z) + d(z, y), \end{aligned}$$

where we used the fact that $d \geq 0$.

3.1.1 An interesting characterization for $\mathcal{T}_\mathcal{V}$

Consider now the topology on G induced by d, namely $\mathcal{T}_d$ (see Definition 2.1.3). It turns out that the *Pro*$-\mathcal{V}$ topology on a group is metrizable, as we know from [9] that $\mathcal{T}_\mathcal{V} = \mathcal{T}_d$. To actually prove this result, we will be using the following easy lemma from general topology that characterizes the equivalence between topologies:

Lemma 3.1.9. *Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies defined over X and $\mathcal{B}_1, \mathcal{B}_2$ any respective bases. Then, $\mathcal{T}_1 = \mathcal{T}_2$ if, and only if*

1. $\forall B \in \mathcal{B}_1,\ \forall x \in B,\ \exists W_x \in \mathcal{T}_2$ *such that* $x \in W_x \subseteq B$;
2. $\forall B' \in \mathcal{B}_2,\ \forall y \in B',\ \exists U_y \in \mathcal{T}_1$ *such that* $y \in U_y \subseteq B'$.

Proof. As the direct implication is trivial, we will focus on the inverse one. For this one, as the two inclusions $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{T}_1 \supseteq \mathcal{T}_2$ are seen the same way, we will only prove one of them, and the other is done analogously. So, let $A = \bigcup_{i\in I} B_i,\ B_i \in \mathcal{B}_1$ be any open set in $\mathcal{T}_1$.
Now consider, for every $i \in I$, and for every $x \in B_i$, the set formed by the union of all $W_x \in \mathcal{T}_2$ from Property *1*, that is also in $\mathcal{T}_2$. That is

$$C_i := \bigcup_{x\in B_i} W_x \in \mathcal{T}_2,$$

where $x \in W_x \subseteq B_i$. It is clear that if $z \in C_i$, then it must exist $x \in B_i$ such that $z \in W_x \subseteq B_i$. In a similar way, if $z \in B_i$, then $z \in W_z \subseteq C_i$ and we can conclude $C_i = B_i$. Now, $A = \bigcup_{i\in I} C_i \in \mathcal{T}_2$ and we are done. □

Before establishing and proving the main result of this section, let us see the aspect of a general open ball of center $g_0 \in G$ and radius $r > 0$, using distance d. This will help us in the upcoming proof of Theorem 3.1.11.

Observation 3.1.10. If $r > 1/4$, it is easy to see that $B(g_0, r) = G,\ \forall g_0 \in G$, as a group $V \in \mathcal{V}$ must have at least order 2 to separate any pair of elements of G. A mere rewriting of the definition of $B(g_0, r)$ for $r \leq 1/4$ yields

$$\begin{aligned} B(g_0, r) &= \{g \in G : d(g, g_0) < r\} = \{g \in G : \rho(g, g_0) > -\log_2 r\} \\ &= \{g \in G : \nexists \varphi\colon G \to V \in \mathcal{V} \text{ homomorphism with } |V| \leq -\log_2 r \text{ and } \varphi(g) \neq \\ &\quad\ \varphi(g_0)\} \\ &= \{g \in G : \forall \varphi\colon G \to V \in \mathcal{V} \text{ homomorphism, } |V| > -\log_2 r \text{ or } \varphi(g) = \varphi(g_0)\}. \end{aligned}$$

Theorem 3.1.11. *The Pro–$\mathcal{V}$ topology coincides with the topology given by the metric d, that is $\mathcal{T}_\mathcal{V} = \mathcal{T}_d$.*

Proof. We have to see that $\mathcal{T}_\mathcal{V} = \mathcal{T}_d$, and we will do it using Lemma 3.1.9, where a basis of $\mathcal{T}_\mathcal{V}$ was given in Proposition 3.1.3 and a basis of $\mathcal{T}_d$ is the usual for metric topologies, $\mathcal{B}_d = \{B(g, r) : g \in G,\ r > 0\}$ (that is, the whole set of open balls).

So, let $\varphi^{-1}(v)$ be any element of the basis of $\mathcal{T}_\mathcal{V}$, with $\varphi\colon G \to V_0 \in \mathcal{V}$ an homomorphism, and $v \in V_0$, and also let $g_0 \in \varphi^{-1}(v)$. We see that $B\left(g_0, 2^{-|V_0|}\right) = \{g \in G : \forall \gamma\colon G \to V \in \mathcal{V}$ homomorphism, $|V| \leq |V_0|$ then $\gamma(g) = \gamma(g_0)\} \subseteq \{g \in G : \varphi(g) = \varphi(g_0) = v\} = \varphi^{-1}(v)$. So, we have $g_0 \in B\left(g_0, 2^{-|V_0|}\right) \subseteq \varphi^{-1}(v)$.

In a similar way, let $B(g', r)$, for $g' \in G,\ r > 0$, be any element of the given basis of $\mathcal{T}_d$, and let $g_1 \in B(g', r)$. Consider now $\mathcal{V}_r = \{V \in \mathcal{V} : |V| \leq -\log_2 r\}$. Note that $|\mathcal{V}_r|$

is finite as there is a finite number of groups of order $m \in \mathbb{N}$ and, hence, we can write $\mathcal{V}_r = \{V_1, \ldots, V_n\}$, for some $n \in \mathbb{N}$. Now, for every $1 \leq i \leq n$ and every $v \in V_i$ consider one homomorphism (if it exists)

$$\varphi_i^v \colon G \longrightarrow V_i \text{ such that } \varphi_i^v(g') = v.$$

Note also that, for fixed $1 \leq i \leq n$, $\alpha_i := |\{\varphi_i^v : v \in V_i\}| \leq |V_i| < +\infty$, so $\{\varphi_i^v : 1 \leq i \leq n,\ v \in V_i\}$ is also a finite set. If we now define

$$\begin{array}{rcl} \varphi_0 \colon G & \longrightarrow & V_1^{\alpha_1} \times \cdots \times V_n^{\alpha_n} \in \mathcal{V} \\ g & \longmapsto & (\varphi_i^v(g))_{1\leq i\leq n}^{v\in V_i}, \end{array}$$

it is easy to see that φ_0 is well-defined and indeed an homomorphism, as φ_i^v are all homomorphisms. Finally, as $|V_i| \leq -\log_2 r$, $\forall i$, then $\varphi_i^v(g_1) = \varphi_i^v(g')$, $\forall i$, $\forall v$ which implies that $\varphi_0(g_1) = \varphi_0(g')$ and, hence, $g_1 \in \varphi_0^{-1}(\varphi_0(g'))$. Also, if $g \in \varphi_0^{-1}(\varphi_0(g'))$, then $\varphi_i^v(g) = \varphi_i^v(g')$, $\forall i$, $\forall v$, which means that $g \in B(g', r)$, as all possible images of g' by an homomorphism φ from G to $V \in \mathcal{V}$, with $|V| \leq -\log_2 r$, are represented in the components of φ_0. This proves that $g_1 \in \varphi_0^{-1}(\varphi_0(g')) \subseteq B(g', r)$.
In fact, it is easy to see that the other inclusion is also true, and hence we have $\varphi_0^{-1}(\varphi_0(g')) = B(g', r)$. This means that every open ball can also be expressed as an element from the basis $\mathcal{B}_{\mathcal{V}}$ of $\mathcal{T}_{\mathcal{V}}$, but not in the other way around, so we conclude that $\mathcal{B}_{\mathcal{V}}$ is finer than $\mathcal{B}_d$, $\mathcal{B}_d \subseteq \mathcal{B}_{\mathcal{V}}$. □

As a consequence of this last theorem, we have an alternative definition for the *Pro–$\mathcal{V}$* topology on a group, via a metric function, which will turn out to be very useful. In fact, a first observation we can make here is that $d_a(g, g') := a^{-\rho(g,g')}$ is an equivalent distance function to d, for every $a > 1$. This is reflected in the following result:

Proposition 3.1.12. *For every $a > 1$, $\mathcal{T}_{d_a} = \mathcal{T}_d$.*

Proof. Firstly, and since $a > 1$, it is clear that d_a is a well–defined pseudodistance just as d is. Now, let $B_d(g_0, r)$ be the open ball with center $g_0 \in G$ and radius $r > 0$, using distance d, and let $g_1 \in B_d(g_0, r)$. By Lemma 3.1.9, we have to find an open set $W_{g_1} \in \mathcal{T}_{d_a}$ such that $g_1 \in W_{g_1} \subseteq B_d(g_0, r)$. Consider $W_{g_1} = B_{d_a}(g_1, r')$, where $0 < r' < r^{1/\log_a 2}$. We clearly have that $g_1 \in W_{g_1}$. To see that $W_{g_1} \subseteq B_d(g_0, r)$, let $g \in W_{g_1}$: then, $d(g, g_0) \leq \max\{d(g, g_1),\ d(g_1, g_0)\} = \max\{(d_a(g, g_1))^{\log_a 2},\ d(g_1, g_0)\} < \max\{(r')^{\log_a 2},\ r\} = r$, so $g \in B_d(g_0, r)$ and we are done.
The reciprocal property here is done analogously, as d is a particular case of d_a, so we consider the proof as finished. □

3.2 Further properties and related distances to d

The main objective of this section is to see that the group operation in G is contracting [9, Section 1.1]. Intuitively, this means that operating on G makes the distance between any two pair of points smaller. Thanks to this result, we will be able to define alternative distances based on d that will turn out to be equivalent. This will also help us in the next section, to characterize the *Pro–$\mathcal{V}$* topology on a finite direct product group via the product topology on it.

Definition 3.2.1. Given a function $f \colon X \to Y$ between metric spaces, with respective distance functions d_X and d_Y, we say that f is a contractive function (or simply a contraction) if

$$d_Y(f(x), f(x')) \leq d_X(x, x'),$$

$\forall x, x' \in X$.

So, we now need to define a metric in the group $G^2 := G \times G$. Fortunately, a good approach is based in the fact that we have a distance function directly from the definition of d in G^2, which in fact induces the *Pro–$\mathcal{V}$* topology on G^2. Indeed,

$$\begin{array}{rcl} D\colon G^2 \times G^2 & \longrightarrow & \mathbb{R} \\ ((g_1, g_2), (g_1', g_2')) & \longmapsto & 2^{-\rho((g_1,g_2),(g_1',g_2'))}, \end{array}$$

defines a natural metric on G^2. Before establishing the proposition that proves that the operation in G is actually a contraction, we have to prove the following easy lemma:

Lemma 3.2.2. *For every pair of elements $(g_1, g_2), (g_1', g_2') \in G^2$, one has that*

$$\rho((g_1, g_2), (g_1', g_2')) \leq \min\{\rho(g_1, g_1'),\ \rho(g_2, g_2')\}$$

Proof. If $\varphi_1\colon G \to V_1 \in \mathcal{V},\ \varphi_2\colon G \to V_2 \in \mathcal{V}$ are homomorphisms from the definitions of $\rho(g_1, g_1')$ and $\rho(g_2, g_2')$ respectively, consider now

$$\begin{array}{rcl} \overline{\varphi}_1\colon G \times G & \longrightarrow & V_1 \in \mathcal{V} \\ (g, g') & \longmapsto & \varphi_1(g) \end{array} \qquad \begin{array}{rcl} \overline{\varphi}_2\colon G \times G & \longrightarrow & V_2 \in \mathcal{V} \\ (g, g') & \longmapsto & \varphi_2(g') \end{array}$$

It is easy to see that $\overline{\varphi}_1, \overline{\varphi}_2$ are well–defined homomorphisms both separating (g_1, g_2) and (g_1', g_2'), so the result follows. □

Now, we can establish and prove the following result, that relates the distance between elements of G before and after operating:

Proposition 3.2.3. *Operating in G is a contraction. That is,*

$$d(g_1g_2, g_1'g_2') \leq D((g_1, g_2), (g_1', g_2')),$$

$\forall g_1, g_2, g_1', g_2' \in G$.

Proof. If $d(g_1g_2, g_1'g_2') = 0$ there is nothing to do. If not, and from the fact that $\rho((g_1, g_2), (g_1', g_2')) \leq \min\{\rho(g_1, g_1'),\ \rho(g_2, g_2')\}$ (see last lemma), one directly has that

$$D((g_1, g_2), (g_1', g_2')) \geq \max\{d(g_1, g_1'),\ d(g_2, g_2')\}.$$

Now, let $\varphi\colon G \to V \in \mathcal{V}$ be an homomorphism from the definition of $\rho(g_1g_2, g_1'g_2')$. If we had $\varphi(g_1) = \varphi(g_1')$ and $\varphi(g_2) = \varphi(g_2')$, then $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2) = \varphi(g_1')\varphi(g_2') = \varphi(g_1'g_2')$, which is a contradiction. Without loss of generality, suppose $\varphi(g_1) \neq \varphi(g_1')$. Then,

$$d(g_1g_2, g_1'g_2') \leq d(g_1, g_1') \leq \max\{d(g_1, g_1'),\ d(g_2, g_2')\} \leq D((g_1, g_2), (g_1', g_2')),$$

which is what we wanted to see, and the proof is done. □

A direct and classic consequence we can establish from this is that multiplication in G is uniformly continuous and, hence, continuous. This comes from the fact that the operation in G is 1–*Lipschitz*, and *Lipschitz* implies uniform continuity (Definition 2.4.2).
There are more examples of contracting functions. In fact, a useful result states that homomorphisms are also contracting:

Proposition 3.2.4. *Let G and G' be groups equipped with the Pro–$\mathcal{V}$ topology, and let $\varphi\colon G \to G'$ be a group homomorphism. Then, φ is contracting and, hence, uniformly continuous.*

Proof. If $\gamma\colon G' \to V \in \mathcal{V}$ is an homomorphism that separates $\varphi(g)$ and $\varphi(g')$, then $\gamma \circ \varphi\colon G \to V \in \mathcal{V}$ is an homomorphism that separates g and g' and, hence,

$$\rho(\varphi(g), \varphi(g')) \geq \rho(g, g').$$

This directly implies that $d(\varphi(g), \varphi(g')) \leq d(g, g')$, which is what we wanted to see. $\square$

Note that in the proof of Proposition 3.2.3 it appeared the known metric $d_\infty((g_1, g_2), (g_1', g_2')) \coloneqq \max\{d(g_1, g_1'),\ d(g_2, g_2')\}$, and we saw that

$$d_\infty((g_1, g_2), (g_1', g_2')) \leq D((g_1, g_2), (g_1', g_2')).$$

The following results prove the reversed inequality, $d_\infty((g_1, g_2), (g_1', g_2')) \geq D((g_1, g_2), (g_1', g_2'))$, which lead us to conclude that $d_\infty = D$.

Lemma 3.2.5. *Let G, H be arbitrary groups, and let $(g_1, g_2), (g_1', g_2') \in G \times G$ be any elements of the product group. Let $\varphi\colon G \times G \to H$ be an homomorphism that separates (g_1, g_2) and (g_1', g_2'). Consider now the homomorphisms*

$$\begin{array}{rcl} \overline{\varphi}_1\colon G & \longrightarrow & H \\ g & \longmapsto & \varphi(g, e) \end{array} \qquad \begin{array}{rcl} \overline{\varphi}_2\colon G & \longrightarrow & H \\ g & \longmapsto & \varphi(e, g). \end{array}$$

Then, either $\overline{\varphi}_1$ separates g_1 and g_1' or $\overline{\varphi}_2$ separates g_2 and g_2'.

Proof. Firstly, it is clear that both $\overline{\varphi}_1$ and $\overline{\varphi}_2$ are well-defined and in fact homomorphisms. Now, if neither $\overline{\varphi}_1$ separates g_1 and g_1' nor $\overline{\varphi}_2$ separates g_2 and g_2', then

$$\varphi(g_1, g_2) = \varphi((g_1, e)(e, g_2)) = \varphi(g_1, e)\varphi(e, g_2) = \overline{\varphi}_1(g_1)\overline{\varphi}_2(g_2) = \overline{\varphi}_1(g_1')\overline{\varphi}_2(g_2') =$$

$$= \varphi(g_1', e)\varphi(e, g_2') = \varphi((g_1', e)(e, g_2')) = \varphi(g_1', g_2'),$$

which is a contradiction since φ separates (g_1, g_2) and (g_1', g_2'). $\square$

Corollary 3.2.6. *For every $(g_1, g_2), (g_1', g_2') \in G^2$,*

$$D((g_1, g_2), (g_1' g_2')) \leq d_\infty((g_1, g_2), (g_1', g_2')).$$

Proof. Directly from last lemma we have that

$$\rho((g_1, g_2), (g_1', g_2')) \geq \min\{\rho(g_1, g_1'),\ \rho(g_2, g_2')\},$$

which implies that

$$D((g_1, g_2), (g_1', g_2')) \leq \max\{d(g_1, g_1'),\ d(g_2, g_2')\} = d_\infty((g_1, g_2), (g_1' g_2')),$$

which is what we wanted to see, and we are done. $\square$

The last two results proved that, indeed, $D = d_\infty$. But d_∞ is not the only usual metric we can define in $G \times G$: in fact,

$$d_1((g_1, g_2), (g_1', g_2')) \coloneqq d(g_1, g_1') + d(g_2, g_2'),$$

$$d_2((g_1, g_2), (g_1', g_2')) \coloneqq \sqrt{d(g_1, g_1')^2 + d(g_2, g_2')^2},$$

are also natural metrics induced by the distance d in G. A known and classic result is that d_1, d_2 and d_∞ are equivalent metrics, regardless of the distance d considered. This comes from the fact that

$$d_\infty \leq d_1 \leq 2 \cdot d_\infty,$$

$$d_\infty \le d_2 \le \sqrt{2} \cdot d_\infty,$$

which also implies $\frac{1}{2}d_1 \le d_2 \le \sqrt{2} \cdot d_1$. So, d_1, d_2, d_∞ (and D) are strongly equivalent[4], and hence $\mathcal{T}_{d_1} = \mathcal{T}_{d_2} = \mathcal{T}_{d_\infty} = \mathcal{T}_D$[5], which is exactly the *Pro–$\mathcal{V}$* topology on G^2 (via Theorem 3.1.11). Note that this result can be generalized with the notion of *Minkowski* distances, $d_p((g_1, g_2), (g_1', g_2')) \coloneqq ((d(g_1, g_2))^p + (d(g_1' g_2'))^p)^{1/p}$ and the fact that $d_\infty \le d_p \le 2^{1/p} \cdot d_\infty$.

3.3 *Pro–$\mathcal{V}$* topologies of product groups

Note that the previous construction to see that the usual metrics yield the same topology was initially motivated by the fact that we wanted to prove that the operation on G was a contraction, and hence we considered the group G^2. But what if we considered the more general group $G_1 \times \cdots \times G_n$, where G_i are possibly different groups? The next theorem answers this question.

Theorem 3.3.1. *The natural distance function on the product group $G_1 \times \cdots \times G_n$,*

$$\begin{array}{rcl} D^n \colon (G_1 \times \cdots \times G_n) \times (G_1 \times \cdots \times G_n) & \longrightarrow & \mathbb{R} \\ ((g_1, \ldots, g_n), (g_1', \ldots g_n')) & \longmapsto & 2^{-\rho((g_1,\ldots,g_n),(g_1',\ldots g_n'))}, \end{array}$$

coincides with $d_\infty^n((g_1, \ldots, g_n), (g_1', \ldots g_n')) \coloneqq \max\{d_1(g_1, g_1'), \ldots, d_n(g_n, g_n')\}$, where

$$\begin{array}{rcl} d_i \colon G_i \times G_i & \longrightarrow & \mathbb{R} \\ (g, g') & \longmapsto & 2^{-\rho_i(g,g')}, \end{array}$$

is the natural distance function on G_i, $\forall i$. Moreover, and considering the natural metrics

$$d_1^n((g_1, \ldots, g_n), (g_1', \ldots g_n')) \coloneqq \sum_{i=1}^{n} d_i(g_i, g_i'),$$

$$d_2^n((g_1, \ldots, g_n), (g_1', \ldots g_n')) \coloneqq \sqrt{\sum_{i=1}^{n} (d_i(g_i, g_i'))^2},$$

one has that $\mathcal{T}_{d_1^n} = \mathcal{T}_{d_2^n} = \mathcal{T}_{d_\infty^n} = \mathcal{T}_{D^n}$, and they do all coincide with the Pro–$\mathcal{V}$ topology on $G_1 \times \cdots \times G_n$.

Proof. The first part of the Theorem is proved by induction in $n \in \mathbb{N}$:
The base case is for $n = 2$: as one can see, in the above construction to prove that $D = d_\infty$ we did not use that $G^2 = G \times G$ was the product of the same group G. In fact, the same construction adapted for a product of two arbitrary groups $G_1 \times G_2$ works, so the base case is proved.
For the inductive step, consider $G_1 \times \cdots \times G_n = (G_1 \times \cdots \times G_{n-1}) \times G_n \eqqcolon H_{n-1} \times G_n$. Thus, we have a product of two groups, and hence

$$D^n((g_1, \ldots, g_n), (g_1', \ldots, g_n')) = D((h_{n-1}, g_n), (h_{n-1}', g_n')) =$$

$$\max\{d_{H_{n-1}}(h_{n-1}, h_{n-1}'),\ d_n(g_n, g_n')\} = \max\{D^{n-1}((g_1, \ldots, g_{n-1}), (g_1', \ldots, g_{n-1}')),\ d_n(g_n, g_n')\},$$

where $h_{n-1} = (g_1, \ldots, g_{n-1})$, $h_{n-1}' = (g_1', \ldots, g_{n-1}')$ and $d_{H_{n-1}}$ is the natural distance function on H_{n-1}, which coincides with D^{n-1}. Now, applying the inductive hypothesis on D^{n-1} we get that the last expression equals

$$\max\{\max\{d_1(g_1, g_1'), \ldots, d_{n-1}(g_{n-1}, g_{n-1}')\},\ d_n(g_n, g_n')\} = \max\{d_1(g_1, g_1'), \ldots, d_n(g_n, g_n')\}$$

[4]Definition 2.4.3.
[5]Definition 2.1.3.

$$= d_\infty^n((g_1 \dots g_n), (g_1', \dots g_n')),$$

which is what we wanted to see. The general inequalities

$$d_\infty^n \le d_1^n \le n \cdot d_\infty^n, \qquad d_\infty^n \le d_2^n \le \sqrt{n} \cdot d_\infty^n,$$
$$\frac{1}{n} \cdot d_1^n \le d_2^n \le \sqrt{n} \cdot d_1^n,$$

directly prove that all the metrics are strongly equivalent, and hence $\mathcal{T}_{d_1^n} = \mathcal{T}_{d_2^n} = \mathcal{T}_{d_\infty^n} = \mathcal{T}_{D^n}$. As this last topology coincides with the *Pro–$\mathcal{V}$* topology on $G_1 \times \cdots \times G_n$ (again, via Theorem 3.1.11), the final result is proved. □

We have considered the *Pro–$\mathcal{V}$* topology on the group $G \coloneqq G_1 \times \cdots \times G_n$. An alternative approach to study this product group could have been considering the *Pro–$\mathcal{V}$* topology in each of the groups $G_1, \dots, G_n$, and then constructing the product topology (see Definition 2.1.17) induced by all of them. The following result states that this two constructions are equivalent:

Theorem 3.3.2. *Let $G \coloneqq G_1 \times \cdots \times G_n$ be a finite product of arbitrary groups. Then, the Pro–$\mathcal{V}$ topology on G coincides with the product topology on G.*

Proof. We will use Lemma 3.1.9 again. A basis of the *Pro–$\mathcal{V}$* topology on G is given by the set of open balls using distance $D^n = d_\infty^n$, namely

$$\mathcal{B}_{d_\infty^n} = \{B_{d_\infty^n}(g, r) : g = (g_1, \dots, g_n) \in G,\ r > 0\}.$$

On the other hand, a basis of the product topology of $G_1 \times \cdots \times G_n$ is given by[6]

$$\mathcal{B}_{d_1 \times \cdots \times d_n} = \{B_{d_1}(g_1, r_1) \times \cdots \times B_{d_n}(g_n, r_n) : g_i \in G_i,\ r_i > 0,\ \forall i\}.$$

We first see that $\mathcal{B}_{d_\infty^n} \subseteq \mathcal{B}_{d_1 \times \cdots \times d_n}$. Indeed, for any $h = (h_1, \dots, h_n) \in G$ and $r > 0$, we have that

$$B_{d_\infty^n}(h, r) = B_{d_1}(h_1, r) \times \cdots \times B_{d_n}(h_n, r).$$

Take $g = (g_1, \dots, g_n) \in B_{d_\infty^n}(h, r)$. Then, $d_\infty^n(g, h) = \max\{d_1(g_1, h_1), \dots, d_n(g_n, h_n)\} < r$. This implies that $d_1(g_1, h_1) < r, \dots, d_n(g_n, h_n) < r$, so $g_1 \in B_{d_1}(h_1, r), \dots, g_n \in B_{d_n}(h_n, r)$ and $g \in B_{d_1}(h_1, r) \times \cdots \times B_{d_n}(h_n, r)$.
Reciprocally, if $g' = (g_1', \dots, g_n') \in B_{d_1}(h_1, r) \times \cdots \times B_{d_n}(h_n, r)$, then $d_1(g_1', h_1) < r, \dots, d_n(g_n', h_n) < r$, and therefore $\max\{d_1(g_1', h_1), \dots, d_n(g_n', h_n)\} = d_\infty^n(g', h) < r$, which means $g' \in B_{d_\infty^n}(h, r)$.

On the other hand, let $B_{d_1}(h_1, r_1) \times \cdots \times B_{d_n}(h_n, r_n)$ be any element of $\mathcal{B}_{d_1 \times \cdots \times d_n}$, with $(h_1, \dots, h_n) =: h \in G$, and $r_i > 0$, $\forall i$, and let $g = (g_1, \dots, g_n) \in B_{d_1}(h_1, r_1) \times \cdots \times B_{d_n}(h_n, r_n)$.
Consider now $r_{min} \coloneqq \min\{r_1, \dots, r_n\}$. We claim that

$$g \in B_{d_\infty^n}(g, r_{min}) \subseteq B_{d_1}(h_1, r_1) \times \cdots \times B_{d_n}(h_n, r_n).$$

As $r_i > 0$, $\forall i$, $r_{min} > 0$ and therefore $g \in B_{d_\infty^n}(g, r_{min})$. Take now $g' = (g_1', \dots, g_n') \in B_{d_\infty^n}(g, r_{min})$. Then, $d_\infty^n(g', g) = \max\{d_1(g_1', g_1), \dots, d_n(g_n', g_n)\} < r_{min}$. Using that, for every i, d_i is an ultrametric distance, we have that

$$d_i(g_i', h_i) \le \max\{d_i(g_i', g_i),\ d_i(g_i, h_i)\} < \max\{r_{min},\ r_i\} = r_i \quad \forall i,$$

and so $g' = (g_1', \dots, g_n') \in B_{d_1}(h_1, r_1) \times \cdots \times B_{d_n}(h_n, r_n)$.

This proves that the *Pro–$\mathcal{V}$* topology on G coincides with the product of the *Pro–$\mathcal{V}$* topologies on $G_1, \dots, G_n$, which is what we wanted to see. In fact, we have also proved that $\mathcal{B}_{d_1 \times \cdots \times d_n}$ is a finer basis than $\mathcal{B}_{d_\infty^n}$, since every element from $\mathcal{B}_{d_\infty^n}$ can be expressed as an element of $\mathcal{B}_{d_\infty^n}$, but not conversely. □

[6]Proposition 2.1.18.

3.4 Topological properties of subgroups of G

Our goal is now to study the topological properties of the subgroups $H \leqslant G$. We want to know under which conditions is H, for example, an open set, or whether the restriction of $\mathcal{T}_\mathcal{V}$ to H is equal to the *Pro–$\mathcal{V}$* topology on H. We will answer all these questions and more in this section (see also [9, Section 1.3]). First of all, let us verify this simple proposition from group theory:

Proposition 3.4.1. *Let G be a group, and $H \leqslant G$ a subgroup. Then, $H_G = \bigcap_{g\in G} g^{-1}Hg$ is the greatest normal subgroup of G contained in H. That is, if $N \trianglelefteqslant G$ is another normal subgroup of G, with $N \leqslant H$, then $N \leqslant H_G$. H_G is usually called the core of H.*

Proof. As $g^{-1}Hg$ is a subgroup of G, for every $g \in G$, and the arbitrary intersection of subgroups is also a subgroup, we easily get that H_G is indeed a subgroup of G. The normality of H_G comes directly from the fact that

$$k^{-1}H_Gk = k^{-1}\left(\bigcap_{g\in G} g^{-1}Hg\right)k = \bigcap_{g\in G}(gk)^{-1}H(gk) = \bigcap_{z\in G} z^{-1}Hz = H_G.$$

If $N \leqslant H$ is another normal subgroup of G, let $a \in N$. Then, $\forall g \in G,\ a = g^{-1}(gag^{-1})g \in g^{-1}Hg$, since $gag^{-1} \in N \leqslant H$, by the normality of N in G. This directly implies that $a \in \bigcap_{g\in G} g^{-1}Hg = H_G$, which is what we wanted to see. □

We are now under the conditions of establishing the first theorem of this section:

Theorem 3.4.2. *Let G be a group equipped with the Pro–$\mathcal{V}$ topology, and let $H \leqslant G$ be a subgroup of G. Then, the following are equivalent:*

a) H is open;

b) H is clopen;

c) H has finite index and $G/H_G \in \mathcal{V}$.

Proof. We will first see that *c)* implies *b)*. Consider the natural homomorphism $\pi\colon G \twoheadrightarrow G/H_G \in \mathcal{V},\ g \mapsto [g]$, which is indeed continuous ($[g]$ denotes the class of g in G/H_G). We claim that

$$H = \pi^{-1}(\pi(H)) = \bigcup_{h\in H} \pi^{-1}(\pi(h)).$$

The second equality is clear, so let's see the first one: the inclusion $H \subseteq \pi^{-1}(\pi(H))$ is trivial by function properties, so we can focus on the inverse one. That way, let $a \in \pi^{-1}(\pi(H))$. Then, $\pi(a) = aH_G \in \pi(H) = \{hH_G : h \in H\}$. So, there exists $h' \in H$ such that $aH_G = h'H_G$. The implications

$$H_G = a^{-1}h'H_G \underset{e\in H_G}{\Longrightarrow} a^{-1}h' \in H_G \subseteq H \Longrightarrow a^{-1}h'h'^{-1} = a^{-1} \in H \Longrightarrow a \in H,$$

finally show the inverse inclusion, so $H = \bigcup_{h\in H} \pi^{-1}(\pi(h))$ is a union of open sets, and hence a open set. As $G = \bigcup_{g\in G} \pi^{-1}(\pi(g))$, then

$$H^c = \bigcup_{g\in G\setminus H} \pi^{-1}(\pi(g)),$$

which is again a union of open sets and hence an open set. This proves that H is also closed and, hence, clopen.

As *b)* clearly implies *a)*, we are now left to prove *a)* $\Longrightarrow$ *c)*. If H is a subgroup of G, then $e_G \in H$ and, since H is open, then there exists an element from $\mathcal{B}_\mathcal{V}$, say $\varphi^{-1}(v)$, with $\varphi\colon G \to V \in \mathcal{V}$ homomorphism, and $v \in V$, such that $e_G \in \varphi^{-1}(v) \subseteq H$. As $\varphi(e_G) = e_V$, then $v = e_V$ and $\varphi^{-1}(v) = \ker(\varphi) \trianglelefteqslant G$. Also, from Proposition 3.4.1 we have that $\ker(\varphi) \leqslant H_G$. From the fact that $\mathrm{Im}(\varphi) \cong G/\ker(\varphi)$ is a subgroup of V, we get that $G/\ker(\varphi) \in \mathcal{V}$, and using the third Isomorphism Theorem,

$$G/H_G \cong G/\ker(\varphi) \Big/ H_G/\ker(\varphi) \in \mathcal{V},$$

as $H_G/\ker(\varphi) \trianglelefteqslant G/\ker(\varphi) \in \mathcal{V}$. From the inclusion $\ker(\varphi) \leqslant H$ and the fact that $[G : \ker(\varphi)] = |\mathrm{Im}(\varphi)| \leq |V| < +\infty$, we get that $[G : H] < +\infty$, which is what we wanted to see. Note that in this proof we actually used the properties that define the pseudovariety $\mathcal{V}$ (see Definition 3.1.1). □

Given a subset $X \subseteq G$, we denote by $Cl(X)$ its topological closure under the *Pro–*$\mathcal{V}$ topology on G. The following result characterizes the form of $Cl(X)$ when X is a subgroup of G:

Theorem 3.4.3. *Let G be a group equipped with the Pro–$\mathcal{V}$ topology, and let $H \leqslant G$ be a subgroup of G. Then,*

$$Cl(H) = \bigcap_{\substack{K \text{ open subgroup of } G \\ H \leqslant K}} K = \bigcap_{\substack{\varphi\colon G \to V \in \mathcal{V} \\ \varphi \text{ homomorphism}}} \varphi^{-1}(\varphi(H)) \tag{1}$$

Proof. A classical topological result establishes that $Cl(A) = \bigcap_{\substack{D^c \in \mathcal{T} \\ A \subseteq D}} D$ (Proposition 2.1.11). Directly from this,

$$Cl(H) = \bigcap_{\substack{K^c \in \mathcal{T}_\mathcal{V} \\ H \subseteq K}} K \subseteq \bigcap_{\substack{K \text{ open subgroup of } G \\ H \leqslant K}} K,$$

since open subgroups of G are also closed by Theorem 3.4.2. From the fact that $K \subseteq \varphi^{-1}(\varphi(K))$ and that $\varphi^{-1}(\varphi(H))$ is an open subgroup of G we get that

$$\bigcap_{\substack{K \text{ open subgroup of } G \\ H \leqslant K}} K \subseteq \bigcap_{\substack{\varphi\colon G \to V \in \mathcal{V} \\ \varphi \text{ homomorphism}}} \varphi^{-1}(\varphi(H)).$$

Finally, let $g \notin Cl(H)$. By the definition of topological closure, then there exists a basic neighborhood $\varphi_g^{-1}(v_g) \in \mathcal{B}_\mathcal{V} \subseteq \mathcal{T}_\mathcal{V}$ of g such that $\varphi_g^{-1}(v_g) \cap H = \emptyset$, where $\varphi_g\colon G \to V_g \in \mathcal{V}$ is an homomorphism and $v_g \in V_g$ (note that both depend on g). Then, $\varphi_g(g) \notin \varphi_g(H)$ and

$$g \notin \varphi_g^{-1}(\varphi_g(H)) \supseteq \bigcap_{\substack{\varphi\colon G \to V \in \mathcal{V} \\ \varphi \text{ homomorphism}}} \varphi^{-1}(\varphi(H)).$$

This concludes that $\bigcap_{\substack{\varphi\colon G \to V \in \mathcal{V} \\ \varphi \text{ homomorphism}}} \varphi^{-1}(\varphi(H)) \subseteq Cl(H)$, which means that all the inclusions we have seen are, in fact equalities, and the Theorem is proven. □

As the arbitrary intersection of subgroups is again a subgroup (see Definition 2.2.5 in the preliminaries), an immediate corollary we get from the last theorem is that $Cl(H) \leqslant G$. We will use this result on the next chapter.

Corollary 3.4.4. *Let G be a group equipped with the Pro–$\mathcal{V}$ topology, and let H be a subgroup of G. Then, $Cl(H)$ is also a subgroup of G.*

We can now characterize closed subgroups $H \leqslant G$ with the next corollary, that directly follows from Theorem 3.4.3:

Corollary 3.4.5. *Let G be a group equipped with the Pro–$\mathcal{V}$ topology, and let $H \leqslant G$ be a subgroup of G. Then, the following are equivalent:*

a) H is closed;

b) H is an intersection of open subgroups;

c) H is the intersection of the open subgroups containing it;

d) $H = \bigcap_{\substack{\varphi\colon G\to V\in\mathcal{V} \\ \varphi \text{ homomorphism}}} \varphi^{-1}(\varphi(H))$.

Proof. The implication *d)* $\Longrightarrow$ *c)* is obvious using (1), and *c)* $\Longrightarrow$ *b)* is trivial. From the fact that a subset X is closed if and only if $X = Cl(X)$ and using again (1), we get *a)* $\Longrightarrow$ *d)*, so we have only left to prove *b)* $\Longrightarrow$ *a)*: so, suppose $H = \bigcap_{i\in I} H_i$, where H_i are all open subgroups of G. Then,

$$Cl(H) \subseteq \bigcap_{i\in I} Cl(H_i) = \bigcap_{i\in I} H_i = H,$$

and as $H \subseteq Cl(H)$ trivially, it must be $Cl(H) = H$, and H is a closed subgroup of G. Note that $Cl(H_i) = H_i$, $\forall i$ since all open subgroups of G are also closed by Theorem 3.4.2. We also used in the proof that, for any collection of subsets A_i, $i \in I$, of a topological space (where I is an arbitrary set), one has that

$$Cl\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} Cl(A_i).$$

□

3.4.1 *Pro–$\mathcal{V}$* topologies of subgroups

We want now to study the *Pro–$\mathcal{V}$* topology on a subgroup $H \leqslant G$. As we did in the case of the product group, we can consider the restriction to H of the *Pro–$\mathcal{V}$* topology on G or directly the *Pro–$\mathcal{V}$* topology on H, considered as a group itself. The following result shows that, if we do not require any additional properties to H, then this two topologies do not necessarily coincide ([9, Proposition 1.5]):

Proposition 3.4.6. *Let G be a group equipped with the Pro–$\mathcal{V}$ topology, and let $H \leqslant G$ be a subgroup of G. Then, the restriction to H of the Pro–$\mathcal{V}$ topology on G, $\mathcal{T}_{\mathcal{V}}^{G}|_H$, is contained in the Pro–$\mathcal{V}$ topology on H, $\mathcal{T}_{\mathcal{V}}^{H}$, and they do not coincide, in general.*

Proof. The form of the elements of $\mathcal{T}_{\mathcal{V}}^{G}|_H$ is $U \cap H$, where $U \in \mathcal{T}_{\mathcal{V}}^{G}$. Expressing U with its decomposition in elements of the natural basis $\mathcal{B}_{\mathcal{V}}$, we get that

$$U \cap H = \left(\bigcup_{i\in I} \varphi_i^{-1}(v_i)\right) \cap H = \bigcup_{i\in I} \left(\varphi_i^{-1}(v_i) \cap H\right) = \bigcup_{i\in I} \varphi_i|_H^{-1}(v_i) \in \mathcal{T}_{\mathcal{V}}^{H},$$

where $\varphi_i|_H\colon H \to V_i \in \mathcal{V}$ is the restriction to H of $\varphi_i\colon G \to V_i \in \mathcal{V}$. As φ_i are all well–defined homomorphisms from G to $V_i \in \mathcal{V}$, $\varphi_i|_H$ are also well defined homomorphisms from H to $V_i \in \mathcal{V}$, so the result follows. □

This inclusion can be strict, as shows the following example:

Example 3.4.7. Take $G = S_3$, $H = A_3 = \{Id, (1\ 2\ 3), (1\ 3\ 2)\} \leqslant G$ and $\mathcal{V} = \mathcal{V}_{Ab}$. Since $A_3 \cong \mathbb{Z}/3\mathbb{Z} \in \mathcal{V}_{Ab}$, one has that $\mathcal{T}_{\mathcal{V}_{Ab}}^{A_3} = \mathcal{P}(A_3)$ is the discrete topology.
On the other hand, we claim that $\{(1\ 2\ 3)\} \notin \mathcal{T}_{\mathcal{V}_{Ab}}^{S_3}|_{A_3}$. Indeed, a first observation we make is that for every homomorphism $\varphi\colon S_3 \to V \in \mathcal{V}_{Ab}$ one has that

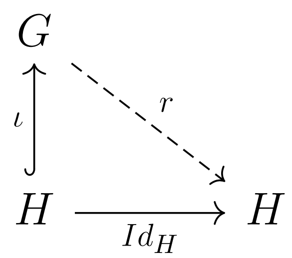
$$\varphi((1\ 2\ 3)) = \varphi((1\ 3)(1\ 2)) = \varphi((1\ 3)) + \varphi((1\ 2)) = \varphi((1\ 2)) + \varphi((1\ 3)) = \varphi((1\ 2)(1\ 3)) =$$
$$= \varphi((1\ 3\ 2)).$$

Now, if $U \cap A_3$ is an arbitrary open set of $\mathcal{T}_{\mathcal{V}_{Ab}}^{S_3}|_{A_3}$ containing $(1\ 2\ 3)$, with $U \in \mathcal{T}_{\mathcal{V}_{Ab}}^{S_3}$, then $(1\ 2\ 3) \in U$. But, let's see that if $(1\ 2\ 3) \in U$, then $(1\ 3\ 2) \in U$: as U is open, there exists an homomorphism $\overline{\varphi}$ from S_3 to a group $V \in \mathcal{V}_{Ab}$ and an element $v \in V$ such that $(1\ 2\ 3) \in \overline{\varphi}^{-1}(v) \subseteq U$. The previous observation directly implies $(1\ 3\ 2) \in \overline{\varphi}^{-1}(v) \subseteq U$ and, hence, $\{(1\ 2\ 3)\} \notin \mathcal{T}_{\mathcal{V}_{Ab}}^{S_3}|_{A_3}$.

A sufficient condition we can require from a subgroup $H \leqslant G$ to make the two topologies be equivalent is to be a retract. The formal definition of this new concept is presented below:

Definition 3.4.8. Let G be a group and $H \leqslant G$ a subgroup. Consider the natural inclusion homomorphism $\iota\colon H \to G$. We say that H is a *retract* of G if there exists an homomorphism $r\colon G \to H$ such that $r \circ \iota = Id_H$ or, equivalently, $r|_H = Id_H$. We will also usually refer to r as a retraction.

Visually, the definition of retract lies essentially in asking the following diagram to be commutative:



With this extra condition, we can establish the following important result ([9, Proposition 1.6]):

Proposition 3.4.9. *Let G be a group equipped with the Pro–$\mathcal{V}$ topology, and let $H \leqslant G$ be a subgroup of G. Then, if H is a retract of G, the restriction to H of the Pro–$\mathcal{V}$ topology on G, $\mathcal{T}_\mathcal{V}^G|_H$, is equal to the Pro–$\mathcal{V}$ topology on H, $\mathcal{T}_\mathcal{V}^H$.*

Proof. By Theorem 3.4.6 we have the inclusion $\mathcal{T}_\mathcal{V}^G|_H \subseteq \mathcal{T}_\mathcal{V}^H$. We will use Lemma 3.1.9 to see the reverse inclusion: that way, let $\varphi^{-1}(v) \in \mathcal{B}_\mathcal{V}^H$, with $\varphi\colon H \to V \in \mathcal{V}$ an homomorphism, $v \in V$ and $h \in \varphi^{-1}(v)$, and let $r\colon G \to H$ be a retraction. Then, $\varphi \circ r\colon G \to V \in \mathcal{V}$ is a well–defined homomorphism from G to a group in $\mathcal{V}$, and one has that

$$h \in (\varphi \circ r)^{-1}(v) \cap H = \varphi^{-1}(v).$$

Since $(\varphi \circ r)^{-1}(v) \in \mathcal{B}_\mathcal{V}^G$, we have that $(\varphi \circ r)^{-1}(v) \cap H \in \mathcal{B}_\mathcal{V}^G|_H$ and the theorem is proved. Note that, here, the two basis considered, $\mathcal{B}_\mathcal{V}^H$ and $\mathcal{B}_\mathcal{V}^G|_H$, coincide. □

Contrary to what one could think, the reciprocal of the last Theorem is not true, in general. The following example shows a particular case of a subgroup $H \leqslant G$ that is not a retract of G, even though $\mathcal{T}_\mathcal{V}^G|_H = \mathcal{T}_\mathcal{V}^H$:

Example 3.4.10. Take $\mathcal{V} = \mathcal{V}_{Ab}$, $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and $H = \{(0,0),\ (2,0)\} \leqslant G$. As $G, H \in \mathcal{V}_{Ab}$, it is clear that $\mathcal{T}^H_{\mathcal{V}_{Ab}} = \mathcal{P}(H) = \mathcal{T}^G_{\mathcal{V}_{Ab}}|_H$ are both the discrete topology.
Now, if H was a retract of G, then there would exist an homomorphism $r\colon G \to H$ such that $r|_H = Id_H$, and

$$(2,0) = r(2,0) = r((1,0) + (1,0)) = r(1,0) + r(1,0),$$

but defining either $r(1,0) \coloneqq (0,0) \in H$ or $r(1,0) \coloneqq (2,0) \in H$ we get a contradiction, since then $r(1,0) + r(1,0) = (0,0) \neq (2,0)$.

Note that, to get this counterexample, we have chosen the pseudovariety $\mathcal{V}$ that worked at convenience. But what if we asked the two topologies to coincide for every possible pseudovariety $\mathcal{V}$? Would it then necessarily have to exist a retraction between the group G and the subgroup $H \leqslant G$? The following result answers this question negatively:

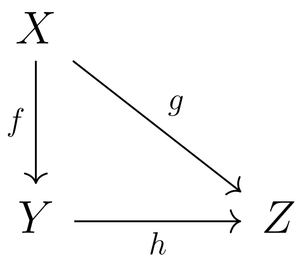
Example 3.4.11. Consider the *Thompson* groups T, V. They are examples of infinite simple groups that are finitely presented, with $T < V$. If we consider an homomorphism φ from T to any element of a pseudovariety $\mathcal{V}$, its kernel can not be trivial as T is infinite. Therefore, $ker(\varphi) = T,\ \forall \varphi$, and the only homomorphisms between T and elements of $\mathcal{V}$ are trivial ones. We conclude that the *Pro*$-\mathcal{V}$ topologies on T and V are both the trivial ones and, hence, the restriction to T of the *Pro*$-\mathcal{V}$ topology on V is also the trivial one (Proposition 3.4.6). So, the two topologies coincide for every pseudovariety $\mathcal{V}$.
Now, as V has strictly greater order than T, take $v \in V \backslash T$. Then, if there existed a retraction $r\colon V \to T$, we would have $r(v) = t \in T$, but also $r(t) = t$, and then $r(vt^{-1}) = 1$, with $vt^{-1} \neq 1$ as $v \in V\backslash T,\ t \in T$. This proves that $ker(r) \neq \{1\}$ and, hence, $ker(r) = V$ (because V is simple), which means that r is the trivial homomorphism and no such retraction exists. This concludes the proof.
A formal proof of the existence of *Thompson* groups and the verification of their properties can be found on [1].

3.5 *Pro–$\mathcal{V}$* topologies of quotient groups

To finish the chapter, we want to study the case of *Pro*$-\mathcal{V}$ topologies of quotient groups. As happened with product groups and subgroups, given a group G and a normal subgroup $H \trianglelefteq G$, we may consider two natural topologies: on the one hand, and as the quotient G/H is again a group, we might think about $\mathcal{T}^{G/H}_{\mathcal{V}}$. On the other hand, it is also natural to consider $\mathcal{T}^G_{\mathcal{V}}/H$, the quotient topology of $\mathcal{T}^G_{\mathcal{V}}$ by H. The most natural question to ask here is: are these two topologies equivalent? A quick thought regarding the last section could involve introducing the concept of *section* to characterize a particular case where the two topologies coincide, as we did in the case of subgroups with the notion of *retraction*. Even though this reasoning would not be completely wrong, the correct answer is a bit more subtle.

Let us begin the section by proving a simple lemma on set theory that will help us in the main theorem below.

Lemma 3.5.1. *Let $f\colon X \to Y,\ g\colon X \to Z,\ h\colon Y \to Z$ be any three mappings such that $h \circ f = g$ (see the commutative diagram below).*



Then, the correspondent preimages also commute, $f^{-1} \circ h^{-1} = g^{-1}$.

Proof. Let $C \subseteq Z$ be any subset. Then,

$$C \supseteq g(g^{-1}(C)) = h(f(g^{-1}(C))),$$

and so $f^{-1}(h^{-1}(C)) \supseteq g^{-1}(C)$. On the other hand, and as

$$C \supseteq h(f(f^{-1}(h^{-1}(C)))) = g(f^{-1}(h^{-1}(C))),$$

we easily get that $g^{-1}(C) \supseteq f^{-1}(h^{-1}(C))$, and we are done. Note that here we used Fact 2.4.1. □

We are now ready to establish and prove the theorem that characterizes the general relation between the two natural topologies involving quotients we can consider.

Theorem 3.5.2. *Let G be a group equipped with the Pro–$\mathcal{V}$ topology and let $H \trianglelefteq G$ be a normal subgroup of G. Then, the Pro–$\mathcal{V}$ topology on the group G/H, $\mathcal{T}_{\mathcal{V}}^{G/H}$, is equivalent to the quotient topology on G by H, $\mathcal{T}_{\mathcal{V}}^{G}/H$.*

Proof. We will first see that $\mathcal{T}_{\mathcal{V}}^{G/H} \subseteq \mathcal{T}_{\mathcal{V}}^{G}/H$. With that aim, let $\varphi^{-1}(v)$ be any element of the common basis $\mathcal{B}_{\mathcal{V}}^{G/H}$, with $\varphi\colon G/H \to V \in \mathcal{V}$ an homomorphism and $v \in V$. Note that we only have to check whether $\pi^{-1}(\varphi^{-1}(v)) \in \mathcal{T}_{\mathcal{V}}^{G}$, where $\pi\colon G \twoheadrightarrow G/H,\ g \mapsto [g]$ is the canonical projection of G into G/H. But, as

$$\pi^{-1}(\varphi^{-1}(v)) = (\varphi \circ \pi)^{-1}(v) \in \mathcal{T}_{\mathcal{V}}^{G},$$

because $\varphi \circ \pi$ is an homomorphism from G to a group of $\mathcal{V}$, the result follows.
On the other hand, let $U \in \mathcal{T}_{\mathcal{V}}^{G}/H$. Then, as $\pi^{-1}(U) \in \mathcal{T}_{\mathcal{V}}^{G}$, we can express it in terms of elements of $\mathcal{B}_{\mathcal{V}}^{G}$, this is

$$\pi^{-1}(U) = \bigcup_{i \in I} \varphi_i^{-1}(v_i),$$

where $\varphi_i\colon G \to V_i \in \mathcal{V}$ are homomorphisms, and $v_i \in V_i$. First note that, as $Im(\varphi_i) \leqslant V_i$, then $Im(\varphi_i) \in \mathcal{V}$, and we may assume that all homomorphisms φ_i are exhaustive. Now, thanks to this first observation, $\varphi_i(H) \trianglelefteq V_i$ and, hence, $V_i/\varphi_i(H) \in \mathcal{V}$, $\forall i$. Let us now define

$$\begin{array}{rcl} \overline{\varphi}_i\colon G/H & \longrightarrow & V_i/\varphi_i(H) \\ {[g]} & \longmapsto & \pi_i(\varphi_i(g)), \end{array}$$

where g is any element of G satisfying $\pi(g) = [g]$, and $\pi_i\colon V_i \twoheadrightarrow V_i/\varphi_i(H),\ v_i \mapsto [v_i]_i$ are the canonical projections of V_i onto $V_i/\varphi_i(H)$. We first have to verify that $\overline{\varphi}_i$ is well–defined, for every $i \in I$: indeed, if g and g' are elements of G satisfying $\pi(g) = \pi(g')$, then $g'g^{-1} \in H$, which implies $\pi_i(\varphi_i(g'g^{-1})) = 1$. This directly means that $\pi_i(\varphi_i(g')) = \pi_i(\varphi_i(g))$, and we conclude that $\overline{\varphi}_i$ is well–defined, $\forall i$. Moreover, and as $\overline{\varphi}_i$ is a composition of homomorphisms, it is also an homomorphism. We now have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi_i\ } & V_i \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_i} \\ G/H & \underset{\overline{\varphi}_i}{\dashrightarrow} & V_i/\varphi_i(H) \end{array}$$

Which yields

$$\pi^{-1}(U) = \bigcup_{i\in I} \varphi_i^{-1}(v_i) \subseteq \bigcup_{i\in I} \varphi_i^{-1}(\pi_i^{-1}([v_i]_i)) = \pi^{-1}\left(\bigcup_{i\in I} \overline{\varphi}_i^{-1}([v_i]_i)\right) = \pi^{-1}\left(\bigcup_{i\in I} \overline{\varphi}_i^{-1}(\pi_i(v_i))\right),$$

where we used last lemma. We now want to see that, for every $i \in I$, $\overline{\varphi}_i^{-1}(\pi_i(v_i)) = \pi(\varphi_i^{-1}(v_i))$. Firstly, the inclusion $\overline{\varphi}_i^{-1}(\pi_i(v_i)) \supseteq \pi(\varphi_i^{-1}(v_i))$ is trivial because of the commutativity of the diagram above. To see the contrary inclusion, let $[y] \in \overline{\varphi}_i^{-1}(\pi_i(v_i))$. Then, $\overline{\varphi}_i([y]) = \pi_i(v_i)$. Note that it is sufficient to find $x \in \pi^{-1}([y])$ such that $\varphi_i(x) = v_i$, as then $[y] = \pi(x) \in \pi(\varphi_i^{-1}(v_i))$.

That being said, let x' be any element of $\pi^{-1}([y])$ (note that such x' must exist as π is exhaustive). Then, again by the commutativity of the diagram above, one has that

$$\pi_i(\varphi_i(x')) = \overline{\varphi}_i(\pi(x')) = \overline{\varphi}_i([y]) = \pi_i(v_i),$$

and so $\varphi_i(x') = v_i\varphi_i(h)$, for some $h \in H$. Then, $\varphi_i(x'h^{-1}) = v_i$. Defining $x \coloneqq x'h^{-1}$ we have that, as $h^{-1} \in H$ (and, hence, $\pi(h^{-1}) = 1$), $\pi(x) = \pi(x') = [y]$ and, by definition, $\varphi_i(x) = v_i$, so we are done. Now, finally,

$$\pi^{-1}\left(\bigcup_{i\in I} \overline{\varphi}_i^{-1}(\pi_i(v_i))\right) = \pi^{-1}\left(\bigcup_{i\in I} \pi(\varphi_i^{-1}(v_i))\right) = \pi^{-1}(\pi(\pi^{-1}(U))) = \pi^{-1}(U),$$

and so we conclude that $\pi^{-1}(U) = \pi^{-1}\left(\bigcup_{i\in I} \overline{\varphi}_i^{-1}([v_i]_i)\right)$, which directly implies that

$$U = \bigcup_{i\in I} \overline{\varphi}_i^{-1}([v_i]_i) \in \mathcal{T}_{\mathcal{V}}^{G/H},$$

as π is exhaustive (Fact 2.4.1). That is, U is open in the *Pro*$-\mathcal{V}$ topology on G/H, and we are finally done. □

Chapter 4

The case of free groups and $\mathcal{V}_p$

In this chapter we focus on the case $G = \mathbb{F}_A$ and $\mathcal{V} = \mathcal{V}_p$ (the pseudovariety of all finite $p-$groups, where p is a prime number). For (needed) previous results about free groups and *Stallings* automata theory, the reader is referred to Section 2.3 and Section 2.3.1, respectively.
Our main objectives here are, firstly, to prove that the closure of a finitely generated subgroup of the free group is also finitely generated. To accomplish this goal, it will be necessary to ask the pseudovariety $\mathcal{V}$ an extra condition, which $\mathcal{V}_p$ will satisfy. Secondly, and when fixing $\mathcal{V} = \mathcal{V}_p$, we will apply the previous result to develop an algorithm to effectively compute generators of the closure of a finitely generated subgroup of $\mathbb{F}_A$. The main references used in this chapter are [14] and [9].

4.1 On free factors

This section is dedicated to introduce and understand the notion of a free factor of a free group, key in the whole chapter, specially in the succeeding section, where we will prove stronger results. First of all, let us present its formal definition.

Definition 4.1.1. Let $\mathbb{F}_A$ be a free group, and $H \leqslant \mathbb{F}_A$ be a subgroup of $\mathbb{F}_A$. We say that H is a *free factor* of $\mathbb{F}_A$, and denote it $H \leqslant_{ff} \mathbb{F}_A$, if some basis[1] B of H can be extended to a basis $B' \supseteq B$ of $\mathbb{F}_A$. It is easy to see that, if some basis of H extends to a basis of $\mathbb{F}_A$, then any basis of H also extends to a basis of $\mathbb{F}_A$.

Example 4.1.2. Directly from the definition, it is clear that any free group is a free factor of itself, and that the trivial group is a free factor of every free group.
Taking $\mathbb{F}_{\{a,b\}}$, $H = \langle ab \rangle$, a basis of H is given by $B_H = \{ab\}$. As we can extend B_H to a basis of $\mathbb{F}_{\{a,b\}}$ by $B = \{ab, a\}$, for example, we conclude that $H = \langle ab \rangle$ is a free factor of $\mathbb{F}_{\{a,b\}}$.
On the other hand, if we consider $\mathbb{F}_{\{a\}} \cong \mathbb{Z}$, the subgroups of $\mathbb{Z}$ are those of the form $H_n = n\mathbb{Z} = \{nz : z \in \mathbb{Z}\}$, for $n \in \mathbb{N} \cup \{0\}$. So, $H_n = \langle n \rangle$, and a basis of H_n is trivially given by $B_n = \{n\}$. As the only bases of $\mathbb{Z}$ are $\{1\}$ and $\{-1\}$, we conclude that B_n can not be extended to a basis of $\mathbb{Z}$, for $n \geq 2$, and, hence, $\mathbb{Z}$ has no free factors except for the trivial ones.

A first observation we make is that $\leqslant_{ff}$ is transitive. That is, if $K \leqslant H \leqslant \mathbb{F}_A$ are groups satisfying $K \leqslant_{ff} H$, $H \leqslant_{ff} \mathbb{F}_A$, then $K \leqslant_{ff} \mathbb{F}_A$. This is clear since, we can extend a basis of K to obtain a basis of H which, in turn, we can extend to get a basis of $\mathbb{F}_A$, finally obtaining a basis of $\mathbb{F}_A$ from an initial basis of K.

[1] Definition 2.3.1.

A usual characterization of the notion of free factor lies in the concept of *free product*:

Definition 4.1.3. We say that a free group $\mathbb{F}_A$ is the *free product* of the subgroups $H, K \leqslant \mathbb{F}_A$, and denote it by $\mathbb{F}_A = H * K$, if the following conditions are satisfied:

1. Any element $g \in \mathbb{F}_A$ can be written as a finite product of elements from H and K;
2. Any finite product of non-identity elements alternating between H and K is not the identity on $\mathbb{F}_A$.

Proposition 4.1.4. *Let $\mathbb{F}_A$ be a free group, and $H \leqslant \mathbb{F}_A$ be a subgroup of $\mathbb{F}_A$. Then, H is a free factor of $\mathbb{F}_A$ if and only if there exists a subgroup $K \leqslant \mathbb{F}_A$ such that $\mathbb{F}_A = H * K$.*

Proof. If H is a free factor of $\mathbb{F}_A$, let B be a basis of H and let $B' \supseteq B$ be a basis of $\mathbb{F}_A$. Consider the subgroup K generated by the elements of $B' \backslash B$. Then, $\mathbb{F}_A = H * K$.

Conversely, if $\mathbb{F}_A = H * K$, let B be a basis of H, and consider a basis $\widetilde{B}$ of K. Then, a basis of $\mathbb{F}_A$ is given by $B' = B \cup \widetilde{B}$. □

It is easy to see that this characterization implies that every element $g \in \mathbb{F}_A,\ g \neq 1$ has a unique representation $g = h_1 k_1 \cdots h_n k_n$, where $h_1, \ldots, h_n \in H,\ k_1, \ldots, k_n \in K$, and all are non-identity elements. This observation yields the next result:

Proposition 4.1.5. *Consider a free group $\mathbb{F}_A$, and let H be a free factor of $\mathbb{F}_A$. Then, H is also a retract of $\mathbb{F}_A$.*

Proof. First, let $K \leqslant \mathbb{F}_A$ such that $\mathbb{F}_A = H * K$. Now, it is sufficient to construct an homomorphism $r \colon \mathbb{F}_A \to H$ such that $r|_H = Id_H$. In that way, given $g \in \mathbb{F}_A$, consider its (unique) representation in terms of alternating elements of H and K, that is, $g = h_1 k_1 \cdots h_n k_n$, where $h_1, \ldots, h_n \in H,\ k_1, \ldots, k_n \in K$, and all are non-identity elements. Defining $r(g) \coloneqq h_1 \cdots h_n$ we easily get that r is a well-defined homomorphism satisfying $r|_H = Id_H$, and we are done. □

Now, and in order to prove that the closure of a finitely generated subgroup of the free group is also finitely generated, we will first have to prove that any free factor of a closed subgroup of $\mathbb{F}_A$ is also closed. Even though, to prove this last result we will have to ask the pseudovariety $\mathcal{V}$ a crucial extra condition, which we will introduce in the following section.

4.2 Previous results

This intermediate section is dedicated to present some needed technical results (see [9, Sections 1-2]) that will lead us to prove the main theorem stated before, on Section 4.2.1. We begin by establishing a useful result that will help us in a more technical proof later on.

Proposition 4.2.1. *Let G be a residually–$\mathcal{V}$ group equipped with the Pro–$\mathcal{V}$ topology, and let H be a retract of G. Then, H is closed in G.*

Proof. We first note that, as G is residually$-\mathcal{V}$, d is a natural distance function, and $d(x, y) = 0$ implies $x = y$ (see Section 3.1). We also let $r \colon G \to H$ be a retraction.
Now, let $h \in Cl(H)$: by the definition of closure via a metric (Proposition 2.1.11) we have that, $\forall \varepsilon > 0$, there exists $\overline{x} = \overline{x}(\varepsilon) \in H$ such that $d(x, \overline{x}) < \varepsilon$. Using now that homomorphisms are contracting (Proposition 3.2.4), one easily obtains

$$d(r(x), r(\overline{x})) \leq d(x, \overline{x}) < \varepsilon,$$

but, as $\overline{x} \in H$, then $r(\overline{x}) = \overline{x}$, and we get that $d(r(x), \overline{x}) < \varepsilon$. Finally, using that d is an ultrametric distance,

$$d(r(x), x) \leq \max\{d(r(x), \overline{x}),\ d(\overline{x}, x)\} < \varepsilon.$$

As this is valid for every $\varepsilon > 0$ (and x does not depend on ε), we conclude that $d(r(x), x) = 0$ and, hence, $x = r(x) \in H$, which is what we wanted to see. □

Now, and as mentioned before, in this chapter we will mainly consider a special type of pseudovarieties, which is defined next.

Definition 4.2.2. We say that a pseudovariety $\mathcal{V}$ is *extension–closed* if the following statement holds: if K is a finite group and $N \trianglelefteq K$ is a normal subgroup of K satisfying $N, K/N \in \mathcal{V}$, then $K \in \mathcal{V}$.

Under this new type of pseudovarieties we can prove the following classic easy lemma [10, Lemma 2.2], that will help us later on.

Lemma 4.2.3. *Let $\mathcal{V}$ be a non–trivial extension–closed pseudovariety. Then, every free group $\mathbb{F}_A$ is residually–$\mathcal{V}$.*

Proof. If $\mathcal{V}$ is non-trivial, there exists $V \in \mathcal{V},\ V \neq \{1\}$. By Cauchy's Theorem (Theorem 2.2.17), there exists a prime number p such that $C_p \leqslant V$, so $C_p \in \mathcal{V}$.[2]
Now, let us prove that every $p-$group belongs to $\mathcal{V}$: so, consider G_r, a group with $|G_r| = p^r$, and let us proceed by induction in $r \geq 1$:
If $r = 1$, $G_1 = C_p \in \mathcal{V}$ and we are done. For $r > 1$, it is well known that there exists a composition series, namely

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{r-1} \trianglelefteq G_r,$$

with $|G_k| = p^k$. Now, $G_j \in \mathcal{V}$, for $0 \leq j \leq r-1$, and $G_{j+1}/G_j = C_p \in \mathcal{V}$. Taking $j = r-1$ and using that $\mathcal{V}$ is extension–closed we easily get that $G_r \in \mathcal{V}$, which is what we wanted.
We now have that $\mathcal{V}_p \subseteq \mathcal{V}$. Another well–known result [18, Theorem 6] states that every free group $\mathbb{F}_A$ is residually$-\mathcal{V}_p$. In our case, this directly implies that $\mathbb{F}_A$ is also residually$-\mathcal{V}$. □

Under this new type of pseudovarieties, and when $H \leqslant G$ is a clopen subgroup, the *Pro*$-\mathcal{V}$ topology on H and the restriction to H of the *Pro*$-\mathcal{V}$ topology on G coincide, as the next result shows:

Proposition 4.2.4. *Let $\mathcal{V}$ be an extension–closed pseudovariety, and let G be a group equipped with the Pro$-\mathcal{V}$ topology. If H is a clopen subgroup of G, then the Pro$-\mathcal{V}$ topology on H coincides with the restriction to H of the Pro$-\mathcal{V}$ topology on G, $\mathcal{T}_{\mathcal{V}}^H = \mathcal{T}_{\mathcal{V}}^G|_H$.*

Proof. Note that, by Proposition 3.4.6, we just have to see the inclusion $\mathcal{T}_{\mathcal{V}}^H \subseteq \mathcal{T}_{\mathcal{V}}^G|_H$. That way, let $\varphi^{-1}(v)$ be a basic open set of $\mathcal{T}_{\mathcal{V}}^H$, with $\varphi\colon H \to V \in \mathcal{V}$ an homomorphism and $v \in V$. Let's first see that we can assume $v = 1$: if $g_0 \in \varphi^{-1}(v) \subseteq H$, we have that $\varphi^{-1}(v) = g_0\varphi^{-1}(1)$. If we now define the homeomorphism[3] $\alpha\colon G \to G,\ g \mapsto g_0^{-1}g$ (note that α is continuous because of Proposition 3.2.3), then there exists $W \in \mathcal{T}_{\mathcal{V}}^G$ satisfying

$$\varphi^{-1}(v) = \alpha^{-1}(\varphi^{-1}(1)) = \alpha^{-1}(H \cap W) = \alpha^{-1}(H) \cap \alpha^{-1}(W) = H \cap \alpha^{-1}(W) \in \mathcal{T}_{\mathcal{V}}^G|_H,$$

[2] C_p denotes the cyclic group of order p.
[3] Definition 2.1.2.

where we used the fact that $g_0 \in H$ and $\alpha^{-1}(W) \in \mathcal{T}_{\mathcal{V}}^{G}$. This proves that, if the argument holds for $v = 1$, then we can extend it to any $v \in V$. So, we just have to prove that $U := \varphi^{-1}(1) \in \mathcal{T}_{\mathcal{V}}^{G}|_H$ but, as $\varphi^{-1}(1) = \varphi^{-1}(1) \cap H$, it is sufficient to prove $\varphi^{-1}(1) \in \mathcal{T}_{\mathcal{V}}^{G}$. Using Theorem 3.4.2, we will see that $U = ker(\varphi) \trianglelefteq H$ has finite index in G and that $G/U_G \in \mathcal{V}$. Indeed, U has finite index in H and, by hypothesis, H has finite index in G, so U has finite index in G. Now, $U_G \trianglelefteq H_G \trianglelefteq G$ and, by the third Isomorphism Theorem,

$$G/U_G \Big/ H_G/U_G \cong G/H_G \in \mathcal{V},$$

again by hypothesis. Note that, as $\mathcal{V}$ is extension–closed, it is sufficient to see that $H_G/U_G \in \mathcal{V}$. Now, H has finite index in G, so there exist $g_1, \ldots, g_r \in G$ such that $H_G = \bigcap_{i=1}^{r} g_i^{-1} H g_i$. The same way, U has finite index in H, so there exist $h_1, \ldots, h_s \in H$ such that

$$U_G = \bigcap_{\substack{i=1,\ldots,r\\ j=1,\ldots,s}} g_i^{-1} h_j^{-1} U h_j g_i = \bigcap_{i=1}^{r} g_i^{-1} U g_i,$$

where we used that $U = ker(\varphi) \trianglelefteq H$. Finally, consider the homomorphism

$$\begin{array}{rcl}\sigma\colon H_G = \bigcap_{i=1}^{r} g_i^{-1} H g_i & \longrightarrow & g_1^{-1} H g_1 \big/ g_1^{-1} U g_1 \times \cdots \times g_r^{-1} H g_r \big/ g_r^{-1} U g_r \cong (H/U)^r \\ h & \longmapsto & ([h]_1, \ldots, [h]_r),\end{array}$$

where $[h]_i$ denotes the class of h modulo $g_i^{-1} U g_i$. It is clear that $ker(\sigma) = U_G$, so

$$H_G/U_G = H_G/ker(\sigma) \cong Im(\sigma) \leqslant (H/U)^r \in \mathcal{V}, \tag{1}$$

where we used the first Isomorphism Theorem and the fact that $H/U \cong Im(\varphi) \leqslant V \in \mathcal{V}$ implies $H/U \in \mathcal{V}$, and so $(H/U)^r \in \mathcal{V}$. In turn, we finally get from (1) that $H_G/U_G \in \mathcal{V}$, which is what we wanted to see. □

Before establishing the main results of the section, here we introduce a basic property involving automaton morphisms (Definition 2.3.18). The results introduced in the preliminaries presenting *Stallings* automata theory will be key from now on.

Lemma 4.2.5. *Let $\varphi\colon \Gamma \to \Gamma'$ be an automaton morphism. Then, if φ is one–to–one, $\langle \Gamma \rangle$ is a free factor of $\langle \Gamma' \rangle$. Moreover, if $\Gamma \subseteq \Gamma'$, then φ is one–to–one.*

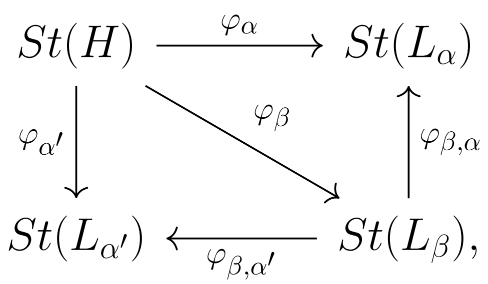
Proof. As φ is one–to–one, any spanning tree of Γ can be extended to a spanning tree of Γ'. From the basis construction described on Proposition 2.3.17 we then get that a basis B of Γ can be extended to a basis $B' \supseteq B$ of Γ' and, hence, $\langle \Gamma \rangle \leqslant_{ff} \langle \Gamma' \rangle$.
The final statement directly follows from the definition of automaton morphism and its uniqueness (see Proposition 2.3.19). □

Proposition 4.2.6. *Let $\mathcal{V}$ be an extension–closed pseudovariety, and consider the free group $\mathbb{F}_A$, equipped with the Pro–$\mathcal{V}$ topology. Now, a finitely generated subgroup $H \leqslant \mathbb{F}_A$ is closed if and only if there exists some clopen subgroup $K \leqslant \mathbb{F}_A$ such that $H \leqslant_{ff} K$. Moreover, one has that $\mathcal{T}_{\mathcal{V}}^{H} = \mathcal{T}_{\mathcal{V}}^{\mathbb{F}_A}|_H$.*

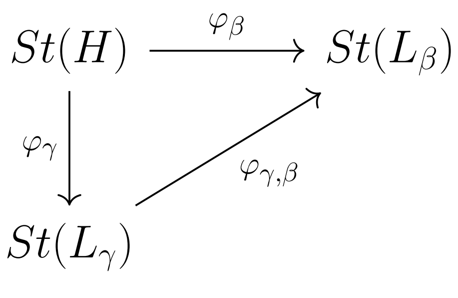
Proof. Firstly, as H is closed we have that (Corollary 3.4.5)

$$H = \bigcap_{\substack{L_\alpha \text{ clopen}\\ H \leqslant L_\alpha \leqslant \mathbb{F}_A}} L_\alpha. \tag{1}$$

We first observe that, if L_α and L_β are both clopen subgroups of the intersection in (1), then $L_\alpha \cap L_\beta$ is again a clopen subgroup intersecting in (1): indeed, the finite intersection of clopen subgroups is clearly also a clopen subgroup (as the intersection of subgroups is again a subgroup, and the finite intersection of clopen sets is also a clopen set, via the definition of topology), and if $H \leqslant L_\alpha$, $H \leqslant L_\beta$, then also $H \leqslant L_\alpha \cap L_\beta$, so the result follows.
Now, consider, for every α, the (unique) automaton morphism $\varphi_\alpha \colon St(H) \to St(L_\alpha)$. We claim that, for every $p, q \in V(St(H))$, $p \neq q$, such that $\varphi_\alpha(p) = \varphi_\alpha(q)$ there exists an $L_{\alpha'}$ such that $\varphi_{\alpha'}(p) \neq \varphi_{\alpha'}(q)$. Indeed, $p \neq q$ implies that $u_p u_q^{-1} \notin H$, where u_p denotes the label of a reduced path starting at the base point of $St(H)$, and ending at p (this is because of the determinism of *Stallings* automata). Now, there must exist α' such that $u_p u_q^{-1} \notin L_{\alpha'}$, which implies that $\varphi_{\alpha'}(p) \neq \varphi_{\alpha'}(q)$, again by the determinism of *Stallings automata*. Now, consider the intersection $L_{\alpha'} \supseteq L_\alpha \cap L_{\alpha'} =: L_\beta$ (that must be again an element from (1) by the initial observation we made), and let us now see that $\varphi_\beta(p) \neq \varphi_\beta(q)$ too. Indeed, we have the following diagram (note that it commutes because of the uniqueness of automata morphisms and because the composition of automata morphisms is again an automaton morphism):



As $\varphi_{\beta,\alpha'}(\varphi_\beta(p)) = \varphi_{\alpha'}(p) \neq \varphi_{\alpha'}(q) = \varphi_{\beta,\alpha'}(\varphi_\beta(q))$, it must be $\varphi_\beta(p) \neq \varphi_\beta(q)$.
Note that, as the intersecting groups L_α are open, they have finite index and, hence, they are finitely generated. This directly implies that $St(L_\alpha)$ is finite, $\forall \alpha$. Note also that we can iterate the previous process: if $St(L_\beta)$ has two vertices p' and q' with $p' \neq q'$ and $\varphi_\beta(p') = \varphi_\beta(q')$, we can find γ such that $L_\gamma \subseteq L_\beta$ and $\varphi_\gamma(p') \neq \varphi_\gamma(q')$. Note also that, still, $\varphi_\gamma(p) \neq \varphi_\gamma(q)$ (see the commutative diagram below).



A similar argument as before yields $\varphi_{\gamma,\beta}(\varphi_\gamma(p)) = \varphi_\beta(p) \neq \varphi_\beta(q) = \varphi_{\gamma,\beta}(\varphi_\gamma(q))$, so $\varphi_\gamma(p) \neq \varphi_\gamma(q)$. Iterating, we will eventually end up with a clopen subgroup $L_\delta \leqslant \mathbb{F}_A$ of the intersection (1) satisfying that $\varphi_\delta \colon St(H) \hookrightarrow St(L_\delta)$ is injective (note that we will obtain L_δ in finite time as $St(L_\alpha)$ is finite, $\forall \alpha$). By last lemma, H is a free factor of $K := L_\delta$.

Conversely, and as K is a subgroup of $\mathbb{F}_A$, it is also free and, by Lemma 4.2.3, it is residually$-\mathcal{V}$. Also, from Proposition 4.1.5 we get that H is a retract of K. We are now under the conditions of applying Proposition 4.2.1 and conclude that H is closed in the *Pro*$-\mathcal{V}$ topology on K, $\mathcal{T}_\mathcal{V}^K$. By Proposition 4.2.4, $\mathcal{T}_\mathcal{V}^K = \mathcal{T}_\mathcal{V}^{\mathbb{F}_A}|_K$, so H is also closed in

the induced topology on $\mathbb{F}_A$ by K. Finally, and using that K is a clopen subgroup of $\mathbb{F}_A$ and, hence, open, we get that H is closed on $\mathcal{T}_{\mathcal{V}}^{\mathbb{F}_A}$ (Proposition 2.1.15).
If H is a closed subgroup and, hence, there exists a clopen subgroup $K \leqslant \mathbb{F}_A$ such that $H \leqslant_{ff} K$, by Proposition 4.1.5, H is also a retract of K. Now, applying Proposition 3.4.9 we get that $\mathcal{T}_{\mathcal{V}}^{H} = \mathcal{T}_{\mathcal{V}}^{K}|_H$. Again, by Proposition 4.2.4, $\mathcal{T}_{\mathcal{V}}^{K} = \mathcal{T}_{\mathcal{V}}^{\mathbb{F}_A}|_K$, and so $\mathcal{T}_{\mathcal{V}}^{K}|_H = \left(\mathcal{T}_{\mathcal{V}}^{\mathbb{F}_A}|_K\right)|_H = \mathcal{T}_{\mathcal{V}}^{\mathbb{F}_A}|_H$ which finally leads us to the desired result. □

Now, the result we wanted to prove in the previous section is a mere corollary from last proposition.

Theorem 4.2.7. *Let $\mathcal{V}$ be an extension–closed pseudovariety, and consider the free group $\mathbb{F}_A$, equipped with the Pro$-\mathcal{V}$ topology. Let H be a closed finitely generated subgroup of $\mathbb{F}_A$, and let $K \leqslant_{ff} H$ be any free factor of H. Then, K is also closed under $\mathcal{T}_{\mathcal{V}}^{\mathbb{F}_A}$.*

Proof. By the last proposition, and as H is closed, there exists a clopen subgroup $\widetilde{H} \leqslant \mathbb{F}_A$ such that $H \leqslant_{ff} \widetilde{H}$. Now, by the transitivity of $\leqslant_{ff}$ we have that $K \leqslant_{ff} \widetilde{H}$ and, using again Proposition 4.2.6, we get that K is closed, which is what we wanted to see. □

4.2.1 Closure of a finitely generated subgroup of $\mathbb{F}_A$

All the results we proved in the past sections directed us to the theorem we are about to prove. With it, we will be able to ensure that the closure of a finitely generated subgroup of the free group is also finitely generated. This result will make things much easier in the following sections, when we will develop an algorithm to compute generators of the closure of a finitely generated subgroup of $\mathbb{F}_A$. The theorem derives from [14, Proposition 3.4].

Theorem 4.2.8. *Let $\mathcal{V}$ be an extension–closed pseudovariety, and consider the free group $\mathbb{F}_A$ equipped with the Pro$-\mathcal{V}$ topology. Now, let H be a finitely generated subgroup of $\mathbb{F}_A$. Then, $Cl(H)$ is also a finitely generated subgroup of $\mathbb{F}_A$.*

Proof. As $H \leqslant \mathbb{F}_A$, we know by Corollary 3.4.4 that $Cl(H)$ is also a subgroup of $\mathbb{F}_A$. Then, by Proposition 2.3.19 there exists an automaton morphism $\varphi\colon St(H) \to St(Cl(H))$. Note that, by Proposition 2.3.17, we just have to see that $St(Cl(H))$ is finite. As $St(H)$ is finite (because H is finitely generated), it is sufficient to see that φ is exhaustive. Indeed, consider the *Stallings* automaton defined by the image of φ, $\Delta \coloneqq Im(\varphi)$. By the definition of $\Delta \subseteq St(Cl(H))$, the (unique) automaton morphism between Δ and $St(Cl(H))$ is one–to–one and, $\langle\Delta\rangle \leqslant_{ff} Cl(H)$ (Lemma 4.2.5). We have now

$$H \leqslant \langle\Delta\rangle \leqslant_{ff} Cl(H),$$

but $\langle\Delta\rangle$ is a free factor of a closed subgroup and, by Theorem 4.2.7 it is also closed in the *Pro*$-\mathcal{V}$ topology on $\mathbb{F}_A$. By the definition of topological closure we finally conclude that $Cl(H) = \langle\Delta\rangle$ which directly implies $St(Cl(H)) = \Delta$, and φ is exhaustive. □

4.3 Deciding p–denseness

From now on, we will only be considering $\mathcal{V}_p$, the pseudovariety of all finite $p-$groups (where p is a prime number). Let us begin the section by showing that $\mathcal{V}_p$ is extension–closed, so all the previous results will also apply here.

Lemma 4.3.1. *Let p be a prime number. Then, $\mathcal{V}_p$ is extension–closed.*

Proof. Let G, N, H be finite groups such that $N \trianglelefteqslant G$, $H = G/N$ and $N, H \in \mathcal{V}_p$. We then have that $|N| = p^r$, $|H| = p^s$, for some natural numbers r, s and, by a direct corollary of *Lagrange's* Theorem (see Definition 2.2.8), $|G| = |H| \cdot |N| = p^{r+s}$, so G is also a $p-$group, $G \in \mathcal{V}_p$ and $\mathcal{V}_p$ is extension–closed. Note that we could apply *Lagrange's* Theorem as G, N and H were all finite groups. □

We now present the following technical lemma that will help us in a proof of an important result below:

Lemma 4.3.2. *Let p be a prime number, and consider the free group $\mathbb{F}_A$ equipped with the Pro–$\mathcal{V}_p$ topology. Now, let H be a clopen proper subgroup of $\mathbb{F}_A$. Then, there exists an exhaustive homomorphism $\varphi\colon \mathbb{F}_A \to \mathbb{Z}/p\mathbb{Z}$ such that $H \subseteq ker(\varphi)$.*

Proof. As H is a clopen subgroup of $\mathbb{F}_A$, $\mathbb{F}_A/H_{\mathbb{F}_A} \in \mathcal{V}_p$, so $|\mathbb{F}_A/H_{\mathbb{F}_A}| = p^r$, for some r. Considering the canonical projection $\pi\colon \mathbb{F}_A \twoheadrightarrow \mathbb{F}_A/H_{\mathbb{F}_A}$ and using Theorem 3.4.2, one has that $H = \pi^{-1}(\pi(H))$ and, since H is a proper subgroup of $\mathbb{F}_A$, $\pi(H)$ is also a proper subgroup of $\mathbb{F}_A/H_{\mathbb{F}_A}$. Now let N be a maximal proper subgroup of $\mathbb{F}_A/H_{\mathbb{F}_A}$ containing $\pi(H)$. We then have that $|N| = p^{r-1}$, $\pi(H) \leqslant N$ and that, in fact, $N \trianglelefteqslant \mathbb{F}_A/H_{\mathbb{F}_A}$ (this is because $\mathbb{F}_A/H_{\mathbb{F}_A}$ is a $p-$group). The following homomorphism

$$\varphi\colon \mathbb{F}_A \overset{\pi}{\twoheadrightarrow} \mathbb{F}_A/H_{\mathbb{F}_A} \overset{\pi_N}{\twoheadrightarrow} \mathbb{F}_A/H_{\mathbb{F}_A}\Big/N \cong \mathbb{Z}/p\mathbb{Z},$$

is exhaustive and satisfies $H \subseteq ker(\varphi)$. Note that $\mathbb{F}_A/H_{\mathbb{F}_A}\Big/N \cong \mathbb{Z}/p\mathbb{Z}$ as $\left|\mathbb{F}_A/H_{\mathbb{F}_A}\Big/N\right| = |\mathbb{F}_A/H_{\mathbb{F}_A}|\Big/|N| = p$, and $\mathbb{Z}/p\mathbb{Z}$ is the only group (modulo isomorphism) of order p. □

Given a prime number p, an alphabet A and the free group $\mathbb{F}_A$, consider the "natural" onto homomorphism

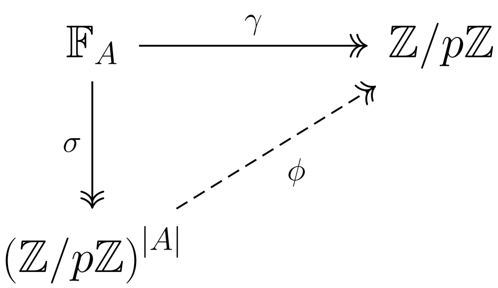
$$\sigma\colon \mathbb{F}_A \overset{\varphi}{\twoheadrightarrow} \mathbb{Z}^{|A|} \overset{\pi_p}{\twoheadrightarrow} \mathbb{Z}^{|A|}/p\mathbb{Z}^{|A|} \cong (\mathbb{Z}/p\mathbb{Z})^{|A|},$$

where π_p is the canonical projection and, given a reduced word $u = a_1^{r_1} \cdots a_n^{r_n} \in \mathbb{F}_A$, with $a_i \in A$, $r_i \in \mathbb{Z}$, $\forall i$, φ "counts" the number of appearances of each letter $a_i \in A$ (this is, in fact, the abelianization map). Here we give some examples with $p = 5$ and $A = \{a, b, c\}$:

- $\sigma(a^3ba^2c^{-1}) = \pi_5(5, 1, -1) = (0, 1, 4)$;
- $\sigma(a^2ba^{-2}cb^{-1}) = \pi_5(0, 0, 1) = (0, 0, 1)$;
- $\sigma(cb^{-3}a^4b) = \pi_5(4, -2, 1) = (4, 3, 1)$;
- $\sigma(cb^7) = \pi_5(0, 7, 1) = (0, 2, 1)$.

It is easy to see that φ is indeed an onto homomorphism, so $\sigma = \pi_p \circ \varphi$ is well–defined and exhaustive. We can now establish the following technical lemma that will help us in the proof of the main theorem of the section below:

Lemma 4.3.3. *Every non–trivial homomorphism $\gamma\colon \mathbb{F}_A \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ factorizes via σ. That is, there exists an homomorphism $\phi\colon\ (\mathbb{Z}/p\mathbb{Z})^{|A|} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ such that $\gamma = \phi \circ \sigma$.*



Proof. We first see that, given $x, y \in \mathbb{F}_A$, $\gamma(xyx^{-1}y^{-1}) = 0$, so the commutator of $\mathbb{F}_A$, $[\mathbb{F}_A, \mathbb{F}_A]$, is contained in $ker(\gamma)$. Now, we define

$$\begin{array}{rcl} \phi' \colon \mathbb{Z}^{|A|} \cong \mathbb{F}_A/[\mathbb{F}_A, \mathbb{F}_A] & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \\ [w] & \longmapsto & \gamma(w). \end{array}$$

We have that ϕ' is well defined because, if w and w' are elements of $\mathbb{F}_A$ satisfying $[w] = [w']$, then $w'w^{-1} \in [\mathbb{F}_A, \mathbb{F}_A] \subseteq ker(\gamma)$, which directly implies that $\gamma(w) = \gamma(w')$. Moreover, and as $\phi'(pw) = p\phi'(w) = 0$, we also get that $p\mathbb{Z}^{|A|} \subseteq ker(\phi')$. Finally, defining

$$\begin{array}{rcl} \phi\colon \left(\mathbb{Z}/p\mathbb{Z}\right)^{|A|} \cong \mathbb{Z}^{|A|}/p\mathbb{Z}^{|A|} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \\ \{v\} & \longmapsto & \phi'(v), \end{array}$$

where $\{u\}$ denotes the class of u, we have that, if v and v' are elements of $\mathbb{Z}^{|A|}$ satisfying $\{v\} = \{v'\}$, then $v'v^{-1} \in p\mathbb{Z}^{|A|} \subseteq ker(\phi')$, which implies $\phi'(v) = \phi'(v')$ and ϕ is well–defined. Now,

$$\phi(\sigma(w)) = \phi(\pi_p(\varphi(w))) = \phi'(\varphi(w)) = \gamma(w),$$

which is what we wanted to see, and we are done. □

Theorem 4.3.4 ([9, Corollary 3.5]). *Let p be a prime number, and consider the free group $\mathbb{F}_A$ equipped with the Pro$-\mathcal{V}_p$ topology. Now let $H \leqslant \mathbb{F}_A$ be a finitely generated subgroup. Then,*

$$H \text{ is } p-\text{dense} \iff \sigma(H) = \left(\mathbb{Z}/p\mathbb{Z}\right)^{|A|}.$$

Proof. For the direct implication, let H be $p-$dense and suppose that $\sigma(H)$ is a proper subgroup of $\left(\mathbb{Z}/p\mathbb{Z}\right)^{|A|}$. Then, $\sigma^{-1}(\sigma(H))$ is also a proper subgroup of $\mathbb{F}_A$, and

$$\sigma^{-1}(\sigma(H)) = \bigcup_{g \in \sigma(H)} \sigma^{-1}(g) = \left(\bigcup_{g \notin \sigma(H)} \sigma^{-1}(g) \right)^c,$$

so $(\sigma^{-1}(\sigma(H)))^c$ is open and, hence, $\sigma^{-1}(\sigma(H))$ is closed. We then have $H \leqslant \sigma^{-1}(\sigma(H))$, and as $\sigma^{-1}(\sigma(H))$ is a closed proper subgroup of $\mathbb{F}_A$ we get a contradiction, since H was dense.

Conversely, as $\sigma(H) = \left(\mathbb{Z}/p\mathbb{Z}\right)^{|A|}$, using last lemma we get that $\gamma(H) = \mathbb{Z}/p\mathbb{Z}$, for every non–trivial homomorphism $\gamma\colon \mathbb{F}_A \to \mathbb{Z}/p\mathbb{Z}$. Now, by Theorem 3.4.3, we have that

$$Cl(H) = \bigcap_{\substack{L \text{ clopen} \\ H \leqslant L}} L. \tag{1}$$

Suppose that there exists L intersecting in (1) such that L is a proper subgroup of $\mathbb{F}_A$. Then, by Lemma 4.3.2, there must exist a non–trivial homomorphism φ such that $L \subseteq ker(\varphi)$. But, then $H \leqslant Cl(H) \leqslant L \leqslant ker(\varphi)$, contradicting the fact that $\gamma(H) = \mathbb{Z}/p\mathbb{Z}$, for every non–trivial homomorphism γ. So, every L in (1) is equal to $\mathbb{F}_A$ and H is $p-$dense. □

To practically decide whether $\sigma(H) = (\mathbb{Z}/p\mathbb{Z})^{|A|}$ or not, we have the following easy corollary that directly follows from last theorem:

Corollary 4.3.5 ([9, Corollary 3.6]). *Let H be a finitely generated subgroup of the free group $\mathbb{F}_A$, and let $h_1, \ldots, h_r$ be a set of generators of H. Considering the matrix $M_p(H) = (\sigma_j(h_i))_{\substack{i=1,\ldots,r \\ j=1,\ldots,|A|}} \in \mathcal{M}^{r\times|A|}(\mathbb{Z}/p\mathbb{Z})$, we have that*

$$H \text{ is } p-dense \iff rk(M_p(H)) = |A|.$$

Note that we have found a characterization of $p-$denseness for subgroups of the free group in terms of a basic linear algebra problem. We will strongly use this result in the next section, eventually ending with a full algorithm to compute generators of the closure of a finitely generated subgroup of the free group (in the *Pro*$-\mathcal{V}_p$ topology, of course).

4.4 The algorithm

Here we describe the algorithm proposed by Margolis, Sapir and Weil in [9, Section 3.2] to compute generators of the closure of finitely generated subgroups of the free group, along with some examples. The described algorithm is a slight modification of a first version proposed by Ribes and Zalesskii in [14, Section 4]. Even though we will not discuss the complexity of the algorithm here, Margolis et al. showed [9, Section 3.3] that, if n is the sum of the lengths of the words of a basis of a finitely generated subgroup $H \leqslant \mathbb{F}_A$, then its algorithm terminates in polynomial time with respect to n, a considerable progress compared to the first version from Ribes and Zalesskii.

First note that the idea of trying to compute generators of the closure of a finitely generated subgroup has now sense because of Theorem 4.2.8.
Now, we suppose here that we are given a finitely generated subgroup of the free group $H \leqslant \mathbb{F}_A$, with basis $B^H = \{u_1, \ldots, u_n\}$, its *Stallings* automaton, $St(H)$, and, for every $p \in V(St(H))$, a reduced label of a path starting at $\odot \in V(St(H))$ and ending at p, u_p. We also set $H_0 \coloneqq \mathbb{F}_A$ (and we trivially know $St(H_0)$ and a basis of H_0, $B^{H_0} = A$). From now on, we will also denote by $Cl(H)^G$ the closure of H under $\mathcal{T}^G_{\mathcal{V}_p}$, and by $Cl(H)^G_K$ the closure of H under $\mathcal{T}^G_{\mathcal{V}_p}|_K$. The algorithm is as follows:

[1] Starting at $i = 0$, we first compute a basis B^{H_i} of H_i using Proposition 2.3.17. We also let A_i be a set in bijection with that basis, and $\kappa_i\colon \mathbb{F}_{A_i} \to \mathbb{F}_A \supseteq H_i$ the natural one–to–one homomorphism (that maps each letter $a_j \in A_i$ to the corresponding element of the basis B^{H_i} according to the bijective relation between B^{H_i} and A_i).

[2] We rewrite the basis B^H in terms of B^{H_i}. This is equivalent to computing a basis of the subgroup $\kappa_i^{-1}(H)$ of $\mathbb{F}_{A_i}$. Using again Proposition 2.3.17, this is done by running the elements of B^H in $St(H_i)$: we start in $\odot$ and, when we find an edge, e, not in the considered spanning tree of $St(H_i)$, T, we go through it and return to $\odot$ (with the unique reduced path, γ_e, inside T). This is an element of B^{H_i}. We then return to e via γ_e^{-1} and continue the process (note that we also finish in $\odot$, as desired, because $H \leqslant H_i$, see next steps). Finally, we convert the combination of elements of B^{H_i} into a combination of elements of A_i, again via the bijective correspondence between the two sets.

[3] Note that $Im(\kappa_i) = H_i$ and, hence, $\kappa_i\colon \mathbb{F}_{A_i} \to H_i$ is an homeomorphism[4]. Then, H is dense in H_i if and only if $k_i^{-1}(H)$ is dense in $\mathbb{F}_{A_i}$. That way, and considering

[4]Definition 2.1.2

$\sigma_i\colon \mathbb{F}_{A_i} \to (\mathbb{Z}/p\mathbb{Z})^{|A_i|}$ as defined in Section 4.3, we can easily decide whether H is $p-$dense in H_i using Corollary 4.3.5. As H_i will always be a closed set (see next steps), by Proposition 4.2.6, if H is $p-$dense in H_i, then $Cl(H)^{\mathbb{F}_A}_{H_i} = Cl(H)^{H_i} = H_i$, and it is easy to see that $Cl(H)^{\mathbb{F}_A} = H_i$ too.

[4] If H is not $p-$dense in H_i, then we compute H_{i+1} as follows: By Theorem 4.3.4, $\sigma_i^{-1}(\sigma_i(\kappa_i^{-1}(H)))$ is a closed subgroup properly contained in $\mathbb{F}_{A_i}$ and, as κ_i is an homeomorphism, $K \coloneqq \kappa_i(\sigma_i^{-1}(\sigma(\kappa_i^{-1}(H))))$ is also a closed subgroup properly contained in H_i (closed, of course, under $\mathcal{T}^{H_i}_{\mathcal{V}_p}$, and containing H). Applying Proposition 4.2.6 we get that K is also closed under $\mathcal{T}^{\mathbb{F}_A}_{\mathcal{V}_p}|_{H_i}$ and, hence, under $\mathcal{T}^{\mathbb{F}_A}_{\mathcal{V}_p}$ too. We have then $H \leqslant K < H_i$.
Unfortunately, there is no clear way to find a basis of K, as we do not have $St(K)$. For that reason, we will define H_{i+1} via its *Stallings* automaton, constructed using $St(H)$ and the definition of K. That way, we define $St(H_{i+1})$ via the following process:

1. First set $\Gamma \coloneqq St(H)$.
2. Now, for every pair of vertices $p, q \in V(\Gamma)$, if $u_p u_q^{-1} \in K$, we then identify the vertices p and q in Γ. Note that this takes finite time as $St(H)$ is finite.
3. Finally, we define $St(H_{i+1})$ as the *Stallings* automaton obtained from the previous step doing the corresponding folldings (note that $St(H_{i+1})$ is computed in finite time and that, now, H_{i+1} is completely determined, modulo isomorphism). Note also that, with this construction, every reduced closed path (starting and ending in $\odot$) of $St(H)$ is also a reduced closed path (starting and ending in $\odot$) of $St(H_{i+1})$, so $H \leqslant H_{i+1}$. By construction, we also have that $H_{i+1} \leqslant K$.

Now, let φ be the (unique) automaton morphism between $St(H_{i+1})$ and $St(K)$, and let $p, q \in V(St(H_{i+1}))$ such that $\varphi(p) = \varphi(q)$. Then, as automata morphisms preserve labels, $u_p u_q^{-1}$ translates to a label of a reduced closed path, starting and ending in $\odot$, of $St(K)$, so $u_p u_q^{-1} \in K$ and, hence $p = q$ (we must have identified the two vertices in step 2 of the previous process). This proves that φ is injective and, hence (Lemma 4.2.5) H_{i+1} is a free factor of K. As K was a closed set, H_{i+1} is also a closed set (via Theorem 4.2.7). Note that we have $H \leqslant H_{i+1} \leqslant K < H_i \leqslant \mathbb{F}_A$, $\forall i$. Finally, we return to step [1].

Remark 4.4.1. Note that, in step [4] of the algorithm, to construct $St(H_{i+1})$ we needed to verify whether $u_p u_q^{-1} \in K = \kappa_i(\sigma_i^{-1}(\sigma_i(\kappa_i^{-1}(H))))$. We have that $u_p u_q^{-1} \in K$ if and only if $u_p u_q^{-1} \in H_i$ and $\sigma_i(\kappa_i^{-1}(u_p u_q^{-1})) \in \sigma_i(\kappa_i^{-1}(H))$. To check if $u_p u_q^{-1} \in H_i$ we run $u_p u_q^{-1}$ in $St(H_i)$, requiring the path to start and end at $\odot$. To compute $\sigma_i(\kappa_i^{-1}(u_p u_q^{-1}))$ in that case, we first express $\kappa_i^{-1}(u_p u_q^{-1})$ in terms of elements of A_i (see step [2]), and then apply σ_i. As we can easily compute a basis of $\sigma_i(\kappa_i^{-1}(H))$ (using the basis of $\kappa_i^{-1}(H)$, see again step [2]), it is now easy to verify whether $\sigma_i(\kappa_i^{-1}(u_p u_q^{-1})) \in \sigma_i(\kappa_i^{-1}(H))$ (as we are now in the known vector field $(\mathbb{Z}/p\mathbb{Z})^{|A_i|}$).

Remark 4.4.2. Note that, as $Cl(H)^{\mathbb{F}_A}$ is the smallest closed subgroup that contains H, we have that $Cl(H)^{\mathbb{F}_A} \leqslant H_i$, $\forall i$. A natural question that can now arise is: does the process described above always end? The answer has to be positive and, of course, it is. The reason of it lies on the geometrical interpretation of the subgroups H_i in terms of its corresponding *Stallings* automata, $St(H_i)$.

The *Stallings* automaton of H_i, $St(H_i)$, has been constructed identifying vertices of $St(H)$ (and, then, doing the corresponding foldings). As $St(H)$ is finite (because H is finitely generated), there is a finite number of automata determined by doing vertex identifications on $St(H)$, so there is a finite number of possible H_i (because they are all different, $H_{i+1} \subsetneq H_i$). This concludes the argument and the algorithm described above always terminates, as desired.

Remark 4.4.3. Now we know that the algorithm always terminates, but it is not difficult to see that the above description clearly implies that $\exists j$ such that $Cl(H)^{\mathbb{F}_A} = H_j$. What if we kept computing H_i's and there was no H_j such that H was $p-$dense in H_j? The answer to that question lies in that, if H is not $p-$dense in any of the computed H_i's, we will then eventually reach an H_j constructed by identifying no vertices at all (because $H_{i+1} \subsetneq H$), so $St(H) = St(H_j)$. This implies $H = H_j$ (Theorem 2.3.16) but, as H_j is a closed set, then H is also a closed set, $Cl(H)^{\mathbb{F}_A} = H = H_j$ and H is, indeed, $p-$dense in $H_j = H$.

To finish the section, we present here two interesting examples of finitely generated subgroups of the free group that practically show how does the previous algorithm work.

Example 4.4.4. Consider the free group $\mathbb{F}_{\{a,b\}}$ and the finitely generated subgroup $H \leqslant \mathbb{F}_{\{a,b\}}$ determined by the following *Stallings* automaton:

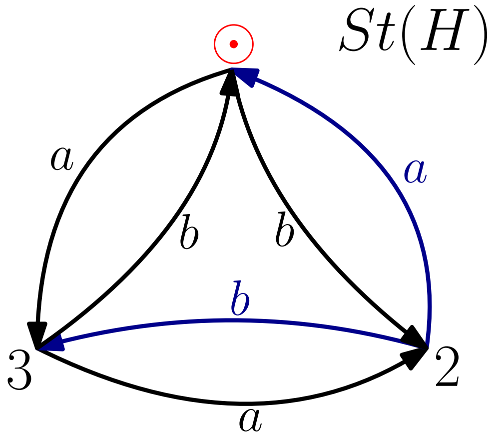


Figure 4.1: *Stallings* automaton of the subgroup H. A spanning tree of $St(H)$ is indicated in blue.

We have that $H = \langle a^{-1}b^{-1},\ a^{-2}b^{-1}a,\ a^{-1}b^2,\ a^{-1}ba^{-1}\rangle$. Note also that $\kappa_0 = Id_{\mathbb{F}_{\{a,b\}}}$, so $\sigma_0(\kappa_0^{-1}(H)) = \sigma_0(H)$ and, using Corollary 4.3.5 we have that

$$M_p(\kappa_0^{-1}(H)) = M_p(H) = \begin{pmatrix} -1 & -1 \\ -1 & -1 \\ -1 & 2 \\ -2 & 1 \end{pmatrix}.$$

This matrix is easily seen to have rank 2 if $p \neq 3$, so $Cl(H)^{\mathbb{F}_{\{a,b\}}} = \mathbb{F}_{\{a,b\}}$ for $p \neq 3$. If $p = 3$, then $rk(M_3(H)) = 1$, and we have to compute H_1. One has that $\sigma_0(\kappa_0^{-1}(H)) = \langle(1,1)\rangle$. Now, taking the labels of the reduced paths, u_p, using the given spanning tree of $St(H)$, we have that $u_\odot = 1,\ u_2 = a^{-1},\ u_3 = a^{-1}b$. Firstly, it is clear that $u_p u_q^{-1} \in H_0 = \mathbb{F}_{\{a,b\}},\ \forall p, q$. Now, it is not difficult to see that, for all possible combinations of distinct vertices $p \neq q$, $\sigma_0(\kappa_0^{-1}(u_p u_q^{-1})) = \sigma_0(u_p u_q^{-1})$ does not belong to the subspace $\langle(1,1)\rangle$. Indeed, we have that $\sigma_0(u_\odot u_2^{-1}) = (1,0),\ \sigma_0(u_\odot u_3^{-1}) = (1,2),\ \sigma_0(u_2 u_3^{-1}) = (0,2) \notin \langle(1,1)\rangle$.

So we do not identify any vertices here and $St(H_1) = St(H)$, $H_1 = H$. As H_1 is a closed set, we directly conclude that, for $p = 3$, $Cl(H)^{\mathbb{F}_{\{a,b\}}} = H_1 = H$.

The following example consists of a slight modification of [9, Example 3.9], to make it simpler:

Example 4.4.5 ([9, Example 3.9])**.** Consider again the free group $\mathbb{F}_{\{a,b\}}$ and the finitely generated subgroup $H \leqslant \mathbb{F}_{\{a,b\}}$ defined by the following *Stallings* automaton:

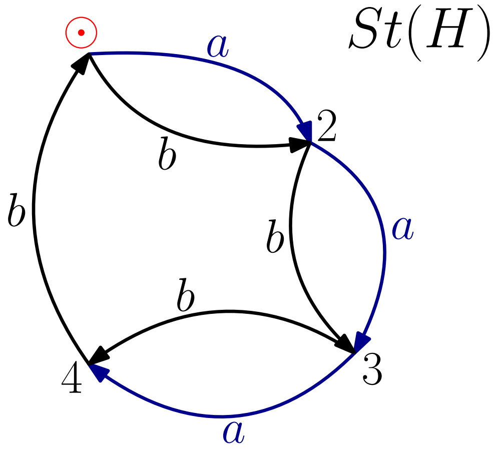


Figure 4.2: *Stallings* automaton of the subgroup H. A spanning tree of $St(H)$ is indicated in blue.

We have that $H = \langle ab^{-1},\ a^2b^{-1}a^{-1},\ a^3b^{-1}a^{-2},\ a^3b\rangle$, and

$$M_p(\kappa_0^{-1}(H)) = M_p(H) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \\ 3 & 1 \end{pmatrix}.$$

This matrix is easily seen to have rank 2 if $p \neq 2$, so $Cl(H)^{\mathbb{F}_{\{a,b\}}} = \mathbb{F}_{\{a,b\}}$ for $p \neq 2$. If $p = 2$, then $rk(M_2(H)) = 1$, and we have to compute H_1. One has that $\sigma_0(\kappa_0^{-1}(H)) = \sigma_0(H) = \langle(1,1)\rangle$ and, taking the labels of the reduced paths, u_p, using the spanning tree of $St(H)$, we have that

$\boldsymbol{p}$	$\odot$	2	3	4
$\boldsymbol{u_p}$	1	a	a^2	a^3

It is easy to see that $\sigma_0(u_\odot u_3^{-1}), \sigma_0(u_2u_4^{-1}) = (0,0) \in \langle(1,1)\rangle$, $\sigma_0(u_\odot u_2^{-1}) = (1,0) \notin \langle(1,1)\rangle$ (and it is clear that $u_pu_q^{-1} \in H_0,\ \forall p,q$), so we only identify the vertices $\odot$, 3 and 2, 4. We can now easily compute the *Stallings* automaton of H_1, which is represented in the figure below:

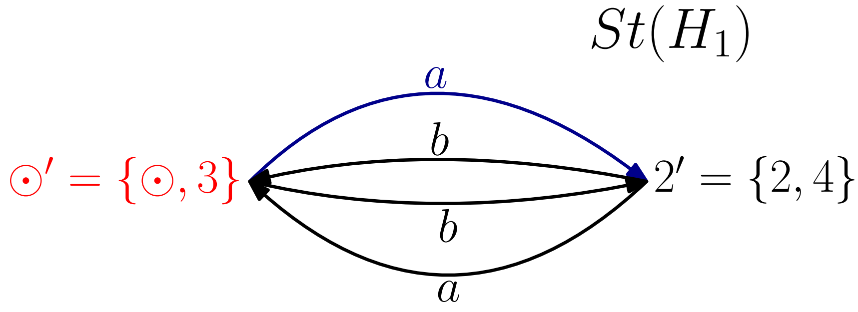


Figure 4.3: *Stallings* automaton of the subgroup H_1. A spanning tree of $St(H_1)$ is indicated in blue.

Now, $H_1 = \langle ab^{-1},\ ab,\ a^2\rangle \cong \mathbb{F}_{\{x,y,z\}}$. Rewriting the initial basis of H in terms of $A_1 = \{x, y, z\}$ one has that

$$ab^{-1} = x,\ a^2b^{-1}a^{-1} = (a^2)(ab)^{-1} = zy^{-1},\ a^3b^{-1}a^{-2} = (a^2)(ab^{-1})(a^2)^{-1} = zxz^{-1},$$

$$a^3b = (a^2)(ab) = zy,$$

and so $\kappa_1^{-1}(H) = \langle x,\ zy^{-1},\ zxz^{-1},\ zy\rangle$. Applying Corollary 4.3.5 to $\kappa_1^{-1}(H)$ one easily gets

$$M_2(\kappa_1^{-1}(H)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

But, this matrix has rank 2, so H is not 2−dense either in H_1 and we have to compute H_2. Using Remark 4.4.1, we will identify vertices $p, q \in V(St(H)) \iff u_pu_q^{-1} \in H_1$ and $\sigma_1(\kappa_1^{-1}(u_pu_q^{-1})) \in \sigma_1(\kappa_1^{-1}(H))$. The only pairs of different vertices that satisfy the first condition are $\odot$, 3 and 2, 4 but, in both cases, $\sigma_1(\kappa_1^{-1}(u_\odot u_3^{-1})) = \sigma_1(\kappa_1^{-1}(u_2u_4^{-1})) = (0, 0, 1) \notin \sigma_1(\kappa_1^{-1}(H)) = \langle(1, 0, 0),\ (0, 1, 1)\rangle$. This concludes that $St(H_2)$ is constructed by identifying no vertices at all (of H) and, hence, $St(H_2) = St(H)$, $H_2 = H$. As in the previous example, H_2 is a closed set (see steps [3] and [4] of the algorithm), and one finally concludes that, for $p = 2$, $Cl(H)^{\mathbb{F}_{\{a,b\}}} = H_2 = H$.

For more interesting examples like these, the reader is referred to [9, Section 3.2].

Chapter 5

Conclusions

In this bachelor thesis we have introduced the notion of a topology on a group, through the concept of the *Pro*$-\mathcal{V}$ topology. We have seen that this abstract idea yield surprising outcomes: for instance, one of the first results we proved is that the *Pro*$-\mathcal{V}$ topology on an arbitrary group G (denoted $\mathcal{T}_{\mathcal{V}}^{G}$) can also be defined via a pseudometric $d_{\mathcal{V}}^{G}$. That is, $\mathcal{T}_{\mathcal{V}}^{G}$ is more natural than one could have thought. With this characterization of $\mathcal{T}_{\mathcal{V}}^{G}$, we were able to prove that the (finite) product topology equals the *Pro*$-\mathcal{V}$ topology of the finite direct product group, an intuitive result.

We then centered in the case of subgroups: after characterizing the form of open and closed subgroups via purely algebraic properties, we focused on the relation between $\mathcal{T}_{\mathcal{V}}^{G}|_{H}$ and $\mathcal{T}_{\mathcal{V}}^{H}$, where $H \leqslant G$. Generally, one has that $\mathcal{T}_{\mathcal{V}}^{G}|_{H} \subsetneq \mathcal{T}_{\mathcal{V}}^{H}$, but there is a known and useful sufficient condition to ensure the equivalence between the two topologies (see Proposition 3.4.9, a result extracted from [9, Section 1.4]). However, there are easy counterexamples which show that condition is not necessary, even if we require it to hold for all pseudovarieties (see Example 3.4.10 and Example 3.4.11). Future work could be intended in that direction, in trying to characterize when do the two topologies coincide, via purely algebraic conditions.

In the last part of Chapter 3 we studied the case of quotient groups and its relation with the quotient topology. Having in mind the previously discussed results, a surprising outcome arosed when we saw that $\mathcal{T}_{\mathcal{V}}^{G}/H = \mathcal{T}_{\mathcal{V}}^{G/H}$ holds in complete generality. The conclusions provided here by Dr. Pedro Silva were absolutely key, and the author wants to thank again its dedication.

In Chapter 4, we focused in the case of free groups: following the steps of [14] and [9], we were able to prove that, under an extension–closed pseudovariety, the closure of a finitely generated subgroup $H \leqslant \mathbb{F}_A$ is also finitely generated. This result led us to the final part of the thesis: we centered in the case $\mathcal{V} = \mathcal{V}_p$, and used known properties of $p-$groups to characterize the problem of determining whether a finitely generated subgroup $H \leqslant \mathbb{F}_A$ is $p-$dense, that is $Cl(H) = \mathbb{F}_A$ (under $\mathcal{T}_{\mathcal{V}_p}^{\mathbb{F}_A}$, of course), in terms of a simpler question of basic linear algebra.

We finally reproduced successfully a known algorithm [9] to effectively compute generators of the closure of a finitely generated subgroup $H \leqslant \mathbb{F}_A$ (under $\mathcal{V}_p$) and saw its applicability with some simple examples.

It is also to be noted that, for the case $\mathcal{V} = \mathcal{V}_{Nil}$, the pseudovariety of all finite nilpotent groups, which is also extension–closed, the closure problem is also solved: one has that the *nil*–closure of a subgroup $H \leqslant \mathbb{F}_A$ equals the intersection, over all prime numbers p, of the $p-$closures of H. It can also be seen that it is sufficient to only compute a finite number of these $p-$closures, so it directly follows that the *nil*$-$closure of a subgroup of

$\mathbb{F}_A$ is also effectively computable (for more details, see [9, Section 4]).

Unfortunately, the case $\mathcal{V} = \mathcal{V}_{Sol}$, the pseudovariety of all finite solvable groups (which is again extension–closed) remains unsolved. One of the reasons of it lies in that, unlike in the cases $\mathcal{V} = \mathcal{V}_p$ and $\mathcal{V} = \mathcal{V}_{Nil}$, no algorithm is yet known to determine whether a finitely generated subgroup $H \leqslant \mathbb{F}_A$ is $sol-$dense. If such an algorithm existed, we would then be able to compute the closure of H under $\mathcal{V}_{Sol}$, just as in the case of $\mathcal{V}_p$ and $\mathcal{V}_{Nil}$. Although, this is an active stream of research: for instance, and for the pseudovariety of all finite *supersolvable* groups, $\mathcal{V}_{SSol} \subseteq \mathcal{V}_{Sol}$, it has been recently shown [11] that the problem of determining whether a finitely generated subgroup of the free group is $Ssol-$dense is decidable. Even though this is a considerable progress, questions such as determining whether a finitely generated subgroup $H \leqslant \mathbb{F}_A$ is $Ssol-$closed, or deciding whether the $Ssol-$closure of H is finitely generated, still remain open.

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