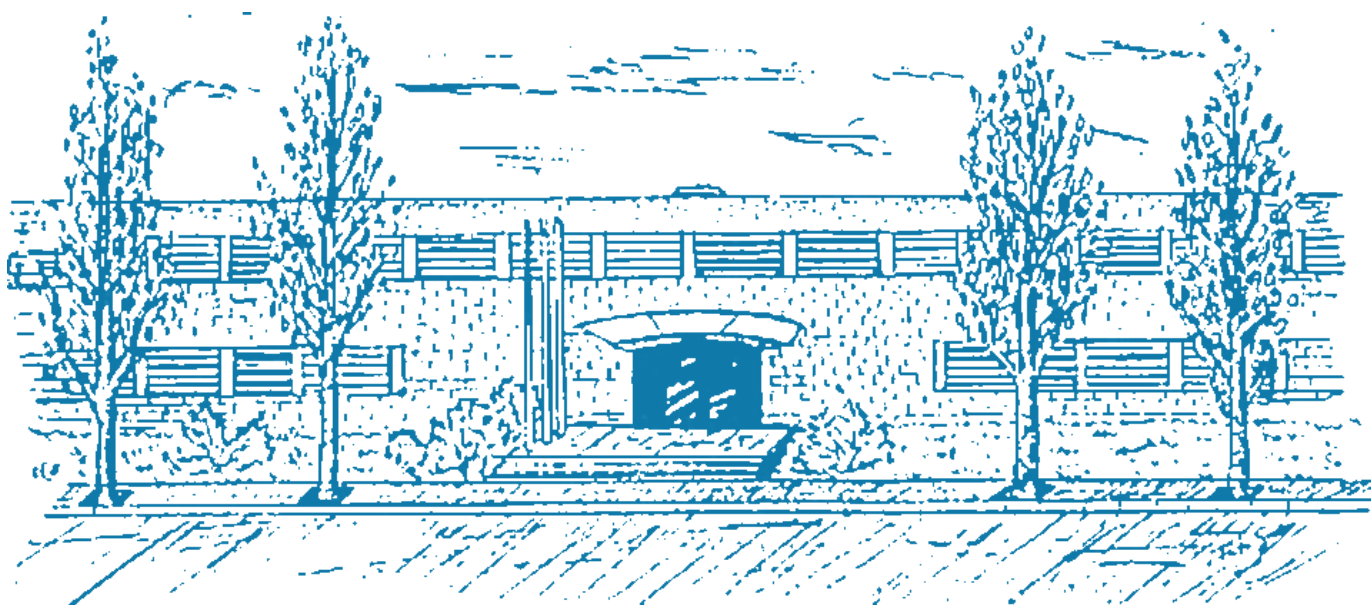


UNIVERSITAT POLITÈCNICA DE CATALUNYA - BarcelonaTech (UPC)
FME - FACULTAT DE MATEMÀTIQUES I ESTADÍSTICA

MASTER IN ADVANCED MATHEMATICS AND MATHEMATICAL ENGINEERING (MAMME)
MASTER'S THESIS

Structural and algorithmic aspects of Droms groups



Author: Bernat Coderch Sendrós

Directors: Enric Ventura Capell, Jordi Delgado Rodríguez

Department: Mathematics

Barcelona, 28 May 2026



UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística



Universitat Politècnica de Catalunya

FACULTAT DE MATEMÀTIQUES I ESTADÍSTICA

STRUCTURAL AND ALGORITHMIC
ASPECTS OF DROMS GROUPS

Author: Bernat Coderch Sendrós

Supervisors: Enric Ventura Capell, Jordi Delgado Rodríguez

May, 2026

Acknowledgements

In the following lines, I would like to take the opportunity to thank all the people who have helped and supported me during the completion of this master's thesis in my native language, Catalan.

En primer lloc, vull agrair als meus dos tutors, Enric Ventura i Jordi Delgado, pel suport incondicional i tot el temps que han dedicat a tutoritzar aquest projecte. Sense la vostra visió experta, els vostres consells i la vostra professionalitat, hagués estat impossible completar aquest treball.

En segon lloc vull donar les gràcies a l'FME i a totes les persones que n'han format part durant aquests 6 meravellosos anys en que he estat formant-me. Tant la qualitat acadèmica de la facultat com les persones que la integren són d'un nivell excepcional.

Per descomptat, gràcies a la meva família: papa, mama i Guillem, gràcies per la vostra paciència, comprensió i pel suport incondicional que m'heu donat en tot moment.

Tampoc no em vull oblidar dels meus amics; als de Figueres i als de Barcelona: gràcies per estar sempre allà i pels moments inoblidables que hem viscut junts. Estic segur que ens queden encara moltes experiències per viure plegats.

Finalment, gràcies Judit; per ser la meva companya de vida i animar-me sempre que ho he necessitat. T'estimo molt.

Abstract

This master's thesis is dedicated to the study of *Droms groups*, a subclass of the wider family of *Right-angled Artin groups*, groups where the only relations are commutativities between generators. After the first part of the manuscript, where we introduce the topic and review a well-known recursive characterization for the class of finitely generated Droms groups, we turn our attention to algorithmic problems within this family. Specifically, we address two types: subgroup and embeddability problems. Here, we make use of the recursive characterization of Droms groups and the powerful framework given by the theory of *Ivanov automata* to deal with subgroups of free products of groups, and we are able to characterize finite index subgroups of Droms groups and prove the solvability of the *Finite Index Problem*, *Subgroup Recognition Problem*, *Embeddability Problem* and *Finite Index Embeddability Problem* in the family of finitely generated Droms groups. In addition, we establish another remarkable result: a graphical generalization of the classical *Schreier's Index Formula* for Droms groups.

Contents

1	Introduction	2
2	Preliminaries	5
2.1	On graphs	5
2.2	On groups	7
2.2.1	On free groups and group presentations	9
2.2.2	On Ivanov automata	11
3	An introduction to RAAGs and Droms groups	17
3.1	The family of Right-angled Artin groups	17
3.2	The subfamily of Droms groups	18
3.2.1	A recursive characterization of $\mathcal{D}_f$	23
4	Algorithmic problems in Droms groups	26
4.1	Subgroup problems in $\mathcal{D}_f$	26
4.1.1	Schreier's Formula for Droms groups	30
4.2	Embeddability problems in $\mathcal{D}_f$	33
5	Conclusions	39
	Bibliography	43

Chapter 1

Introduction

Since they were introduced in the late 1970s by A. Baudisch (see [1] and [2]), *Right-angled Artin groups* (then known as *semifree groups*) have been a recurrent object of study for geometric group theorists due to their simple presentations and rich structure of subgroups.

In the 1980s, C. Droms studied this family of groups (see [15], [16] and [17]), from now on denoted by $\mathcal{R}$, under the name of *graph groups*, as he proved that every element $H \in \mathcal{R}$ could be uniquely represented by a simple undirected graph X_H . This was a very relevant result as Droms allowed us to think of RAAGs in terms of their graphical representation.

In the 1990s, the conjugacy problem was proved to be solvable in $\mathcal{R}$ in linear time (see [25]). This result was then generalized by Crisp, Godelle and Wiest (see [7]) to a large family of subgroups of RAAGs. This was an important advance in the field from the algorithmic point of view.

Recently, these groups have attracted much interest in geometric group theory and the topology of 3-manifolds due to their actions on CAT(0) cube complexes (see [3] for further reading on the topic).

However, the family of RAAGs is not totally well-behaved: for instance, $\mathbb{F}_2 \times \mathbb{F}_2 \in \mathcal{R}$ is seen to have finitely generated subgroups that are not finitely presented, hence, not RAAGs. This result shows that $\mathcal{R}$ is not closed under taking subgroups, justifying the characterization of a subclass of RAAGs whose subgroups are again of this type.

It was in [17] where Droms characterized this class in purely graphical terms, introducing the now called family of *Droms groups*, $G(\mathcal{D})$. This result then led to a recursive characterization of finitely generated Droms groups, which we will use throughout this work.

In the family of finitely generated Droms groups, algorithmic problems have also been studied: for instance Delgado et al. proved in [12] that the *Subgroup Intersection Problem*, *SIP* for short, was solvable in $G(\mathcal{D})_{fg}$. In fact, the main objective of this work is to study algorithmic problems regarding the concepts of index of subgroups of Droms groups and the existence of embeddings between members of $G(\mathcal{D})_{fg}$.

This master's thesis is organized as follows: firstly, Chapter 2 contains preliminary sections on graphs and groups, fixing the notation and presenting basic definitions and results that will be used throughout the manuscript. Particularly, for groups, we include specific units about free groups and group presentations, as well as an introduction to *Ivanov automata* (see [12] and [21]). This last topic will be crucial later in Chapter 4 when we deal with subgroups of free products of groups.

Chapter 3 is dedicated to present the family of RAAGs and the class of Droms groups.

After some preliminary results, where we review the fundamental theorem by Droms in [17], we prove the following recursive characterization for finitely generated Droms groups:

Theorem. *The class of finitely generated Droms groups, $G(\mathcal{D})_{fg}$, is the minimal family satisfying the following properties:*

- 1) $\{1\} \in G(\mathcal{D})_{fg}$;
- 2) $H \in G(\mathcal{D})_{fg} \implies H \times \mathbb{Z} \in G(\mathcal{D})_{fg}$;
- 3) $H_1, H_2 \in G(\mathcal{D})_{fg} \implies H_1 * H_2 \in G(\mathcal{D})_{fg}$.

In the rest of the chapter we analyze this notorious result and provide some interesting examples.

Chapter 4 is dedicated to study new algorithmic problems within the family of Droms groups, and contains the core contributions of this thesis. Specifically, we address two types of problems: subgroup and embeddability problems. The first type contains problems that depend on a group G given by a (finite) presentation and have as input a (finitely generated) subgroup $H \leq G$ given by a (finite) set of generators. An example of a subgroup problem is the so-called *Finite Index Problem*:

Finite Index Problem for $H = \langle A \mid B \rangle$, $FIP(H)$: given a finite subset $S \subseteq H$, decide whether the subgroup generated by S has finite index in H , that is, if $\langle S \rangle \leq_{fi} H$. In the affirmative case, additionally, compute $[H : \langle S \rangle]$.

On the other hand, we have embeddability problems. These are problems that depend on a group given by a (finite) presentation and have as input another group (given by another finite presentation). A simple example of a problem of this type is the so-called *Finite Index Embeddability Problem*:

Finite Index Embeddability Problem, for $H = \langle A \mid B \rangle$, $FIEP(H)$: given a group $K = \langle A' \mid B' \rangle$, decide whether there exists a finite index embedding $K \hookrightarrow_{fi} H$. In the affirmative case, additionally, compute the set $S_{H,K}$ of possible indexes for the embeddings $K \hookrightarrow_{fi} H$.

To address this kind of algorithmic problems, we will continuously use the commented recursive characterization for $G(\mathcal{D})_{fg}$ and the theory of *Ivanov automata* presented in the preliminaries, as we will have to deal with subgroups of free products of groups. As remarkable results, apart from proving the solvability of the *Finite Index Problem*, *Subgroup Recognition Problem*, *Embeddability Problem* and *Finite Index Embeddability Problem* in $G(\mathcal{D})_{fg}$, we are also able to characterize finite index subgroups of Droms groups and to prove the following graphical generalization of the well-known *Schreier's Index Formula*:

Theorem (Schreier's Formula for Droms groups). *Let X be a finite Droms graph and $L \leq_{fi} G(X)$. Then,*

$$(c_X - 1)[G(X) : L] = c_Y - 1,$$

where Y is the Droms graph associated to L , and where c_X and c_Y denote the number of connected components of X and Y , respectively.

Finally, in the last chapter we present the conclusions we have reached, and outline potential avenues for further research.

Chapter 2

Preliminaries

The objective of this chapter is to present the fundamental topics necessary to understand and follow this manuscript, as well as to unify notation, ensuring that the whole thesis is self-contained. In this direction, we include sections on graph theory and on group theory; the latter contains specific parts on free groups, group presentations, and *Ivanov automata*, critical in the development of Chapter 4.

Even though most statements are presented without proof here, we always provide a detailed reference for all results.

2.1 On graphs

In this first section we introduce the concept of graph and present some elementary properties and results. We also set here the notation on the topic, that will be used throughout all the thesis. The main references used here are [14] and [38].

Definition 2.1.1. An *undirected simple graph* (or simply a graph) is a pair $X = (V(X), E(X))$, where $V(X)$ is a set and $E(X)$ is a set of unordered pairs of V . The elements of $V(X)$ are called *vertices* of the graph and the elements of $E(X)$ are called *edges* of the graph. When there is no ambiguity, we will write V and E instead of $V(X)$ and $E(X)$. Throughout this thesis, graphs will be denoted by capital letters X, Y, Z and W , and the set of all graphs will be denoted by $\mathcal{G}$.

Graphs are usually represented by a drawing, where vertices are drawn as dots and edges are depicted as lines connecting two vertices. Below is an example:

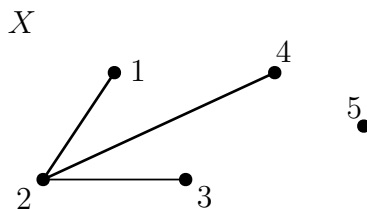


Figure 2.1: Undirected simple graph X with vertex set $V(X) = \{1, 2, 3, 4, 5\}$ and edge set $E(X) = \{\{1, 2\}, \{2, 3\}, \{2, 4\}\}$.

If loops and multiple edges between vertices are allowed, we recover the definition of a non-simple undirected graph. However, unless otherwise stated, all graphs considered in this manuscript are assumed to be simple.

With the notion of graph defined, we can now introduce some more basic concepts:

Definition 2.1.2. The number of vertices of a graph is called the *order* of the graph, and is usually denoted by $|V|$. Similarly, the number of edges of a graph is called the *size* of the graph, and is usually denoted by $|E|$.

Definition 2.1.3. Given a graph $X = (V, E)$, we will say that two vertices $x, y \in V$ are *adjacent* if $\{x, y\} \in E$. The vertices adjacent to a particular vertex $z \in V$ are usually called the *neighbors* of z .

The graph of order n where every pair of vertices is adjacent is called the *complete graph* on n vertices, and is denoted by K_n . Similarly, the graph of order n where no pair of vertices is adjacent is called the *empty graph* on n vertices, and is denoted by E_n (note that $E_0 = (\emptyset, \emptyset)$ is the empty graph).

We can also introduce the concepts of subgraph and induced subgraph, which are particularly relevant in this thesis.

Definition 2.1.4. A *subgraph* of a graph $X = (V, E)$ is another graph $X' = (V', E')$ with $V' \subseteq V$ and where E' is a set of unordered pairs of V' contained in E . In a similar manner, an *induced subgraph* (or *full subgraph*) of X is a graph $X' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$ satisfying

$$x, y \in V' \text{ and } \{x, y\} \in E \iff \{x, y\} \in E'.$$

Given a subset $S \subseteq V$, $X_{(S)}$ denotes the subgraph of X induced by $V \setminus S$.

The degree of a vertex is a very used concept and will be applied in some results of this work. Below, its formal definition:

Definition 2.1.5. Given a graph $X = (V, E)$ and a vertex $x \in V$, we define the *degree* of x , denoted by $\delta(x)$, as the number of edges $e \in E$ which have x as an endpoint. Note that, for any finite simple undirected graph X , the following formula is always satisfied:

$$\frac{1}{2} \sum_{x \in V} \delta(x) = |E|.$$

Apart from K_n and E_n , there are some other special types of graphs deserving a definition. Here we introduce two of them:

Definition 2.1.6. We define the *path graph* and *cycle graph* on n vertices, denoted by P_n and C_n , respectively, as follows:

$$P_n = (\{1, \dots, n\}, \{\{1, 2\}, \dots, \{n-1, n\}\}), \quad C_n = (\{1, \dots, n\}, \{\{1, 2\}, \dots, \{n, 1\}\}).$$

Note that $|E(P_n)| = n - 1$ and $|E(C_n)| = n$. Below, a graphical representation of P_n and C_n , both for $n = 4$:

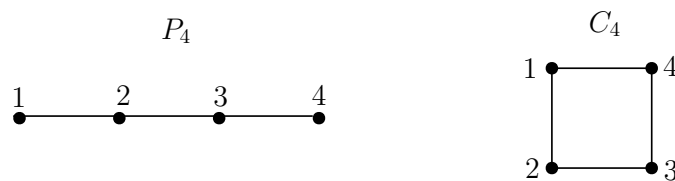


Figure 2.2: Graphical representations of P_4 and C_4 .

In connection with the above, we next define chords and chordal graphs. These notions are essential for the results presented later in this work.

Definition 2.1.7. A *chordal graph* is a graph in which all cycles of four or more vertices have a *chord*, which is an edge that is not part of the cycle but connects two vertices of the cycle.

It is clear that two graphs with different vertex sets but the same graphical representation are closely related. The concept of graph homomorphism and, specifically, the notion of graph isomorphism allows us to formalize this intuition.

Definition 2.1.8. Let $X = (V, E)$ and $X' = (V', E')$ be any two graphs. Then, a map $\varphi: V \rightarrow V'$ is said to be a *graph homomorphism* if, for any $x, y \in V$ with $\{x, y\} \in E$, then $\{\varphi(x), \varphi(y)\} \in E'$. If φ is a bijective function and its inverse, φ^{-1} , is also a graph homomorphism, then X and X' are said to be *isomorphic graphs*, and we denote it by $X \cong X'$.

A very important concept on graph theory is connectivity: the definition that follows summarizes this notion.

Definition 2.1.9. A *walk* in a graph $X = (V, E)$ is a sequence of vertices $Q = (x_0, \dots, x_n)$ such that $\{x_i, x_{i+1}\} \in E$, for $i \in \{0, \dots, n-1\}$. The *length* of the walk is the number of edges it contains (with repetitions), in this case, n .

We say that a graph $X = (V, E)$ is *connected* if every pair of vertices $x, x' \in V$ can be connected by a walk, and *disconnected* otherwise. We also define the concept of *connected component* of a graph X as any maximal connected subgraph of X . The number of connected components of a graph X will be denoted by c_X .

Finally, a *separating set* of a graph $X = (V, E)$ is any subset $S \subseteq V$ such that $X_{(S)}$ has more connected components than X .

We can also construct new graphs from previous ones in many different ways. Two of them are presented next:

Definition 2.1.10. Let $X = (V, E)$ and $X' = (V', E')$ be two graphs. We define the *disjoint union* of X and X' as the graph $X \sqcup X' := (V \sqcup V', E \sqcup E')$. In a similar manner, we define the *join* of X and X' as the graph $X \wedge X' := (V \sqcup V', E \sqcup E' \sqcup \tilde{E})$, where $\tilde{E} = \{\{x, y\} : x \in V, y \in V'\}$. The special case $X \wedge E_1$ is also called the *cone* of X , and is usually denoted by $\text{cone}(X)$.

To conclude this section we may define the concept of bipartite graph, which will appear in the *Ivanov automata* section.

Definition 2.1.11. A graph $X = (V, E)$ is called *bipartite* if there exist $V_1, V_2 \subseteq V$ such that (1) $V_1 \cup V_2 = V$; (2) $V_1 \cap V_2 = \emptyset$; (3) $\forall e \in E, e \cap V_1, e \cap V_2 \neq \emptyset$. A bipartite graph always satisfies the formula

$$\sum_{x \in V_1} \delta(x) = |E| = \sum_{x' \in V_2} \delta(x').$$

2.2 On groups

In this section we present both the basic and more specialized concepts on group theory that will be assumed throughout the thesis. For an introduction to the theory of groups the reader is referred to [24] and [33]. For more specific reading on geometric and combinatorial group theory, the reader is referred to [29] and [30].

We begin the section by giving the basic algebraic definition of the manuscript:

Definition 2.2.1. A *group* is a pair $(G, *)$, where G is a nonempty set and $*$: $G \times G \rightarrow G$ is a *binary operation* satisfying:

1. *Associativity*: $\forall g_1, g_2, g_3 \in G, (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$;
2. *Identity*: there exists $e \in G$ (sometimes denoted by $1 \in G$), the *identity*, such that $e * g = g * e = g, \forall g \in G$;
3. *Inverses*: $\forall g \in G$, there exists $g^{-1} \in G$ (the *inverse* of g) such that $g * g^{-1} = g^{-1} * g = e$.

Moreover, if $\forall g_1, g_2 \in G, g_1 * g_2 = g_2 * g_1$, we will say that the group is *abelian* (or commutative). Finally, and to lighten notation, we will usually refer to simply G as the group.

Example 2.2.2. There are lots of natural examples of groups in all of mathematics: for instance, the set of integers with the usual addition is an (abelian) group. A simple example of non-abelian group is the symmetric group on n symbols, $\mathcal{S}_n$, with the composition of functions. Remember that $\mathcal{S}_n$ is formed by the permutations that can be performed on the n symbols.

Many times, and when the group operation is understood, we will use the convention that, for $g_1, g_2 \in G, g_1 g_2 := g_1 * g_2$. Moreover, we also define $g^n := g * \cdots * g$ (with n factors). Now, we present the concepts of commutator and conjugate, which deserve a definition:

Definition 2.2.3. Let $(G, *)$ be a group and $g, h \in G$. We define the *commutator* of g and h as $[g, h] := g^{-1} h^{-1} g h$ (note that $[g, h] = 1$ if and only if g and h commute). We also define the *conjugate* of g by h as $g^h := h^{-1} g h$.

Definition 2.2.4. We define the *order of a group* $(G, *)$ as the cardinal of G , and we denote it by $|G|$. We say that a group is *finite* if it has finite order, and *infinite* otherwise. Analogously, we define the *order of an element* $g \in G$ as the smallest positive integer n such that $g^n = 1$ (if such n does not exist, we then say that g has infinite order), and we denote it by $|g|$.

Let us now present a fundamental definition for constructing new groups from previous ones.

Definition 2.2.5. Let $(G, *)$ be a group and $H \subseteq G$. We say that H is a *subgroup* of G , and denote it by $H \leq G$, if $(H, *_H)$ is also a group, where $*_H$ is the operation on G restricted to H .

Definition 2.2.6. Let $(G, *)$ be a group and $S \subseteq G$. We define the *subgroup generated* by S as $\langle S \rangle := \{s_1^{r_1} * \cdots * s_n^{r_n} : s_i \in S, r_i \in \mathbb{Z}, \forall i, n \in \mathbb{N}\}$. If there exists $S \subseteq G$, with $|S| < \infty$, such that $\langle S \rangle = G$, we then say that G is *finitely generated*. Otherwise, G is *infinitely generated*.

We also define the *rank* of G as $\text{rk}(G) := \min\{|S| : S \subseteq G, \langle S \rangle = G\} \leq |G|$.

We can now introduce the concept of coset.

Definition 2.2.7. Let G be a group, $H \leq G$ and $g \in G$. Then, the *left coset* of g with respect to H is the set $gH := \{g * h : h \in H\}$. In a similar way, $Hg := \{h * g : h \in H\}$ is the *right coset* of g with respect to H . A classic result states that there are the same number of left and right cosets, so we define the *index* of H in G , $[G : H]$, as the number of cosets of H in G .

Our goal is now to define quotient groups. To be able to do so, we first have to consider a special type of subgroups, which are defined next.

Definition 2.2.8. Let G be a group and $N \leq G$ a subgroup. We say that N is a *normal subgroup* of G , and denote it by $N \trianglelefteq G$, if $gN = Ng$, $\forall g \in G$.

Definition 2.2.9. Let G be a group and $N \trianglelefteq G$ a normal subgroup. We define the *quotient group* of G by N as the set $G/N := \{gN : g \in G\}$ under the operation $(gN)(g'N) := (gg')N$.

It can be seen that this operation is well-defined and turns G/N into a group. In fact, normality of the subgroup N is necessary for this to work.

Now, we want to talk about functions between groups. If we want them to have useful properties, we have to ask them to respect groups operations. This is why we introduce the so-called group homomorphisms.

Definition 2.2.10. Let $(G, \cdot)$ and $(G', *)$ be two arbitrary groups. Then a *group homomorphism* between G and G' is a function $\varphi: G \rightarrow G'$ such that

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) * \varphi(g_2), \quad \forall g_1, g_2 \in G.$$

If the homomorphism is a bijective function, we then say that it is an *isomorphism*. If there exists an isomorphism between two groups G and G' , we say that they are *isomorphic*, and we denote it by $G \cong G'$.

One can think of isomorphic groups as essentially the same, but with different “labels” of their elements. The isomorphism that exists between them tells us how is the relation between these “labels”.

2.2.1 On free groups and group presentations

This section introduces the notions of free groups and group presentations. Free groups are a fundamental concept in group theory, as a classical result states that any group can be expressed as a quotient of a free one. This decomposition leads to a group presentation, which is the standard method for defining groups. The main references used in this section are [9], [29] and [33].

Firstly, we define what it means for a group to be free.

Definition 2.2.11. Let $(G, *)$ be an arbitrary group. We say that $A \subseteq G$ is *free* in G if different reduced products on $A^\pm := A \cup A^{-1}$ yield different elements of G .¹ That is, $a_1 * \dots * a_n = a'_1 * \dots * a'_m \Rightarrow n = m, (a_1, \dots, a_n) = (a'_1, \dots, a'_m), \forall a_1, \dots, a_n, a'_1, \dots, a'_m \in A^\pm$. We also say that A *generates* G if $\langle A \rangle = G$. Finally, we say that A is a *basis* of G if A is free on G and it generates G . If there exists a basis A of G , we will say that G is a *free group* over A .

Elements of free groups can be thought of as finite sequences of symbols from their basis. Next, we provide the intuition for constructing a free group from a given basis A . For a more detailed construction, the reader is referred to [9].

¹Reduced products are those that do not contain subwords of the form $a^{-1}a$ or aa^{-1} , for any $a \in A$.

Definition 2.2.12. Let $A = \{a_1, a_2, \dots\}$ be a set of symbols (usually called an *alphabet*). We say that a *word* on A is a finite sequence of symbols of A , and denote it by $w := a_{i_1} \dots a_{i_n}$. The *length* of a word is defined as the number of symbols of A it contains (counting possible repetitions), and it is denoted by $|w|$. In this case, $|w| = n$. The unique word with no symbols (and length 0) is called the *empty word*, and is denoted by 1. Finally, the set of all words over A is denoted by A^* .

Note that there is a natural operation we can introduce in A^* : given two words $w = a_1 \dots a_n$ and $v = b_1 \dots b_m \in A^*$, we define the *concatenation* of w and v as

$$w \cdot v = wv = a_1 \dots a_n b_1 \dots b_m.$$

Clearly, concatenation is associative but it is not a commutative operation. Moreover, $1w = w1 = w$, $\forall w \in A^*$, and $|wv| = |w| + |v|$, $\forall w, v \in A^*$. Despite this, no word except for the trivial one has an inverse.

We can try to fix this: given an alphabet A we consider $A^\pm := A \cup A^{-1}$, where $A^{-1} := \{a^{-1} : a \in A\}$ is just a set of formal inverses of the symbols in A . As before, we denote by $(A^\pm)^*$ the set of all words over $A^\pm$, and we say that reduced words over $A^\pm$ are those that do not contain consecutive letters being inverses of each other. Now the following theorem can be stated (for a more detailed construction, see [9]):

Theorem 2.2.13. *Let A be an arbitrary alphabet. Then, the set $\mathbb{F}_A$ of reduced words over $(A^\pm)$, together with the operation of concatenating and reducing, is a free group over A .*

The class of isomorphism of a free group only depends on the cardinal of its basis, which is also the rank of the group, as shows next theorem.

Theorem 2.2.14. *Let A be an alphabet and $\mathbb{F}_A$ the free group over A . Then, $rk(\mathbb{F}_A) = |A|$. Moreover, if $\mathbb{F}_{A'}$ is the free group over another alphabet A' , then*

$$\mathbb{F}_A \cong \mathbb{F}_{A'} \iff |A| = |A'|.$$

From now on, $\mathbb{F}_n$ will refer to the free group of rank n , without specifying any basis of it. We can now state the main theorem of this section, which implies that free groups contain all the “information” about every possible group.

Theorem 2.2.15. *For any group G , there exists a cardinal $n \geq 0$ and a normal subgroup $N \trianglelefteq \mathbb{F}_n$ such that $G = \mathbb{F}_n/N$.*

We just have to give one more definition before introducing the concept of group presentation.

Definition 2.2.16. Let G be a group and $R \subseteq G$. The *normal closure* of R , denoted by $\langle\langle R \rangle\rangle$, is the minimal normal subgroup of G that contains R , that is $\langle\langle R \rangle\rangle = \bigcap_{\substack{R \subseteq H \\ H \trianglelefteq G}} H$.

Definition 2.2.17. Given any group G , let S be an alphabet and $R \subseteq \mathbb{F}_S$ such that $G \cong \mathbb{F}_S / \langle\langle R \rangle\rangle$. We then say that the ordered pair (S, R) is a *presentation* for G , and we usually write $G = \langle S \mid R \rangle$. Theorem 2.2.15 tells us that every group admits a (non-unique) presentation.

When we are given a group presentation $\langle S \mid R \rangle$, a map from the generators S to another group H defines a group homomorphism if and only if the images of the generators satisfy the relations in R . This is reflected in the following proposition:

Proposition 2.2.18. *Let $G = \langle S \mid R \rangle$ and G' be any two groups. Then, any map $\varphi: S \rightarrow G'$ extends to a group homomorphism $\tilde{\varphi}: G \rightarrow G'$ if and only if $\varphi'(r) = 1_{G'}$, $\forall r \in R$, where $\varphi': \mathbb{F}_S \rightarrow G'$ is the natural extension of φ to $\mathbb{F}_S$.*

We can now define the concepts of direct product and free product of groups through presentations.

Definition 2.2.19. Let $G = \langle S \mid R \rangle$ and $G' = \langle S' \mid R' \rangle$ be any two groups. We define the *free product* of G and G' as the group with presentation $G * G' = \langle S \sqcup S' \mid R \sqcup R' \rangle$. In a similar way, we define the *direct product* of G and G' as the group with presentation $G \times G' = \langle S \sqcup S' \mid R \sqcup R' \sqcup \tilde{R} \rangle$, where $\tilde{R} = \{[s, s'] : s \in S, s' \in S'\}$ is the set of commutators between all elements of S and all of S' .

To treat general elements in free products and direct products of groups we generally use normal forms. Next proposition establishes the usual normal forms in these two types of group constructions.

Proposition 2.2.20. *Let G and G' be any two groups. Then,*

1. *Any element of $G * G'$ can be written as $g_1 g'_1 \cdots g_n g'_n$, with $g_i \in G$ and $g'_j \in G'$. Moreover, if we require $g_i \neq 1_G$, for $i \in \{2, \dots, n\}$ and $g'_j \neq 1_{G'}$, for $j \in \{1, \dots, n-1\}$, then the representation is unique.*
2. *Any element of $G \times G'$ can be written in a unique way as gg' , where $g \in G$ and $g' \in G'$.*

To end up the section we state the famous *Kurosh Subgroup Theorem*, which characterizes the form of the subgroups of a free product of groups.

Theorem 2.2.21 (Kurosh Subgroup Theorem, [23]). *Let $G * G'$ be an arbitrary free product of groups, and $H \leq G * G'$. Then, there exist $H_i \leq G$, $g_i \in G * G'$, for $i \in I$, $K_j \leq G'$, $g'_j \in G * G'$, for $j \in J$ and $D \subseteq G * G'$ such that*

$$H = \left(\ast_{i \in I} g_i^{-1} H_i g_i \right) * \left(\ast_{j \in J} g'_j{}^{-1} K_j g'_j \right) * \mathbb{F},$$

where $\mathbb{F}$ is the free subgroup of $G * G'$ freely generated by D . Moreover, $H_i = G \cap g_i H g_i^{-1}$, $K_j = G' \cap g'_j H g'_j{}^{-1}$. Also, H is finitely generated if and only if the three sets I , J and D are finite and $H_i \leq_{fg} G$, $K_j \leq_{fg} G'$.

Corollary 2.2.22. *Let $G * G'$ be an arbitrary free product of groups, and $H \leq G * G'$. Then, there exist $H_i \leq G$, $i \in I$, $K_j \leq G'$, $j \in J$ and $r \in \mathbb{Z}^+ \cup \{0\}$ such that*

$$H \cong \left(\ast_{i \in I} H_i \right) * \left(\ast_{j \in J} K_j \right) * \mathbb{F}_r.$$

2.2.2 On Ivanov automata

This section is dedicated to introduce the notion of *Ivanov automaton*, which are graph-based objects that describe subgroups of a free products of groups. We assume here that the reader is familiar with *Stallings automata*, which are the graphical objects that represent subgroups of free groups. For more detailed references on *Ivanov automata* theory, the reader is referred to [12] and [21], and for an introduction to the theory of *Stallings automata*, see [10] and [37].

We begin by defining the types of objects we will be considering:

Definition 2.2.23. Let G_1 and G_2 be two arbitrary groups. A *wedge* (G_1, G_2) –*automaton* is a septuple $\Gamma = (V, E, \iota, \tau, ^{-1}, \ell, \odot)$, where:

1. $\Sigma = (V, E, \iota, \tau, ^{-1})$ is an *involutive digraph* (the *underlying digraph* of Γ) with three types of vertices, $V = V_0 \sqcup V_1 \sqcup V_2$, namely *primary*, *1–secondary* and *2–secondary*; and with all the arcs in E connecting, in either direction, a primary vertex with a secondary one (thus, $E = E_1 \sqcup E_2$, where $e \in E_i$ if e connects a primary vertex with a i –secondary one).
2. ℓ is a twofold *label map*: for $i = 1, 2$, $\ell: E_i \rightarrow G_i$, $e \mapsto \ell_e$ (compatible with involution) and $\ell: V_i \rightarrow \text{Sbg}(G_i)$, $p \mapsto \ell_p$ (secondary vertices are labeled by subgroups of the corresponding group).
3. $\odot$ is a distinguished primary vertex called the *basepoint* of Γ .

Recall that wedge automaton are involutive (as automata). Hence, a wedge automata Γ can always be represented by one of its arc orientations. Then $E(\Gamma) = E^\pm(\Gamma) := E^+(\Gamma) \sqcup E^-(\Gamma)$, where $E^-(\Gamma)$ is the set of inverses of the arcs in $E^+(\Gamma)$.

Below is an example of a wedge $(\mathbb{Z}, \mathbb{Z}^2)$ –automaton, where a denotes the generator of $\mathbb{Z}$ and b, c denote the generators of $\mathbb{Z}^2$. Edges and secondary vertices of the same type are equally colored, and primary vertices are represented with $\bullet$ (note also that the basepoint $\odot$ is colored in green).

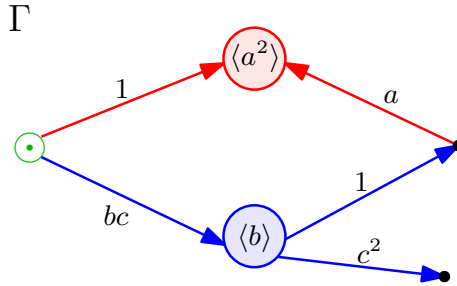


Figure 2.3: Simple example of a wedge $(\mathbb{Z}, \mathbb{Z}^2)$ –automaton Γ .

Note that, by definition, any wedge automaton is primary–secondary bipartite (see Definition 2.1.11). Next, let us adapt some concepts of graph theory (see Section 2.1) to these new objects:

Definition 2.2.24. We say that a wedge automaton Γ is *connected* if its associated underlying digraph is connected. Similarly, we say that Γ is *finite* if its associated underlying digraph is finite. Finally, we say that Γ is of *finite type* if it is finite and all the label subgroups of secondary vertices are finitely generated.

We also extend the notion of a walk to wedge automata in the natural way.

Definition 2.2.25. A *walk* in a wedge automaton Γ is a sequence of alternating and adjacent vertices and arcs, always starting and ending at primary vertices, $\gamma = p_0(e_1^{-1}q_1e_1')p_1(e_2^{-1}q_2e_2')p_2 \cdots p_{r-1}(e_r^{-1}q_re_r')p_r$, where $p_0, \dots, p_r$ are primary vertices, $q_1, \dots, q_r$ are secondary vertices and, for every $i \in \{1, \dots, r\}$, e_i, q_i, e_i' are incident. For $i = 1, 2$, an i –*elementary walk* is a walk of length 2 visiting a secondary vertex of i –type; it is *degenerate* if it consists of two mutually inverse arcs, and *non-degenerate* otherwise.

The *label* of a walk γ of the form described above is the subset $\ell_\gamma = (\ell_{e_1}^{-1}\ell_{q_1}\ell_{e_1'}) (\ell_{e_2}^{-1}\ell_{q_2}\ell_{e_2'}) \cdots (\ell_{e_r}^{-1}\ell_{q_r}\ell_{e_r'}) \subseteq G_1 * G_2$. If we pick the trivial element when visiting every secondary vertex we obtain the so-called *basic label* of γ , denoted by $\ell_\gamma^\bullet = (\ell_{e_1}^{-1}\ell_{e_1'}) (\ell_{e_2}^{-1}\ell_{e_2'}) \cdots (\ell_{e_r}^{-1}\ell_{e_r'}) \in \ell_\gamma$.

Definition 2.2.26. Let Γ be a wedge (G_1, G_2) –automaton and let p, p' be two primary vertices. We define the *coset recognized* by Γ relative to (p, p') as the set $\langle \Gamma \rangle_{(p, p')} := \bigcup_{\gamma} \ell_{\gamma}$, where the union runs over all walks in Γ from p to p' . When $p = p'$, then we abbreviate $\langle \Gamma \rangle_p := \langle \Gamma \rangle_{(p, p')}$. Finally, if $p = p' = \odot$, we simply write $\langle \Gamma \rangle := \langle \Gamma \rangle_{\odot}$ and we call it the *subgroup recognized* by Γ .

As in the case of *Stallings automata*, closed paths in the automaton translate to subgroups, while paths between distinct vertices correspond to cosets. This reasoning justifies the above terminology, and is reflected on the following lemma.

Lemma 2.2.27. *Let Γ be a connected wedge (G_1, G_2) –automaton and let $p, p' \in V_0$. Then,*

- 1) $\langle \Gamma \rangle_p$ is a subgroup of $G_1 * G_2$;
- 2) $\langle \Gamma \rangle_p$ and $\langle \Gamma \rangle_{p'}$ are conjugated to each other, that is $\langle \Gamma \rangle_{p'} = (\langle \Gamma \rangle_p)^g$ for every $g \in \langle \Gamma \rangle_{(p, p')}$;
- 3) $\langle \Gamma \rangle_{(p, p')}$ is a right coset of $\langle \Gamma \rangle_p$, that is, $\langle \Gamma \rangle_{(p, p')} = \langle \Gamma \rangle_p \cdot g$, for every $g \in \langle \Gamma \rangle_{(p, p')}$.

Given any two groups G_1, G_2 and $H \leq G_1 * G_2$, our objective is now to find a wedge (G_1, G_2) –automaton Γ such that $\langle \Gamma \rangle = H$. As happens in the case of *Stallings automata*, this is always possible, and the reasoning is based on the construction of the flower automaton. Indeed, if $H = \langle h_1, h_2, \dots \rangle$, where $h_i = h_{i,1}h'_{i,1} \dots h_{i,n_i}h'_{i,n_i} \in G_1 * G_2$, with $h_{i,j} \in G_1$, $h'_{i,j} \in G_2$, then, for each generator h_i consider the *petal automaton* associated to h_i , $\text{Pt}(h_i)$ (see Figure 2.4).

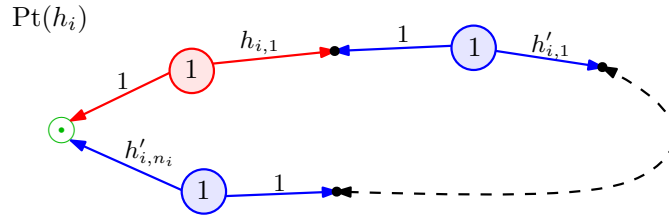


Figure 2.4: Petal automaton associated to h_i . Clearly, $\langle \text{Pt}(h_i) \rangle = \langle h_i \rangle$.

It is not difficult to see that $\langle \text{Pt}(h_i) \rangle = \langle h_i \rangle \leq G_1 * G_2$. Now, if we construct the petal automaton associated to each generator of H , and we identify all the basepoints into a single one, we obtain the so-called *flower automaton* of $\{h_1, h_2, \dots\}$, $\text{Fl}(h_1, h_2, \dots)$. It is clear that $\langle \text{Fl}(h_1, h_2, \dots) \rangle = \langle h_1, h_2, \dots \rangle = H$ and that, if H is finitely generated, $H = \langle h_1, h_2, \dots, h_n \rangle$, then $\text{Fl}(h_1, h_2, \dots, h_n)$ is finite and can be constructed in finite time. However, as it happens in the *Stallings* case, this construction strongly depends on the generating set of H we chose; our goal is now to reduce this redundancy to a level that clarifies the structure of the subgroup (essentially encoding its *Kurosh decomposition*).

Definition 2.2.28. Let G_1 and G_2 be any two groups and let Γ be a wedge (G_1, G_2) –automaton. We say that Γ is *reduced* if the following conditions are satisfied:

- 1) Γ is connected;
- 2) Every primary vertex of Γ is adjacent with at most one 1–secondary vertex, and with at most one 2–secondary vertex;
- 3) No non-degenerate elementary walk in Γ reads the trivial element.

We will use gothic font (i.e. $\mathfrak{J}$, $\mathfrak{A}$) to denote reduced wedge automata.

Directly from the definition of reduced wedge automaton we can derive the following important result, which will be crucial later in Chapter 4.

Lemma 2.2.29. *Let G_1 and G_2 be any two groups and let $\mathfrak{J}$ be a reduced wedge (G_1, G_2) -automaton. Then, different primary vertices $p, q \in V_0(\mathfrak{J})$ yield different cosets of $\langle \mathfrak{J} \rangle$, that is*

$$p \neq q \implies \langle \mathfrak{J} \rangle_{(\odot, p)} \neq \langle \mathfrak{J} \rangle_{(\odot, q)}.$$

Now, we have to ensure that, given a finitely generated subgroup $H = \langle h_1, \dots, h_n \rangle$, we are always able to construct a reduced wedge automaton $\mathfrak{J}$ that recognizes H . As happens with *Stallings automata*, the strategy is starting from a flower automaton $Fl(h_1, \dots, h_n)$ recognizing the subgroup and then applying some elementary transformations on $Fl(h_1, \dots, h_n)$ (that do not change the subgroup recognized by the automaton).

Definition 2.2.30. Let G_1, G_2 be any two groups, and consider the following *elementary transformations* on a wedge (G_1, G_2) -automaton Γ .

1. *Adjustment:* replace the label of any arc e leaving a secondary vertex p by $h \cdot \ell_e$, for any $h \in \ell_p$.

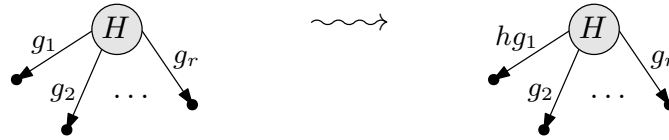


Figure 2.5: Adjustment.

2. *Conjugation:* for $i = 1, 2$, replace, given $g \in G_i$, the label of a i -secondary vertex p by $(\ell_p)^{g^{-1}} = g\ell_pg^{-1}$, and replace the label of every arc e_i leaving p by $g\ell_{e_i}$.

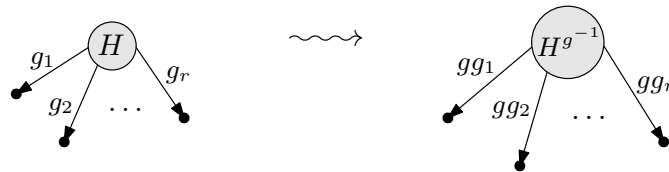


Figure 2.6: Conjugation.

3. *Isolation:* remove from Γ all the connected components not containing the base-point $\odot$.

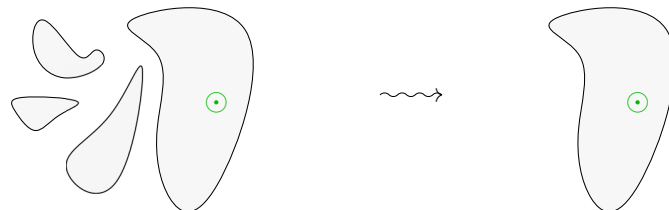


Figure 2.7: Isolation.

4. *Primary open folding*: for $i = 1, 2$, given two i -secondary vertices p_1, p_2 adjacent to the same primary vertex through respective arcs e_1, e_2 with the same label $\ell_{e_1} = \ell_{e_2} = g$, identify p_1 and p_2 into a secondary vertex with label $\langle \ell_{p_1}, \ell_{p_2} \rangle$, and identify the arcs e_1, e_2 into a unique arc e with label $\ell_e = g$.

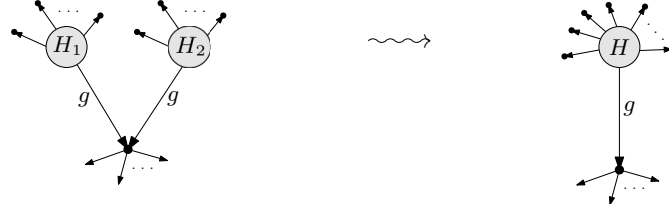


Figure 2.8: Primary open folding, with $H = \langle H_1, H_2 \rangle$.

5. *Secondary open folding*: for $i = 1, 2$, given an i -secondary vertex p adjacent to two different primary vertices p_1, p_2 through arcs e_1, e_2 having the same label $\ell_{e_1} = \ell_{e_2} = g$, identify the vertices p_1, p_2 and the arcs e_1, e_2 into an arc e with label $\ell_e = g$.

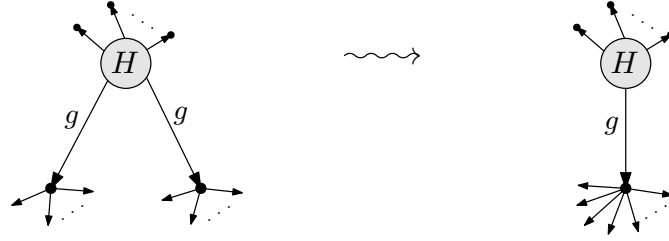


Figure 2.9: Secondary open folding.

6. *Closed folding*: for $i = 1, 2$, given a primary vertex p adjacent to a i -secondary vertex q by two non mutually inverse arcs e_1, e_2 , identify e_1 and e_2 into a unique arc e with label $\ell_e = \ell_{e_1}$, and change the label of q to $\langle \ell_q, \ell_{e_1} \ell_{e_2}^{-1} \rangle$.

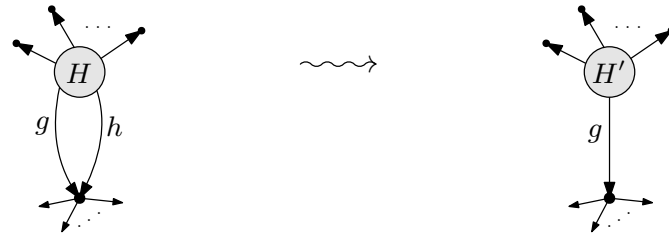


Figure 2.10: Closed folding, with $H' = \langle H, gh^{-1} \rangle$.

With these elementary transformations defined, we can now state the following two important results.

Lemma 2.2.31. *Let Γ be a wedge (G_1, G_2) -automaton, and let $\Gamma \rightsquigarrow \Gamma'$ be any of the elementary transformations presented in Definition 2.2.30. Then, $\langle \Gamma \rangle = \langle \Gamma' \rangle$.*

Theorem 2.2.32 (S. V. Ivanov, [21]). *For any two groups G_1, G_2 , and any finitely generated subgroup $H \leq G_1 * G_2$, there exists a reduced wedge (G_1, G_2) -automaton $\mathfrak{J}$ satisfying $\langle \mathfrak{J} \rangle = H$. Moreover, if both G_1 and G_2 have solvable membership problem, then given a finite set of generators of a subgroup $H \leq_{fg} G_1 * G_2$, one can algorithmically obtain a reduced wedge (G_1, G_2) -automaton of finite type $\mathfrak{J}$ satisfying $\langle \mathfrak{J} \rangle = H$ in finite time.*

One of the main advantages of reduced wedge automata is that they encode the *Kurosh decomposition* of the subgroup they recognize. To see how this new notion is related with this important result, let us first fix the notation.

Notation 2.2.33. Let G_1, G_2 be any two groups and $\mathfrak{J}$ a reduced wedge (G_1, G_2) –automaton. Fix a maximal subtree² T in $\mathfrak{J}^+$, and let $E'^+ = E(\mathfrak{J}^+) \setminus E(T)$. Now, for every two vertices $u, v \in V(\mathfrak{J})$, let $T(u, v)$ denote the unique walk without backtracking from u to v along T . Now, for every $u \in V(\mathfrak{J})$ we define $z_u := \ell_{T(u, \odot)}^\bullet$ and, for every $e \in E' := E'^\pm$ we define $x_e := z_{\iota_e}^{-1} \ell_e z_{\iota_e}$.

Now, the following important theorem can be stated:

Theorem 2.2.34 (Kurosh Subgroup Theorem, [23]). *Let $\mathfrak{J}$ be a reduced (G_1, G_2) –automaton. Then, with the above notation we have that*

$$\langle \mathfrak{J} \rangle = \mathbb{F} * \left(\bigstar_{p \in V_1} z_p^{-1} \ell_p z_p \right) * \left(\bigstar_{q \in V_2} z_q^{-1} \ell_q z_q \right), \quad (1)$$

where $\mathbb{F}$ is the free subgroup of $\langle \mathfrak{J} \rangle$ freely generated by the set $\{x_e : e \in E'\}$. Moreover, if $\mathfrak{J}$ is of finite type, then the subgroup $\langle \mathfrak{J} \rangle$ is finitely generated, and we can algorithmically compute a Kurosh decomposition like (1) for $\langle \mathfrak{J} \rangle$.

Corollary 2.2.35. *Let $\mathfrak{J}$ be a reduced wedge (G_1, G_2) –automaton. Then, $\langle \mathfrak{J} \rangle$ is finitely generated if and only if $\mathfrak{J}$ is of finite type. In this case, a set of generators for $\langle \mathfrak{J} \rangle$ is computable.*

As one could expect, these last two results are easily seen to be false if we substitute reduced wedge automata by general wedge automata.

²A tree is a connected acyclic graph.

Chapter 3

An introduction to RAAGs and Droms groups

This chapter is dedicated to presenting the family of Right-angled Artin groups (RAAGs, for short) and the specific subfamily of Droms groups. After introducing the topic and presenting some known results we will present one of the main theorems of the chapter: the graphical characterization of Droms groups in [17]. This will allow us to, then, prove a recursive characterization for this family, which will be very useful later in Chapter 4. The main references used in this chapter are [4], [5] and [8].

3.1 The family of Right-angled Artin groups

It is well known that subgroups of free abelian groups are also free abelian, with rank less than or equal to the rank of the ambient group.

An analogue of this result also exists for free groups, stating that every subgroup of a free group is also free. This notorious result is known as the *Nielsen-Schreier Theorem* as it was initially proved by Nielsen in the 1920s (the finitely generated case, see [31]), and then generalized by Schreier (see [34]). Note that, in the free case, subgroups do not have any restriction on their rank; in the extreme case, one can find subgroups of $\mathbb{F}_2$ isomorphic to $\mathbb{F}_\infty$ (hence with infinite rank).

In terms of presentations we have, at one extreme, free groups, generated by some elements with no relations between them. If A is an arbitrary set, then $\mathbb{F}_A$ denotes the free group on A , which can be expressed as

$$\mathbb{F}_A = \langle A \mid - \rangle.$$

At the other extreme, free abelian groups contain all the commutative relations among their generators (and no other relation among them), that is, if A is a finite set,

$$\mathbb{Z}^{|A|} = \langle A \mid [a_i, a_j] = 1, \forall a_i, a_j \in A \rangle.$$

Here, we can ask the natural questions: what happens with groups with just *some* of the commutative relations? Do the presentations of their subgroups only have commutative relations too? This family of groups was introduced in the 1970s (see [1] and [2]), has been widely studied and its members are usually called *Right-angled Artin groups* (RAAGs for short) or *graph groups*.

Definition 3.1.1. Let $X = (V, E)$ be a finite graph. We define the RAAG (or graph group) associated to X as the group with presentation

$$G(X) := \langle V \mid [x, y] = 1 \text{ if } \{x, y\} \in E \rangle.$$

We will refer to the family of all RAAGs as $\mathcal{R}$.

Note that, by definition, RAAGs are finitely presented groups. If we extend Definition 3.1.1 to allow infinite graphs, we then obtain the wider family of *partially commutative groups* (PC-groups, for short), $\mathcal{C}$.

With this new notion, the extreme cases, empty graphs and complete graphs, correspond to free groups and free abelian groups, respectively. Moreover, we can also find group operations in terms of graph properties: for instance, if X is a disconnected graph, then $G(X)$ can easily be expressed as a free product of groups, and if X is a join (see Definition 2.1.10), then $G(X)$ can be decomposed into a direct product of groups. The following image illustrates these concepts with simple examples:



Figure 3.1: Cases where X is disconnected or a join, and their correspondence in $G(X)$.

These two graph properties (being disconnected and being a join) and their analogous group operations will be discussed further later in the chapter.

It is also clear that, if X and Y are two isomorphic graphs, then the RAAGs $G(X)$ and $G(Y)$ are also isomorphic. However, the converse is not immediate: it was Droms in [16] who proved it and, that way, not only he established a bijective correspondence between graphs and RAAGs, but he also solved the isomorphism problem in $\mathcal{R}$.

Theorem 3.1.2 (C. Droms, [16]). *The map $X \mapsto G(X)$ is a bijective correspondence between finite graphs (modulo isomorphism) and isomorphism classes of RAAGs.*

The proof of this theorem uses methods that are outside the scope of group theory, so we will not include it in this work.

This notorious result allows us to think of RAAGs just as graphs. However, if we consider, for example, C_4 , the cycle graph of 4 vertices, we have that $G(C_4) \cong \mathbb{F}_2 \times \mathbb{F}_2$ has finitely generated subgroups that are not finitely presented (this is true since $\mathbb{F}_2 \times \mathbb{F}_2$ is not coherent, see [36]) and, hence, not PC-groups.

This reasoning shows that not all elements of $\mathcal{C}$ are well-behaved and reinforces the need for finding a subfamily of PC-groups whose subgroups are again of this type. This subclass of $\mathcal{C}$ will be the so-called family of *Droms groups*, and is the main object of study of the following section.

3.2 The subfamily of Droms groups

As we introduced in last section, the family of *Droms groups* is defined as the subset of PC-groups whose subgroups are again PC-groups. We may similarly define the family of *Droms graphs* as the subset of $\mathcal{G}$ that represent Droms groups. From now on, we

will denote by $\mathcal{D} \subseteq \mathcal{G}$ and $G(\mathcal{D}) \subseteq \mathcal{C}$ the respective families of Droms graphs and Droms groups. As we will usually work with finitely presented groups (in $\mathcal{R}$, equivalent to finitely generated groups), we also denote by $\mathcal{D}_f \subseteq \mathcal{D}$ and $G(\mathcal{D})_{fg} \subseteq G(\mathcal{D})$ the respective families of *finite Droms graphs* and *finitely generated Droms Groups*. Formally, we have that

$$X \in \mathcal{D} \iff G(X) \in G(\mathcal{D}) \iff \forall H \leq G(X), \exists Y \in \mathcal{G} \text{ such that } H = G(Y),$$

and

$$X \in \mathcal{D}_f \iff G(X) \in G(\mathcal{D})_{fg}.$$

However, the plain definition of $\mathcal{D}$ does not give us much information about the form of its associated groups or defining graphs. It was C. Droms in [17] who successfully provided a characterization for the members of $\mathcal{D}_f$ (and, hence, $G(\mathcal{D})_{fg}$) in pure graphical terms. Understanding and applying this characterization is our next objective.

To begin with, we need to prove a pair of technical lemmas purely on graph theory that will help us later when characterizing the elements of $\mathcal{D}_f$. For an introduction to graph theory concepts and the notation used here the reader is referred to [14], [38] and the preliminaries.

Lemma 3.2.1. *Let $X \in \mathcal{G}$ be a finite connected and chordal graph. Then, every minimal separating set S of X induces a complete graph.*

Proof. A first observation here is that, if D is a connected component of $X_{(S)}$, then D is adjacent to all vertices of S , since otherwise the set of neighbors of D in S , a smaller set than S , would separate X , contradicting the minimality of the separating set.

Now, let $x, y \in S$, X_1 and X_2 be two different connected components of $X_{(S)}$ and Q_1, Q_2 two minimal walks (in $X_1 \sqcup \{x, y\}$ and $X_2 \sqcup \{x, y\}$, respectively) connecting x and y (note that Q_1 and Q_2 exist due to the first observation). Defining $C := Q_1 \cup Q_2$, C is a cycle of length at least 4, so it must contain a chord. As Q_1 and Q_2 are minimal and X_1 and X_2 are two different connected components of $X_{(S)}$, such chord must be $\{x, y\}$, and so we are done (see Figure 3.2). $\square$

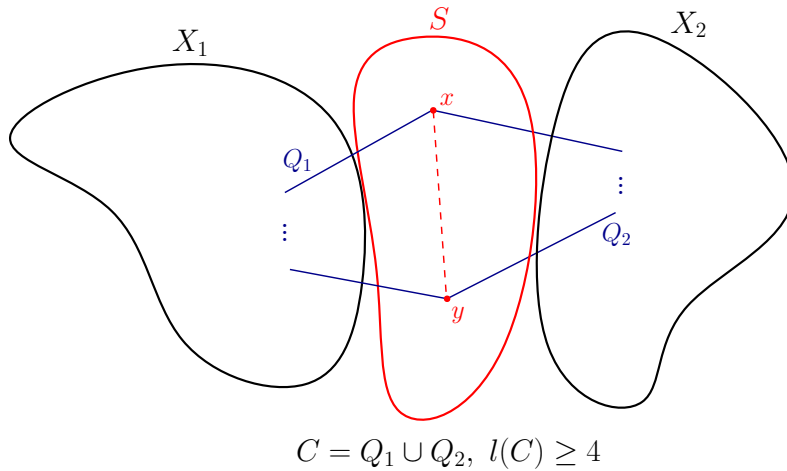


Figure 3.2: Situation for the proof of the lemma.

With this result we can directly state and prove the second lemma on graph theory, which will be the crucial graphical result of the section.

Lemma 3.2.2. *Let $X \in \mathcal{G}$ be a finite connected graph with no full subgraph isomorphic to either C_4 or P_4 . Then there exists at least one vertex of X adjacent to every other vertex of X .*

Proof. If X is a complete graph, the result follows directly. Otherwise, suppose X is not complete: then, let S be a minimal separating set of X . We have that $S \neq \emptyset, V(X)$, since X is a connected non-complete graph.

Clearly, every cycle of length 4 inside X contains a chord, and as X does not contain full subgraphs isomorphic to P_4 either, cycles of length greater than 4 also contain a chord and we are under the conditions of Lemma 3.2.1, so S induces a complete graph.

Now, let $x \in S$. Note that it is sufficient to see that, for any $y \in V(X) \setminus S$, we have that $\{x, y\} \in E(X)$. That way, let $y \in V(X) \setminus S$: then $X_{(S)}$ is disconnected, and we may denote by C_y the connected component of $X_{(S)}$ containing y and by C any other component of $X_{(S)}$.

Suppose now that x and y are not connected. Since S is minimal, there must exist $z \in C$ adjacent to $x \in S$ and a minimal walk $Q_y = (x, y_1, \dots, y_n, y)$, with $y_1, \dots, y_n \in C_y$, connecting x and y , with $n \geq 1$. By the minimality of the walk, there is no edge between vertices of Q_y , and $\{y, z\}, \{y_i, z\} \notin E(X)$ since the vertices belong to different connected components of $X_{(S)}$. So, as Figure 3.3 shows, X contains at least one full subgraph isomorphic to P_4 , a contradiction. Hence, $\{x, y\} \in E(X)$ and we are done. $\square$



Figure 3.3: Walk connecting $z \in C$ with $y \in C_y$. Since $n \geq 1$, the walk contains at least one full subgraph isomorphic to P_4 , a contradiction.

With this technical result, we are now finally under the conditions to characterize the subclass of RAAGs whose subgroups are PC-groups, $G(\mathcal{D})_{fg}$.

Theorem 3.2.3 (C. Droms, [17]). *Let $X \in \mathcal{G}$ be a finite graph. Then, every subgroup of $G(X)$ is again a PC-group if and only if X has no full subgraph isomorphic to either C_4 or P_4 . In this case, $X \in \mathcal{D}_f$ and $G(X) \in G(\mathcal{D})_{fg}$.*

Proof. The implication to the left is done by induction on the number of vertices of X , $|V(X)|$:

If X is empty, the result is clear, and if $|V(X)| = 1$, then $G(X) = \mathbb{Z}$, and every subgroup $H \leq \mathbb{Z}$ is either trivial or isomorphic to $\mathbb{Z}$, hence, again a PC-group. For the inductive step, first fix $n > 1$, suppose the implication is true for graphs with $n-1$ vertices or less, and let us prove it for graphs X with $|V(X)| = n$.

We will distinguish the connected and disconnected cases: firstly, if X is a finite disconnected graph with connected components $X_1, \dots, X_m$, then

$$G(X) = G(X_1) * \dots * G(X_m),$$

and, as X has no full subgraph isomorphic to C_4 or P_4 , neither do $X_1, \dots, X_m$. Now, $X_1, \dots, X_m$ satisfy $|V(X_i)| < n$, for $1 \leq i \leq m$ so, by inductive hypothesis, subgroups of $G(X_i)$, for $1 \leq i \leq m$, are again PC-groups. Kurosh Subgroup Theorem characterizes

the form of subgroups of a free product which, in this case, can be expressed as follows: if H is an arbitrary subgroup of $G(X) = G(X_1) * \cdots * G(X_m)$, then

$$H \cong \mathbb{F}_r * \left(\bigstar_{k \in K} H_k \right),$$

where $0 \leq r \leq \infty$, $H_k \leq G(X_i)$, for some $1 \leq i \leq m$ and K is a possibly infinite set. As $\mathbb{F}_r$ is a PC-group, $\mathbb{F}_r = G(E_r)$ and $H_k = G(Y_k)$ because of the observation made before, we finally conclude that

$$H \cong G(E_r) * \left(\bigstar_{k \in K} G(Y_k) \right) = G \left(E_r \sqcup \left(\bigsqcup_{k \in K} Y_k \right) \right),$$

and H is a PC-group, as desired.

For the connected case, and by Lemma 3.2.2, there exists a vertex $z \in V(X)$ adjacent to all other vertices of X , and we have that

$$G(X) = \langle z \rangle \times G(X_{(z)}).$$

So, we can write the following short exact sequence

$$1 \longrightarrow \langle z \rangle \longrightarrow G(X) \xrightarrow{\rho} G(X_{(z)}) \longrightarrow 1, \quad (1)$$

where $\rho: G(X) \rightarrow G(X_{(z)})$ denotes the projection of $G(X)$ onto the second component. Now, let H be any subgroup of $G(X)$: then, restricting (1) to H , we get

$$1 \longrightarrow H \cap \langle z \rangle \longrightarrow H \xrightarrow{\rho} \rho(H) \longrightarrow 1. \quad (2)$$

We first observe here that, as X contains no full subgraph isomorphic to either C_4 or P_4 , neither does $X_{(z)}$. Moreover, $|V(X_{(z)})| = n - 1$ so, by the inductive hypothesis, subgroups of $G(X_{(z)})$ are again PC-groups. In particular $\rho(H) \leq G(X_{(z)})$ is a PC-group, say $\rho(H) \cong G(Y)$, for some graph $Y \in \mathcal{G}$.

Now, if $H \cap \langle z \rangle$ is trivial, then $H \cong \rho(H) \cong G(Y)$ and we are done. On the other hand, if the previous intersection is infinite cyclic, say $H \cap \langle z \rangle = \langle z^s \rangle$, with $s \geq 1$, we will see that (2) splits: firstly, and by the definition of ρ , it is clear that, for every $h \in H$, $\rho(h)h^{-1} \in \langle z \rangle$. Now, take a generator $y \in V(Y)$ of $\rho(H) = G(Y)$. Then, $y = \rho(h)$, for some $h \in H$ and, by the observation above, there exists $l \in \mathbb{Z}$ such that $yz^l \in H$.

That way, for every generator $y \in V(Y)$ of $\rho(H)$ we may define the homomorphism

$$\begin{aligned} \rho^{-1}: \rho(H) &\longrightarrow H \\ y &\longmapsto yz^l. \end{aligned}$$

To see that ρ^{-1} is well-defined we just have to see that the relations on $\rho(H)$ (of the form $[y, y']$, for some $y, y' \in V(Y)$) are preserved in H (see Proposition 2.2.18). That way, let y, y' be two generators of $\rho(H)$ such that $[y, y'] = 1$. Then,

$$\rho^{-1}([y, y']) = [\rho^{-1}(y), \rho^{-1}(y')] = [yz^l, y'z^{l'}] = [y, y'] = 1,$$

where we used that z commutes with all elements of $G(X)$ (and, hence, of $\rho(H)$). Finally, $\rho(\rho^{-1}(y)) = \rho(yz^l) = y$ for all $y \in V(Y)$, so $\rho \circ \rho^{-1} = \text{Id}$ and (2) splits.

We can now write H as a semidirect product, concretely $H = \langle z^s \rangle \rtimes_{\alpha} G(Y)$. In fact, the semidirect product is actually a direct product because of the form of α ,

$$\begin{aligned} \alpha: G(Y) &\longrightarrow \text{Aut}(\langle z^s \rangle) \\ y &\longmapsto \alpha(y) = C_{\rho^{-1}(y)}|_{\langle z^s \rangle}, \end{aligned}$$

where C_x denotes conjugation with $x \in G(X)$. We have that, as z commutes with every element, $\alpha(y) = \text{Id}_{\langle z^s \rangle}$, $\forall y \in G(Y)$, and, finally,

$$H = \langle z^s \rangle \times G(Y) \cong \mathbb{Z} \times G(Y) \cong G(\text{cone}(Y)).$$

So, we conclude that H is a PC-group, as we wanted, and we are finally done.

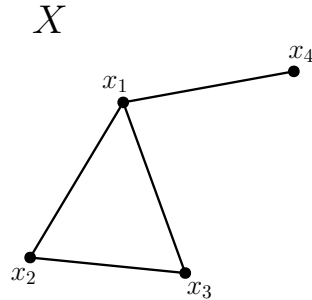
For the implication to the right, note that it is sufficient to find subgroups $H \leq G(C_4)$, $K \leq G(P_4)$ that are not PC-groups. This is because if Y is a full subgraph of X , then the subgroup of $G(X)$ generated by the vertices of Y is isomorphic to $G(Y)$.

The case of $G(C_4) \cong \mathbb{F}_2 \times \mathbb{F}_2$ has already been discussed: since $\mathbb{F}_2 \times \mathbb{F}_2$ is not coherent, it contains finitely generated subgroups that are not finitely presented and, hence, not PC-groups (see [36] for a detailed proof).

The case of $G(P_4)$ is not that direct. Firstly, let us denote by t the unique generator of the finite cyclic group $\mathbb{Z}_2$. Now, consider the homomorphism $\varphi: G(P_4) \rightarrow \mathbb{Z}_2$ defined by $\varphi(v) = t$, for every $v \in V(P_4)$. Defining $K := \ker(\varphi) \leq G(P_4)$, one has that K is not a PC-group. However, the proof of this fact uses methods outside the scope of group theory and will not be included here (a complete reference can be found in [17]). $\square$

A good exercise is to now show how the previous theorem works with a simple example: that is, given a finite graph $X \in \mathcal{D}_f$, study the form of the subgroups $H \leq G(X)$, and verify that they can all be represented by (non necessarily finite) graphs again.

Example 3.2.4. Consider the following graph:



Clearly, X does not contain full subgraphs isomorphic to C_4 nor P_4 , so we are under the conditions of the previous theorem. We can also write the presentation of the RAAG associated to X , which in this case corresponds to

$$G(X) = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_3] \rangle.$$

We can use the procedure of the previous proof to try to list all possible subgroups of $G(X)$. Here, X is clearly connected, and $x_1 \in V(X)$ is the vertex of X adjacent to all the other vertices of the graph. As $X_{(x_1)} \cong K_2 \sqcup E_1$ is disconnected, $G(X_{(x_1)}) = \langle x_2, x_3 \mid [x_2, x_3] \rangle * \langle x_4 \mid - \rangle \cong \mathbb{Z}^2 * \mathbb{Z}$ is a free product and, hence,

$$G(X) = \langle x_1 \mid - \rangle \times \left(\langle x_2, x_3 \mid [x_2, x_3] \rangle * \langle x_4 \mid - \rangle \right) \cong \mathbb{Z} \times (\mathbb{Z}^2 * \mathbb{Z}).$$

Now, subgroups $H \leq \mathbb{Z}^2 * \mathbb{Z}$ are given by the Kurosh Subgroup Theorem, which in this case take the form

$$H \cong \mathbb{F}_r * \left(\bigstar_{k \in K} \mathbb{Z}^2 \right) * \left(\bigstar_{j \in J} \mathbb{Z} \right),$$

where $0 \leq r \leq \infty$ and K and J are possibly infinite sets. So, absorbing the last term into $\mathbb{F}_r$ by adjusting r , we can now conclude that, if $H \leq G(X)$, then

$$\begin{cases} H \cong \mathbb{F}_r * (*_{k \in K} \mathbb{Z}^2) = G(E_r \sqcup (\bigsqcup_{k \in K} K_2)); \\ \text{or} \\ H \cong \mathbb{Z} \times (\mathbb{F}_r * (*_{k \in K} \mathbb{Z}^2)) = G(\text{cone}(E_r \sqcup (\bigsqcup_{k \in K} K_2))), \end{cases}$$

where r and K are as specified before. In terms of graphs, we are looking at these two possible situations:



Figure 3.4: Graph representations of the possible subgroups $H \leq G(X)$ when $r = 5$, $|K| = 3$.

A direct observation we can make here is that, even if X is a finite graph (representing a finitely presented group $G(X)$), there may exist infinitely presented subgroups $H \leq G(X)$ (which are represented by infinite graphs). This is due to the form of the subgroups of a free product of groups, given by the Kurosh Subgroup Theorem, and reinforces the need to distinguish between the definitions of $\mathcal{D}$ and $\mathcal{D}_f$ and, hence, the definitions of $G(\mathcal{D})$ and $G(\mathcal{D})_{fg}$.

3.2.1 A recursive characterization of $\mathcal{D}_f$

In the proof of Theorem 3.2.3 we used two operations on graphs: taking the cone of a graph $X \in \mathcal{G}$, denoted by $\text{cone}(X)$ (or $X \wedge E_1$), which simply adds a vertex to X connected to all the other vertices of the graph, and the disjoint union of two graphs $X, Y \in \mathcal{G}$, denoted by $X \sqcup Y$, that represents the disconnected graph with components X and Y .

The next proposition allows us to recursively characterize the family of finite Droms graphs using just these two graph operations.

Proposition 3.2.5. *The family of finite Droms graphs, $\mathcal{D}_f$, is the minimal subfamily of finite graphs satisfying the following properties:*

- 1) $E_0 \in \mathcal{D}_f$;
- 2) $X \in \mathcal{D}_f \implies \text{cone}(X) \in \mathcal{D}_f$;
- 3) $X_1, X_2 \in \mathcal{D}_f \implies X_1 \sqcup X_2 \in \mathcal{D}_f$.

That is, if $\mathcal{P} \subseteq \mathcal{G}$ is another family of finite graphs satisfying 1), 2) and 3), then $\mathcal{D}_f \subseteq \mathcal{P}$.

Proof. Firstly, we see that $\mathcal{D}_f$ satisfies 1), 2) and 3).

Clearly, $E_0 \in \mathcal{D}_f$. Now, if $X \in \mathcal{D}_f$, then X has no full subgraphs isomorphic to either C_4 or P_4 . Let Y be a full subgraph of $\text{cone}(X)$: then, there exists a full subgraph Z of X such that

$$Y \cong Z \quad \text{or} \quad Y \cong \text{cone}(Z).$$

By hypothesis, Z does not contain induced copies of either C_4 or P_4 , so neither does $\text{cone}(Z)$. Thus, we conclude $\text{cone}(X) \in \mathcal{D}_f$.

On the other hand, suppose $X_1, X_2 \in \mathcal{D}_f$. Then, if Y is a full subgraph of $X_1 \sqcup X_2$, there must exist a full subgraph Y_1 of X_1 and a full subgraph Y_2 of X_2 such that $Y \cong Y_1 \sqcup Y_2$. As neither Y_1 nor Y_2 contain induced copies of C_4 or P_4 , the same holds for Y , so $X \in \mathcal{D}_f$ and we are done.

Now, let $\mathcal{P}$ be another family of finite graphs satisfying 1), 2) and 3) and let's see $\mathcal{D}_f \subseteq \mathcal{P}$. If $X \in \mathcal{D}_f$, we will prove by induction on the number of vertices of X that $X \in \mathcal{P}$: if $|V(X)| = 0$, then directly $X = E_0 \in \mathcal{P}$.

Now, suppose $|V(X)| \geq 1$ and the containment true for graphs with less than $|V(X)|$ vertices. If X is connected, then it must have a vertex $z \in V(X)$ adjacent to all other vertices of X (see Lemma 3.2.2), with $X_{(z)} \in \mathcal{D}_f$ because $X_{(z)}$ is the subgraph of X induced by $V(X) \setminus \{z\}$. Hence, by inductive hypothesis $X_{(z)} \in \mathcal{P}$ and, using 2), $X \cong \text{cone}(X_{(z)}) \in \mathcal{P}$.

On the other hand, if X is disconnected we can then write $X = X_1 \sqcup X_2$, where X_1 and X_2 are both nontrivial (thus, with less vertices than X). Again $X_1, X_2 \in \mathcal{D}_f$ because they are the subgraphs of X induced by $V(X_1)$ and $V(X_2)$, respectively. Hence, and by inductive hypothesis $X_1, X_2 \in \mathcal{P}$ and, using 3), $X = X_1 \sqcup X_2 \in \mathcal{P}$, so we are finally done. $\square$

Due to the minimality of $\mathcal{D}_f$, it follows that every graph $X \in \mathcal{D}_f$ can be obtained from E_0 through a finite sequence of cones and disjoint unions of previously constructed elements. Next, we state the algebraic counterpart of the previous result, obtaining an alternative characterization for the subfamily of finitely generated Droms groups.

Corollary 3.2.6. *The family of finitely generated Droms groups, $G(\mathcal{D})_{fg}$, is the minimal subfamily of finitely generated RAAGs satisfying the following properties:*

- 1) $\{1\} \in G(\mathcal{D})_{fg}$;
- 2) $G(X) \in G(\mathcal{D})_{fg} \implies G(X) \times \mathbb{Z} \in G(\mathcal{D})_{fg}$;
- 3) $G(X_1), G(X_2) \in G(\mathcal{D})_{fg} \implies G(X_1) * G(X_2) \in G(\mathcal{D})_{fg}$.

That is, if $\mathcal{H} \subseteq \mathcal{R}$ is another family of finitely generated RAAGs satisfying 1), 2) and 3), then $G(\mathcal{D})_{fg} \subseteq \mathcal{H}$.

Proof. The proof directly follows from last proposition, where cones in graphs translate to direct products with $\mathbb{Z}$, and disjoint unions of graphs correspond to free products of groups. Also, $G(E_0) = \{1\}$, so the result clearly follows. $\square$

Similarly as for $\mathcal{D}_f$, Corollary 3.2.6 establishes that every group $G(X) \in G(\mathcal{D})_{fg}$ can be obtained from the trivial group through a finite sequence of direct products with $\mathbb{Z}$ and free products of previously constructed elements.

Also, these last two results will help us prove properties for finite Droms graphs and finitely generated Droms groups: by an inductive reasoning, if we want to prove that all members in $\mathcal{D}_f$ satisfy a certain property $\mathcal{P}$, then Proposition 3.2.5 tells us that it is sufficient to prove $\mathcal{P}$ for E_0 , $\text{cone}(X)$ and $Y \sqcup Z$, while assuming that $X, Y, Z \in \mathcal{D}_f$ already satisfy $\mathcal{P}$. The argument for $G(\mathcal{D})_{fg}$ is completely analogous using Corollary 3.2.6.

To conclude the section, let us make a simple observation: by definition, Droms groups and its subgroups are PC-groups; however, we can say more: subgroups of Droms groups are Droms groups again. The following easy result reflects this fact.

Observation 3.2.7. The subgroups of a Droms group are again Droms groups.

Proof. Let $G(X) \in G(\mathcal{D})$, with $X \in \mathcal{D}$, and suppose there exists a subgroup $H \leq G(X)$ such that $H \notin G(\mathcal{D})$. Then, by definition, there would exist a subgroup $K < H$ which is not a PC-group but, as $K < H \leq G(X)$, we get a contradiction, since all subgroups of $G(X)$ were PC-groups, by definition.

However, note that the observation is not true for $G(\mathcal{D})_{fg}$, as there exist infinitely generated subgroups of finitely generated Droms groups (see Example 3.2.4). For instance, $\mathbb{F}_\infty \in G(\mathcal{D}) \setminus G(\mathcal{D})_{fg}$ embeds into $\mathbb{F}_2 \in G(\mathcal{D})_{fg}$, a trivial example. $\square$

Chapter 4

Algorithmic problems in Droms groups

This chapter is dedicated to studying algorithmic problems in the family of Droms groups. In algorithmic group theory, typically, groups are given as inputs by means of a (finite) presentation; note that, in our case, this is equivalent to providing the (finite) Droms graph associated to the group. Hence, we will be able to use the characterizations for finitely presented Droms groups we proved in the previous section. Particularly, Proposition 3.2.5 and Corollary 3.2.6 will be continuously used throughout the whole chapter.

We divide the chapter into two main sections, where we will address two different types of problems: subgroup and embeddability problems. Here, the notion of *Ivanov automata*, presented in the preliminaries and detailed in [12] and [21], will become central, as we will need to work with subgroups of free products of groups.

In that direction, note that to construct a reduced wedge automaton for a subgroup of a free product of groups, $H \leq G_1 * G_2$, *Membership Problem* must be solvable in G_1 and G_2 (see Theorem 2.2.32). Even though not all RAAGs have solvable membership problem ($\mathbb{F}_2 \times \mathbb{F}_2 = G(C_4) \in \mathcal{R}$ is the classical counterexample), the subfamily of finitely generated Droms groups have indeed solvable MP (see [22]; in fact, as shown in [26], finitely generated Droms groups have solvable *Submonoid Membership Problem*, a generalization of MP), so Theorem 2.2.32 will apply in the whole chapter.

4.1 Subgroup problems in $\mathcal{D}_f$

This first section is dedicated to studying algorithmic problems in Droms groups that depend on a single (finitely generated) Droms group $G(X) \in G(\mathcal{D})_{fg}$ and have as input a (finitely generated) subgroup $H \leq G(X)$, given by a (finite) set of generators $S \subseteq G(X)$ such that $H = \langle S \rangle$.

Particularly, we are interested in solving two problems for $G(X) \in G(\mathcal{D})_{fg}$: given $S \subseteq G(X)$, the first one consists on deciding whether the subgroup $\langle S \rangle \leq G(X)$ is of finite index in $G(X)$, and the second one finds a presentation for $\langle S \rangle$. The general problems are formally stated next:

Finite Index Problem for $H = \langle A \mid B \rangle$, $FIP(H)$: given a finite subset $S \subseteq H$, decide whether the subgroup generated by S has finite index in H , that is, if $\langle S \rangle \leq_{fi} H$. In the affirmative case, additionally, compute $[H : \langle S \rangle]$.

Subgroup Recognition Problem for $H = \langle A \mid B \rangle$, $SRP(H)$: given a finite subset $S \subseteq H$, find a presentation for the subgroup generated by S , $\langle S \rangle$.

It is worth mentioning that, for the family of all finitely presented groups, these problems are unsolvable. However, when restricting to $G(\mathcal{D})_{fg}$, we will be able to prove solvability.

We begin with the study of *FIP*. To prove the decidability of the problem we will use the recursive characterization of Droms groups and a graphical characterization of finite index subgroups of a free product of groups in terms of *Ivanov automata*. Analogously to the theory of *Stallings automata*, where the property of having a saturated Stallings automaton characterizes finite index subgroups of free groups, we will identify similar conditions for subgroups of free products that ensure finiteness of the index.

Proposition 4.1.1. *Let $G_1 * G_2$ be an arbitrary free product of groups, $H \leq_{fg} G_1 * G_2$ and $\mathfrak{J}$ a reduced wedge automaton recognizing H . Then, H has finite index in $G_1 * G_2$ if and only if the following conditions are satisfied:*

- 1) *For $\iota = 1, 2$, every label subgroup of a ι -secondary vertex of $\mathfrak{J}$ has finite index in G_ι ;*
- 2) *For $\iota = 1, 2$, every ι -secondary vertex q satisfies $\delta(q) = [G_\iota : \ell(q)]$;*
- 3) *Every primary vertex of $\mathfrak{J}$ has degree 2.*

If that is the case, then

$$[G_1 * G_2 : H] = |V_0(\mathfrak{J})| = \sum_{q \in V_1(\mathfrak{J})} \delta(q) = \sum_{r \in V_2(\mathfrak{J})} \delta(r).$$

Proof. Let us start with the implication to the right, so suppose H has finite index n in $G_1 * G_2$ and let us first see 1):

Indeed, assume, without loss of generality, that there exists a 1-secondary vertex with label $B \leq G_1$ that has infinite index in G_1 . Now, by the Kurosh Subgroup Theorem, there exists $g \in G_1 * G_2$ such that $B = G_1 \cap H^g$. Then, we have that

$$\infty = [G_1 : B] = [G_1 : G_1 \cap H^g] \leq [G_1 * G_2 : H^g] = [G_1 * G_2 : H] < \infty,$$

a contradiction. Note that in the first inequality we exploited that, generally, for $A, C \leq G$, we have $[A : A \cap C] \leq [G : C]$. We also used the fact that the index is preserved under conjugation of subgroups.

To see 3) note that, since $\mathfrak{J}$ is reduced, every primary vertex has degree either 1 or 2. So, suppose $\exists p \in V_0(\mathfrak{J})$ with $\delta(p) = 1$ and, without loss of generality, suppose that p has no G_2 -edge. Then, consider $a \in G_2$, $b \in G_1$ two nontrivial elements and let us attach an infinite branch to p as showed in the following image:

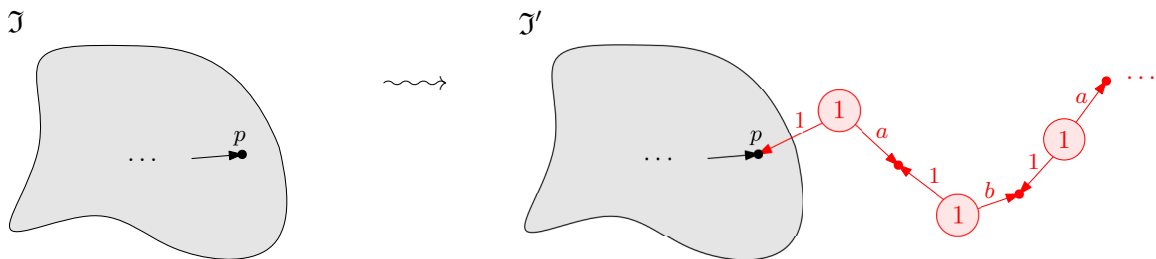


Figure 4.1: Construction of $\mathfrak{J}'$ attaching an infinite branch to the vertex p of $\mathfrak{J}$ that has degree 1.

We observe that $\mathfrak{J}'$ is reduced: indeed, we only have to verify that no non-degenerate elementary walk in $\mathfrak{J}'$ reads the trivial element, but this is clear since they only read the singletons $\{a\} \neq_{G_2} \{1\}$ or $\{b\} \neq_{G_1} \{1\}$ (since the label subgroups are trivial). Moreover, from the construction of $\mathfrak{J}'$, every closed path starting and ending at the basepoint of $\mathfrak{J}$ can only visit added edges in a backtracking way; so, $\langle \mathfrak{J}' \rangle = \langle \mathfrak{J} \rangle$. Finally, using Lemma 2.2.29 we get that $[G_1 * G_2 : H] = [G_1 * G_2 : \langle \mathfrak{J}' \rangle] \geq |V_0(\mathfrak{J}')| = \infty$, a contradiction.

Finally, we see 2) with a similar argument. Note that, as $\mathfrak{J}$ is reduced, then, for $\iota = 1, 2$, and for every ι -secondary vertex q , $\delta(q) \leq [G_\iota : \ell(q)]$: in fact, as a direct consequence of being reduced, labels at the edges incident to q belong to different cosets of $\ell(q)$. On the other hand, if $\delta(q) < [G_\iota : \ell(q)]$, choose an element $g \in G_\iota$ not belonging to any of the cosets of G_ι modulo $\ell(q)$ represented by those labels, and add a new edge incident to q , labeled by g , and going to a new primary vertex (of degree 1). It is clear that this enlargement provides a new reduced wedge atomaton $\tilde{\mathfrak{J}}$ recognizing the same subgroup H and having a primary vertex of degree 1. Connecting with the previous construction, this contradicts again $H \leq_{fi} G_1 * G_2$.

For the implication to the left, suppose conditions 1) – 3) are satisfied and let us see that $H \leq_{fi} G_1 * G_2$. Firstly, using the three conditions and the fact that $\mathfrak{J}$ is reduced and primary-secondary bipartite we directly have that

$$|V_0(\mathfrak{J})| = \sum_{q \in V_1(\mathfrak{J})} \delta(q) = \sum_{r \in V_2(\mathfrak{J})} \delta(r).$$

Now, note that it is sufficient to show that every element $g \in G$ can be read as the label of a path from the basepoint of $\mathfrak{J}$ to a primary vertex. Firstly, since $\mathfrak{J}$ is reduced and 2) holds, for $\iota = 1, 2$, labels of edges adjacent to a ι -secondary vertex p of $\mathfrak{J}$ form a set of coset representatives for G_ι and, hence, all elements of G_ι can be read as the label of an elementary walk from any primary vertex p . Now, since every primary vertex q of $\mathfrak{J}$ has degree 2 (that is, q is incident to an edge of each type, because $\mathfrak{J}$ is reduced), we get that every element of the form $g_1 g'_1 \dots g_m g'_m$, with $g_i \in G_1$, $g'_j \in G_2$ and $m \in \mathbb{Z}^+$ can be read as the label of a path starting at $\odot \in V_0(\mathfrak{J})$. But that is the normal form of elements in $G_1 * G_2$, so we are done. In this case, by Lemma 2.2.29, $[G_1 * G_2 : \langle \mathfrak{J} \rangle] = |V_0(\mathfrak{J})|$. $\square$

With the help of this technical result we now prove the decidability of $FIP(G(X))$, for $X \in \mathcal{D}_f$, which will follow in a quite direct way. Note that, along the proof, we use that the *Intersection Problem* is solvable for Droms groups (see [12]).

Theorem 4.1.2. *For every $X \in \mathcal{D}_f$, $FIP(G(X))$ is solvable.*

Proof. Firstly, the problem is clearly solvable when the graph X has no vertices. Now, we just have to see that the solvability of $FIP(G(X))$ passes through cones and disjoint unions of graphs (see Proposition 3.2.5). That way, suppose $Y \in \mathcal{D}_f$ and assume $FIP(G(Y))$ is solvable. Then, if S is a finite subset of $G(\text{cone}(Y))$ and z is the vertex of the cone, we have that

$$\langle S \rangle \leq_{fi} G(\text{cone}(Y)) \iff \begin{cases} \langle z \rangle \cap \langle S \rangle = \langle z^r \rangle, \text{ for some } r \geq 1, \text{ and} \\ \rho(\langle S \rangle) \leq_{fi} G(Y), \end{cases}$$

where $\rho: \langle z \rangle \times G(Y) \longrightarrow G(Y)$ denotes the projection onto the second component. By [12], the intersection $\langle z \rangle \cap \langle S \rangle$ is computable and, as $\rho(\langle S \rangle) \leq_{fi} G(Y)$ is decidable by assumption, we are done. In the case $\langle S \rangle \leq_{fi} G(\text{cone}(Y))$, if $\langle S \rangle = \langle z^r \rangle \times \rho(\langle S \rangle)$, with $r \geq 1$ and $\rho(\langle S \rangle) \leq_{fi} G(Y)$, then

$$[G(\text{cone}(Y)) : \langle S \rangle] = [\langle z \rangle : \langle z^r \rangle] \cdot [G(Y) : \rho(\langle S \rangle)] = r \cdot [G(Y) : \rho(\langle S \rangle)].$$

Finally, suppose $Y_1, Y_2 \in \mathcal{D}_f$, assume $FIP(G(Y_1))$ and $FIP(G(Y_2))$ are both solvable and let us prove that $FIP(G(Y_1 \sqcup Y_2))$ is also solvable. Following Proposition 4.1.1, if $\mathfrak{J}$ is a reduced wedge automaton recognizing $\langle S \rangle$ (constructible in finite time, see the preliminaries), then $\langle S \rangle \leq_{fi} G(Y_1 \sqcup Y_2) = G(Y_1) * G(Y_2)$ if and only if conditions 1) – 3) from Proposition 4.1.1 are satisfied. Condition 1) is verifiable by assumption, and conditions 2) and 3) are easy checkable in finite time, so $FIP(G(Y_1 \sqcup Y_2))$ is solvable and we are done. In the case $\langle S \rangle \leq_{fi} G(Y_1 \sqcup Y_2)$, then $[G(Y_1 \sqcup Y_2) : \langle S \rangle] = |V_0(\mathfrak{J})|$. $\square$

Note that the algorithm that directly follows from last proof is recursive: at each step of the procedure, multiple instances of the same algorithm (but on graphs with fewer vertices) are started, eventually ending with the naive base cases, which are trivial to solve. This is because we used Proposition 3.2.5, which describes Droms graphs and their associated groups in a recursive way.

Now we turn to study the decidability of the *Subgroup Recognition Problem* on a Droms graph $G(X) \in G(\mathcal{D})_{fg}$, $SRP(G(X))$. In this case, the setting is much simpler, as we can directly prove its solvability without relying on prior results.

Theorem 4.1.3. *For every $X \in \mathcal{D}_f$, $SRP(G(X))$ is solvable.*

Proof. We also proceed using the recursive description of $\mathcal{D}_f$ in Proposition 3.2.5. Firstly, the result is obvious for the unique graph X with no vertices.

Now, take $Y \in \mathcal{D}_f$, assume $SRP(G(Y))$ is solvable and let us see that $SRP(G(\text{cone}(Y)))$ is also solvable. If $z \in V(\text{cone}(Y))$ is one of the vertices adjacent to all the other vertices of $\text{cone}(Y)$ and $\rho: \langle z \rangle \times G(Y) \rightarrow G(Y)$ denotes the projection onto the second component of $G(\text{cone}(Y))$, we then have that

$$\langle z \rangle \cap \langle S \rangle = \begin{cases} \{1\} & (\text{and so, } \langle S \rangle \cong \rho(\langle S \rangle) \leq G(Y)) \text{ or} \\ \langle z^r \rangle, & \text{for some } r \geq 1 \text{ (and so, } \langle S \rangle = \langle z^r \rangle \times \rho(\langle S \rangle), \text{ where } \rho(\langle S \rangle) \leq G(Y)) \end{cases}$$

By [12], we can evaluate the intersection $\langle z \rangle \cap \langle S \rangle$, and so we are left to deal with its remaining two cases: in the first one, as $\langle S \rangle$ and $\rho(\langle S \rangle)$ are isomorphic, we just have to answer $SRP(G(Y))$ on input $\rho(\langle S \rangle)$, which can be done by assumption (note that $\rho(\langle S \rangle) = \langle \rho(S) \rangle$ is also easily evaluated) and, in the second case, $\langle S \rangle \cong G(\text{cone}(Z))$, where $Z \in \mathcal{D}_f$ is such that $\rho(\langle S \rangle) \cong G(Z)$, findable again by assumption.

Finally, suppose now $Y_1, Y_2 \in \mathcal{D}_f$, assume $SRP(G(Y_1))$, $SRP(G(Y_2))$ are both solvable and let us prove that $SRP(G(Y_1 \sqcup Y_2))$ is also solvable. Firstly, as S is finite and $\langle S \rangle$ is a finitely generated subgroup of a free product, we can construct a reduced wedge automaton recognizing $\langle S \rangle$, namely $\mathfrak{J}$, in finite time (see the preliminaries). Now, by Theorem 2.2.34, $\mathfrak{J}$ directly gives us a finite *Kurosh decomposition* for $\langle S \rangle$, say

$$\langle S \rangle = \langle \mathfrak{J} \rangle \cong \mathbb{F}_r * \left(\bigstar_{i=1}^p H_i \right) * \left(\bigstar_{j=1}^q K_j \right) \leq G(Y_1) * G(Y_2),$$

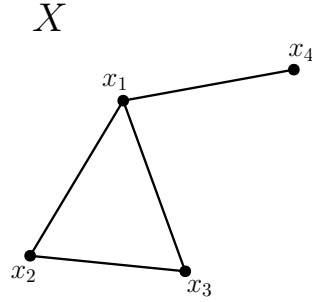
where $0 \leq r < \infty$, $H_i \leq G(Y_1)$, and $K_j \leq G(Y_2)$. Then, we have that

$$\langle S \rangle \cong G \left(E_r \sqcup \left(\bigcup_{i=1}^p Z_i \right) \sqcup \left(\bigcup_{j=1}^q W_j \right) \right),$$

where $Z_i, W_j \in \mathcal{D}_f$ are such that $H_i \cong G(Z_i)$, $K_j \cong G(W_j)$, findable by assumption. This finally ends the proof. $\square$

Even though the algorithm we obtained to solve $SRP(G(X))$, for $X \in \mathcal{D}_f$, is also recursive, we can try to apply it with a simple example.

Example 4.1.4. Consider the same $X \in \mathcal{D}_f$ as in the previous chapter:



We have $G(X) = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_3] \rangle = \langle x_1 \rangle \times G(X_{(x_1)}) \cong \mathbb{Z} \times (\mathbb{Z}^2 * \mathbb{Z})$. Now, let $S_1 = \{x_1x_4, x_4^{-1}x_2x_4, x_2x_3\}$. As X is connected, we have to compute the intersection $\langle x_1 \rangle \cap \langle S_1 \rangle$, and here it is not difficult to see that $\langle x_1 \rangle \cap \langle S_1 \rangle = \{1\}$. Then, $\langle S_1 \rangle$ is isomorphic to $\rho(\langle S_1 \rangle) = \langle \rho(S_1) \rangle$, where $\rho: \langle x_1 \rangle \times G(X_{(x_1)}) \rightarrow G(X_{(x_1)})$ denotes the projection of $G(X)$ onto the second component. Now,

$$\rho(S_1) = \{x_4, x_4^{-1}x_2x_4, x_2x_3\}.$$

But, if $x_4 \in \rho(\langle S_1 \rangle)$, then $x_2 \in \rho(\langle S_1 \rangle)$ and so $x_3 \in \rho(\langle S_1 \rangle)$ too. We then have $\langle S_1 \rangle \cong \rho(\langle S_1 \rangle) = G(X_{(x_1)})$ and $X_{(x_1)} \in \mathcal{D}_f$ is the graph we were looking for. In this case, $\langle S_1 \rangle \cong \langle x_2, x_3, x_4 \rangle = G(X_{(x_1)}) \cong \mathbb{Z}^2 * \mathbb{Z}$.

Consider now $S_2 = \{x_1^4, x_4x_2, x_2x_3\}$. We clearly have $\langle x_1 \rangle \cap \langle S_2 \rangle = \langle x_1^4 \rangle$ and $\rho(S_2) = \{1, x_4x_2, x_2x_3\}$. Here, we can not conclude anything more about the subgroup $\rho(\langle S_2 \rangle)$, so we have to compute a reduced wedge automaton $\mathfrak{J}$ recognizing $\rho(\langle S_2 \rangle)$ (see the preliminaries):

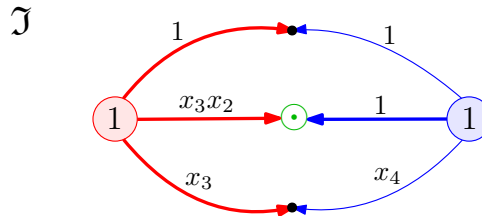


Figure 4.2: A reduced wedge automaton for $\rho(\langle S_2 \rangle)$. A spanning tree of $\mathfrak{J}$ is denoted with thicker edges.

From this construction, and using Theorem 2.2.34, we get that $\rho(\langle S_2 \rangle) \cong \mathbb{F}_2$, and so $\langle S_2 \rangle \cong \mathbb{Z} \times \mathbb{F}_2 \cong G(\text{cone}(E_2))$, which is what we wanted to obtain.

4.1.1 Schreier's Formula for Droms groups

Before turning to study embeddability problems on Droms groups, we will prove an interesting condition that must satisfy finite index subgroups of Droms groups. Let us start by providing some historical background:

In the late 1920s O. Schreier [34] proved the following formula for finite index subgroups H of a free group $\mathbb{F}$ (again free), which related the rank of the subgroup (as a free group) with the rank of the ambient group and the index:

$$(\operatorname{rk}(\mathbb{F}) - 1)[\mathbb{F} : H] = \operatorname{rk}(H) - 1. \quad \left(\begin{array}{l} \text{Schreier's} \\ \text{Formula} \end{array} \right)$$

The following theorem extends this compact result to the realm of finite index subgroups of Droms groups, exchanging the rank of the free groups with the number of connected components of the corresponding Droms graphs.

Theorem 4.1.5 (Schreier's Formula for Droms groups). *Let $X \in \mathcal{D}_f$ be a finite Droms graph and $L \leq_{fi} G(X)$. Then,*

$$(c_X - 1)[G(X) : L] = c_Y - 1, \quad \left(\begin{array}{l} \text{Schreier's} \\ \text{Formula} \\ \text{for } \mathcal{D}_f \end{array} \right)$$

where $Y \in \mathcal{D}_f$ is such that $G(Y) \cong L$.

Proof. We will proceed by induction on the number of connected components of X , c_X .

If $c_X = 1$, then X is connected and $\exists z \in V(X)$ such that $X = \operatorname{cone}(X_{(z)})$, $G(X) \cong G(X_{(z)}) \times \mathbb{Z}$. Now, since L has finite index in $G(X)$, we must have that

$$G(Y) \cong L = H \times \mathbb{Z}, \text{ with } H \leq_{fi} G(X_{(z)}),$$

and so Y is also a cone, hence, connected, and with $c_Y = 1$. Then, the formula trivially holds.

Suppose now $c_X > 1$ and assume the theorem is true for graphs with less than c_X connected components. We have that X can be expressed as $X = X_1 \sqcup X'$, where X_1 is a connected graph and X' is possibly disconnected, with $c_X - 1 \geq 1$ connected components. Then, $G(X) = G(X_1) * G(X')$. Now, let $\mathfrak{J}$ be a reduced wedge automaton recognizing L . Note that, since L has finite index in $G(X)$ we can use the characterization of Proposition 4.1.1 for $\mathfrak{J}$.

If $V_1(\mathfrak{J})$ is the set of 1-secondary vertices of $\mathfrak{J}$ (with labels corresponding to subgroups of $G(X_1)$) and $V_2(\mathfrak{J})$ is the set of 2-secondary vertices of $\mathfrak{J}$ (with labels corresponding to subgroups of $G(X')$), then Theorem 2.2.34 establishes that

$$G(Y) \cong L \cong \left(\bigstar_{p \in V_1(\mathfrak{J})} \ell(p) \right) * \left(\bigstar_{q \in V_2(\mathfrak{J})} \ell(q) \right) * \mathbb{F}_r,$$

where $r = \operatorname{rk}(\mathfrak{J})$. Since $\mathfrak{J}$ is connected (because it is reduced), and using again Proposition 4.1.1, we also have that

$$r = 1 - |V(\mathfrak{J})| + |E(\mathfrak{J})| = 1 - |V_1(\mathfrak{J})| - |V_2(\mathfrak{J})| + [G(X) : L], \quad (1)$$

where we used that $|V_0(\mathfrak{J})| = [G(X) : L]$, $|E(\mathfrak{J})| = 2[G(X) : L]$. On the other hand, we can also count the number of connected components of Y , c_Y :

$$\begin{aligned} c_Y &= r + |V_1(\mathfrak{J})| + \sum_{q \in V_2(\mathfrak{J})} ((c_{X'} - 1)[G(X') : \ell(q)] + 1) = \\ &= r + |V_1(\mathfrak{J})| + |V_2(\mathfrak{J})| + (c_X - 2) \sum_{q \in V_2(\mathfrak{J})} [G(X') : \ell(q)]. \end{aligned} \quad (2)$$

Note that we used here that X_1 is connected (hence, subgroups of $G(X_1)$ of finite index have their associated Droms graph connected too) and the inductive hypothesis (to count the connected components of the Droms graphs associated to the finite index subgroups of $G(X')$). Now, applying Proposition 4.1.1, $[G(X') : \ell(q)] = \delta(q)$ and the sum $\sum_{q \in V_2(\mathfrak{J})} [G(X') : \ell(q)]$ equals $[G(X) : L]$. Finally, combining (1) and (2) and rearranging the terms we get the desired result. $\square$

Schreier's Formula is easily seen to be a particular case of last theorem: when $G(X)$ is a free group (a particular case of Droms groups), then L is also free, and X, Y are empty graphs with as many connected components as the ranks of the respective groups. Hence, in that case, c_X and c_Y directly translate to $\text{rk}(G(X))$ and $\text{rk}(G(Y))$, respectively, and we easily recover the original Schreier's Formula.

Note that *Schreier's Formula for Droms groups* establishes that two isomorphic subgroups of a Droms group whose associated Droms graphs are disconnected can not have different (finite) index. However, it is not clear if, given an arbitrary Droms group $G(X) \neq \{1\}$ and a positive integer $k \in \mathbb{Z}^+$, one can always find a subgroup $H \leq_k G(X)$. The following simple observation answers this question in a positive way.

Observation 4.1.6. Let $X \in \mathcal{D}_f$, with $X \neq E_0$, and $k \in \mathbb{Z}^+$. Then, there exists a computable $H \leq_k G(X)$.

Proof. If $X \neq E_0$, then $G(X) \neq \{1\}$. The base case here is $X = E_1$: in that situation, and if $x \in V(X)$ is the unique vertex of the graph, we have that $G(X) = \langle x \rangle \cong \mathbb{Z}$, and $H := \langle x^k \rangle \cong \mathbb{Z}$ is a subgroup of $G(X)$ of index k .

Now, assume the observation is true for $X \in \mathcal{D}_f$, and let's prove it for $\text{cone}(X)$: we have that there exists $z \in V(\text{cone}(X))$ such that $G(\text{cone}(X)) = G(X) \times \langle z \rangle$ and, defining $H := G(X) \times \langle z^k \rangle$, we have that

$$[G(\text{cone}(X)) : H] = [G(X) : G(X)] \cdot [\langle z \rangle : \langle z^k \rangle] = 1 \cdot k = k.$$

Note that, in this case, $H \cong G(\text{cone}(X))$. On the other hand, suppose now the observation is true for $X_1, X_2 \in \mathcal{D}_f$, $X_1, X_2 \neq E_0$, and let us prove it for $X_1 \sqcup X_2$: firstly, take $H_1 \leq_k G(X_1)$, $H_2 \leq_k G(X_2)$ (which exist and are computable by assumption). Then, define H via a reduced wedge automaton $\mathfrak{J}$ as follows: $\mathfrak{J}$ has k primary vertices of degree 2, each adjacent to 2 secondary vertices (both of degree k) with respective label subgroups H_1 and H_2 (see Figure 4.3). Set also that, for $\iota = 1, 2$, the labels of the ι -edges form a set of (right) coset representatives of H_ι . That way, it is clear that $\mathfrak{J}$ is reduced (see Definition 2.2.28).

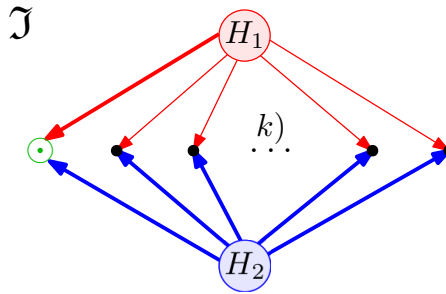


Figure 4.3: Reduced wedge automaton for H , $\mathfrak{J}$ (note that, here, the situation of the basepoint $\odot$ is irrelevant). A spanning tree of $\mathfrak{J}$ is denoted with thicker edges.

Moreover, following Proposition 4.1.1, we easily have that $H \leq_{fi} G(X_1 \sqcup X_2)$, and that

$$[G(X_1 \sqcup X_2) : H] = |V_0(\mathfrak{J})| = k.$$

In this case, the rank r of the free part of the Kurosh decomposition of H is easily seen to be $r = k - 1$, so here we conclude that $H \cong H_1 * H_2 * \mathbb{F}_{k-1}$. $\square$

Note that, combining this last result with the decidability of $SRP(G(X))$ of last section, for $X \in \mathcal{D}_f$, we easily get that, for any $X \in \mathcal{D}_f$, $X \neq E_0$, and $k \in \mathbb{Z}^+$ we can find $Y \in \mathcal{D}_f$ such that $G(Y) \hookrightarrow_k G(X)$.

4.2 Embeddability problems in $\mathcal{D}_f$

In this second section we are going to study algorithmic problems in Droms groups that depend on a single (finite) Droms graphs $X \in \mathcal{D}_f$ and have as input another (finite) Droms graph $Y \in \mathcal{D}_f$. Here, and as we can not formally say that $G(Y)$ is a subgroup of $G(X)$, we will have to work in terms of embeddings, that is, injective morphisms $G(Y) \hookrightarrow G(X)$.

Similarly as in the previous section, we are now interested in two problems involving the existence of embeddings between Droms groups and the conditions under which these embeddings have finite index. The two general problems are formally stated next:

Embeddability Problem, for $H = \langle A \mid B \rangle$, $EP(H)$: given a group $K = \langle A' \mid B' \rangle$, decide whether there exists an embedding $K \hookrightarrow H$.

Finite Index Embeddability Problem, for $H = \langle A \mid B \rangle$, $FIEP(H)$: given a group $K = \langle A' \mid B' \rangle$, decide whether there exists a finite index embedding $K \hookrightarrow_{fi} H$. In the affirmative case, compute the set $S_{H,K}$ of possible indexes for the embeddings $K \hookrightarrow_{fi} H$.

We will begin by studying the solvability of $EP(G(X))$, for $X \in \mathcal{D}_f$. Before doing so, let us prove the following easy lemma:

Lemma 4.2.1. *For any non-complete graph $X \in \mathcal{G}$ and every group H ,*

$$H \hookrightarrow G(X) \implies H \hookrightarrow_{\infty} G(X).$$

Proof. Firstly, let us define $G(X)^2 := \langle x^2 : x \in V(X) \rangle \leq G(X)$. Now, we claim that $G(X)^2 \leq_{\infty} G(X)$: indeed, as X is non-complete, there exist $y, z \in V(X)$ such that $\{y, z\} \notin E(X)$. Then, $\langle y, z \rangle \leq G(X)$ is isomorphic to $\mathbb{F}_2$, and we have that

$$\langle y^2, z^2 \rangle = G(X)^2 \cap \langle y, z \rangle \leq G(X) \cap \langle y, z \rangle = \langle y, z \rangle.$$

But, as $\langle y^2, z^2 \rangle \leq_{\infty} \langle y, z \rangle$ (construct, for example, the *Stallings automaton* of $\langle y^2, z^2 \rangle$ inside $\langle y, z \rangle \cong \mathbb{F}_2$ and see that it is not saturated), we must have $G(X)^2 \leq_{\infty} G(X)$ and the claim is complete. Finally, $G(X)^2 \cong G(X)$, so $H \hookrightarrow G(X)$ implies $H \hookrightarrow G(X)^2 \leq_{\infty} G(X)$, and we are done. $\square$

We can now prove solvability of $EP(X)$, for $X \in \mathcal{D}_f$, as next theorem shows:

Theorem 4.2.2. *For every $X \in \mathcal{D}_f$, $EP(G(X))$ is decidable.*

Proof. Clearly, $EP(\{1\})$ is solvable as $G(Y) \hookrightarrow \{1\} \iff Y \cong E_0$. Now, suppose $Y \in \mathcal{D}_f$ and assume $EP(G(Y))$ is solvable. Then,

$$G(Z) \hookrightarrow G(\text{cone}(Y)) \iff \begin{cases} G(Z) \hookrightarrow G(Y), \text{ or} \\ \exists W \in \mathcal{D}_f \text{ such that } Z \cong \text{cone}(W) \text{ and } G(W) \hookrightarrow G(Y). \end{cases}$$

The first case is solvable by assumption and, for the second one, we just have to check whether Z is a cone (which is easily doable by counting the degrees of its vertices) and use the assumption again.

On the other hand, suppose now $Y_1, Y_2 \in \mathcal{D}_f$, assume both $EP(G(Y_1))$ and $EP(G(Y_2))$ are solvable, and let us see that $EP(G(Y_1 \sqcup Y_2))$ is also solvable. Given $Z \in \mathcal{D}_f$, look at its connected components $Z_1, \dots, Z_m$, and we claim that

$$G(Z) \hookrightarrow G(Y_1 \sqcup Y_2) \iff \text{for every } i \in \{1, \dots, m\}, \begin{cases} Z_i \cong E_1, \text{ or} \\ G(Z_i) \hookrightarrow G(Y_1), \text{ or} \\ G(Z_i) \hookrightarrow G(Y_2). \end{cases} \quad (1)$$

The implication to the right is due to the form of the subgroups of a free product of groups given by Kurosh Subgroup Theorem.

The implication to the left is not that direct, and we need a more detailed argument. Leaving appart the isolated vertices, decompose Z as the disjoint union of its connected components $Z = Z_1 \sqcup \dots \sqcup Z_{m'} \sqcup E_r$. By hypothesis each Z_i embeds in either $G(Y_1)$ or $G(Y_2)$; denote by $Z_{1,1}, \dots, Z_{1,p}$ those embedable into $G(Y_1)$ and by $Z_{2,1}, \dots, Z_{2,q}$ the remaining ones (embedable then into $G(Y_2)$); we have $p, q \geq 0$, $p + q = m'$.

Denote by $L_{1,1}, \dots, L_{1,p} \leq_M G(Y_1)$ (resp., by $L_{2,1}, \dots, L_{2,q} \leq_M G(Y_2)$) some subgroups of $G(Y_1)$ (resp., $G(Y_2)$) isomorphic to the $G(Z_{1,i})$'s (resp., $G(Z_{2,j})$'s), and having big enough index, $M \gg 0$ (this condition on the index can be assumed by Lemma 4.2.1 in the case Y_1 is not complete, and by the existence of copies of $\mathbb{Z}^n \leq_M \mathbb{Z}^n$ with arbitrary index). Adding trivial groups to one of the two lists $L_{1,1}, \dots, L_{1,p}$ or $L_{2,1}, \dots, L_{2,q}$, if necessary, we can assume $q = p + 1$. Now, consider the wedge automaton $\mathfrak{J}_0$ from the following figure:

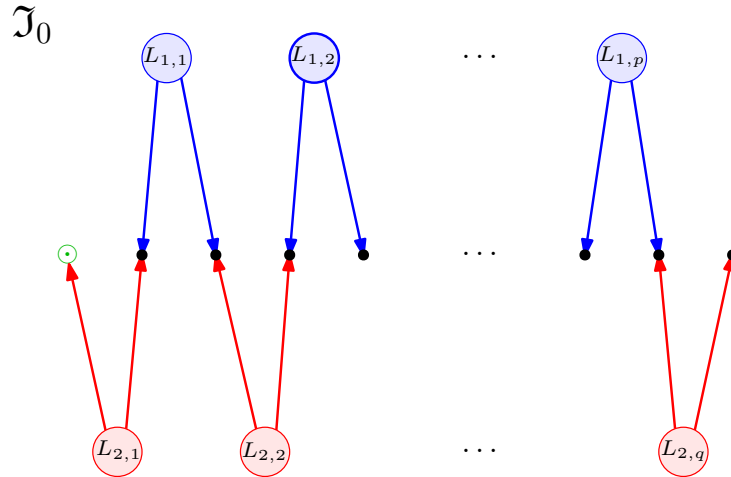


Figure 4.4: $\mathfrak{J}_0$ has $2q$ primary vertices, $p = q - 1$ 1-secondary vertices (those in blue), q 2-secondary vertices (those in red) and it is a tree (in fact, a path of length $4q - 2$).

Clearly, $\mathfrak{J}_0$ is connected; moreover, it is reduced after labeling the edges incident to each secondary vertex with elements belonging to different cosets modulo the corresponding subgroup (which is possible because $M \geq 2$). By Theorem 2.2.34 (and since the underlying graph is a tree), we have that

$$G(Y_1 \sqcup Y_2) \geq \langle \mathfrak{J}_0 \rangle \cong \left(\bigstar_{i=1}^p L_{1,i} \right) * \left(\bigstar_{j=1}^q L_{2,j} \right) \cong G(Z_1) * \dots * G(Z_{m'}).$$

Finally, to gain the extra free complement required, of rank r , add r many extra primary vertices, each one of degree 2 and adjacent to the 1-secondary labeled with $L_{1,1}$ and to the 2-secondary vertex labeled with $L_{2,1}$ (see Figure 4.5); the labels of these new edges can be defined without losing the reduction property since $L_{1,1} \leq_M G(Y_1)$ and $L_{2,1} \leq_M G(Y_2)$ with big enough M .

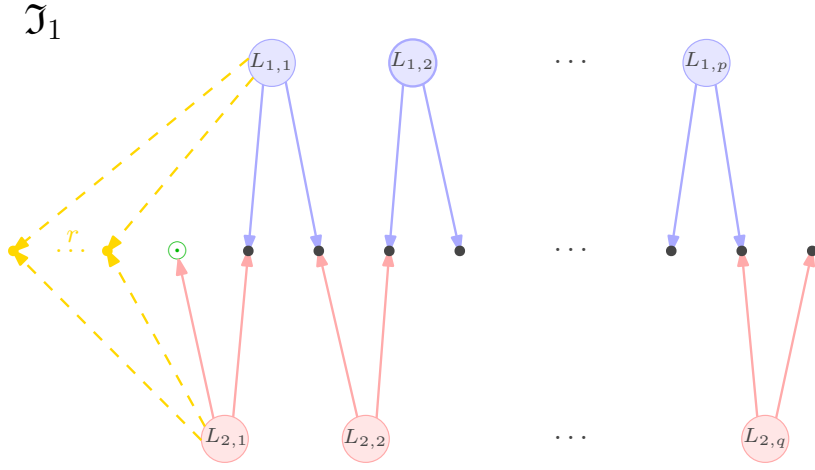


Figure 4.5: $\mathfrak{J}_1$ is constructed by adding r new primary vertices to $\mathfrak{J}_0$, and connecting each one to both $L_{1,1}$ and $L_{2,1}$.

The wedge automaton $\mathfrak{J}_1$ obtained this way is, again, reduced, and satisfies

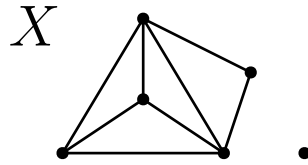
$$G(Y_1 \sqcup Y_2) \geq \langle \mathfrak{J}_1 \rangle = \left(\bigstar_{i=1}^p L_{1,i} \right) * \left(\bigstar_{j=1}^q L_{2,j} \right) * \mathbb{F}_r \cong G(Z_1) * \cdots * G(Z_{m'}) * \mathbb{F}_r \cong G(Z),$$

completing the proof of the claim.

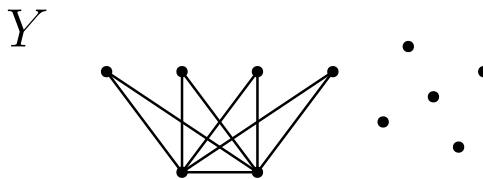
Finally, we can clearly check the conditions on the right hand side of (1): verifying whether $Z_i \cong E_1$ is trivial, and the remaining two cases are solvable by assumption, so we conclude, by Proposition 3.2.5, that $EP(G(X))$ is solvable for every $X \in \mathcal{D}_f$, which is what we wanted to prove. $\square$

As we have commented in the previous section, the algorithm for solving $EP(G(X))$, for $X \in \mathcal{D}_f$, that directly follows from last proof is recursive and can be a little confusing. The following example tries to clarify its application with an easy case of Droms graph $X \in \mathcal{D}_f$ and simple inputs $Y \in \mathcal{D}_f$.

Example 4.2.3. Consider the following graph X , and let us solve $EP(G(X))$ for different input graphs Y :



Firstly, one can easily verify that $X \in \mathcal{D}_f$. In fact, $G(X) \cong (\mathbb{Z}^2 \times (\mathbb{Z}^2 * \mathbb{Z})) * \mathbb{Z} \in G(\mathcal{D})_{fg}$. Now, consider the following input graph $Y \in \mathcal{D}_f$:

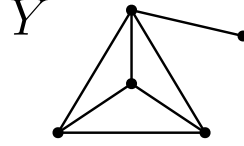


It is not difficult to see that $G(Y) \cong (\mathbb{Z}^2 \times \mathbb{F}_4) * \mathbb{F}_5$, but just from this information it is not trivial to decide if $G(Y)$ embeds into $G(X)$. Following the steps of last theorem, and

as X is disconnected, to check whether $G(Y) \hookrightarrow G(X)$ we just have to verify if the only nonempty connected component of Y embeds into $\mathbb{Z}^2 \times (\mathbb{Z}^2 * \mathbb{Z})$. Indeed, as $\mathbb{F}_4 \hookrightarrow \mathbb{Z}^2 * \mathbb{Z}$, which implies $G(E_4) \hookrightarrow G(K_2 \sqcup E_1)$, we have that

$$G(E_4) \hookrightarrow G(K_2 \sqcup E_1) \implies G(E_4 \wedge E_1) \hookrightarrow G((K_2 \sqcup E_1) \wedge E_1) \implies G(Y) \hookrightarrow G(X),$$

and so, in this case, the algorithm answers **yes**. On the other hand, and for the same $X \in \mathcal{D}_f$, consider now the input graph $Y \in \mathcal{D}_f$ defined as follows:



In this case, $G(Y) \cong \mathbb{Z} \times (\mathbb{Z}^3 * \mathbb{Z})$, and the following chain of implications now shows that $G(Y)$ does not embed into $G(X)$.

$$\left. \begin{array}{l} G(Y) \not\hookrightarrow G(K_2 \sqcup E_1) \\ G(K_3 \sqcup E_1) \not\hookrightarrow G(K_2 \sqcup E_1) \end{array} \right\} \implies G(Y) \not\hookrightarrow G((K_2 \sqcup E_1) \wedge E_1) \left. \begin{array}{l} \\ G(K_3 \sqcup E_1) \not\hookrightarrow G(K_2 \sqcup E_1) \implies G(K_3 \sqcup E_1) \not\hookrightarrow G((K_2 \sqcup E_1) \wedge E_1) \end{array} \right\} \implies G(Y) \not\hookrightarrow G(X),$$

so the algorithm answers **No** here.

Before focusing on solving *FIEP* let us point that, for embedded subgroups, *Schreier's Formula for Droms graphs* applies: that is, if $G(Y) \hookrightarrow_k G(X)$, then $(c_X - 1)k = c_Y - 1$. This necessary condition for finite index embedded subgroups of a Droms groups will be strongly used next in the study of the decidability of *FIEP*.

Theorem 4.2.4. *For every $X \in \mathcal{D}_f$, $FIEP(G(X))$ is solvable, with the set $S_{H,K}$ being just a singleton $S_{H,K} = \{k\}$ (when X is disconnected) or a set of the form $S_{H,K} = k\mathbb{Z}$ (when X is connected).*

Proof. The proof follows the same lines as that of Theorem 4.2.2, with a bit more delicate arguments to accommodate the finite index condition. Firstly, $FIEP(\{1\})$ is clearly solvable as $G(Y) \hookrightarrow_{fi} G(\{1\}) \iff Y = E_0$; in this case, the set of possible indices is just $\{1\}$.

Now, let $Y \in \mathcal{D}_f$, assume $FIEP(G(Y))$ is solvable and let us see that $FIEP(G(\text{cone}(Y)))$ is also solvable. Indeed, we have seen that

$$G(Z) \hookrightarrow_{fi} G(\text{cone}(Y)) \iff \exists W \in \mathcal{D}_f \text{ s.t. } \text{cone}(W) = Z \text{ and } G(W) \hookrightarrow_{fi} G(Y).$$

Both conditions are clearly algorithmically checkable (the second one by assumption), so we conclude that $FIEP(G(\text{cone}(Y)))$ is solvable. In the case $G(Z) \hookrightarrow_{fi} G(\text{cone}(Y))$, the set $S_{\text{cone}(Y),Z}$ of possible indexes of $G(Z)$ inside $G(\text{cone}(Y))$ is precisely the set of multiples of the elements of $S_{Y,W}$, namely a set of the form $k\mathbb{Z}$.

On the other hand, let now $Y_1, Y_2 \in \mathcal{D}_f$ be nontrivial, assume $FIEP(G(Y_1))$ and $FIEP(G(Y_2))$ are both solvable and let us prove that $FIEP(G(Y_1 \sqcup Y_2))$ is also solvable. Firstly, let us define $Y := Y_1 \sqcup Y_2$ and, for a given input graph $Z \in \mathcal{D}_f$, let $k := \frac{c_Z - 1}{c_Y - 1}$. We know that, if $k \notin \mathbb{Z}$, then clearly $G(Z) \not\hookrightarrow_{fi} G(Y)$ and the algorithm must answer **No**. If

$k \in \mathbb{Z}$, we look at the connected components of Z , say $Z_1, \dots, Z_m$, and consider the full list of decompositions of the form

$$Z = \left(\bigsqcup_{i=1}^p C_{1,i} \right) \sqcup \left(\bigsqcup_{j=1}^q C_{2,j} \right) \sqcup E_r,$$

where the $C_{1,i}$'s and $C_{2,j}$'s are disjoint unions of Z_i 's; there are certainly finitely many such decompositions and we can enumerate all of them. For each such decomposition, use the hypotheses to eliminate from the list those not satisfying that $G(C_{1,i}) \hookrightarrow_{fi} G(Y_1)$ for every $i = 1, \dots, p$ and $G(C_{2,j}) \hookrightarrow_{fi} G(Y_2)$ for every $j = 1, \dots, q$. If the remaining list is empty, the answer is **No**. Otherwise, we have the list of all possible decompositions satisfying that, for every $i = 1, \dots, p$, $G(C_{1,i}) \hookrightarrow_{fi} G(Y_1)$ with index in $\{k_{1,i}\}$ or $k_{1,i}\mathbb{Z}$; and that, for every $j = 1, \dots, q$, $G(C_{2,j}) \hookrightarrow_{fi} G(Y_2)$ with index in $\{k_{2,j}\}$ or $k_{2,j}\mathbb{Z}$. (Observe that, in this argument, not just Z_i 's but disjoint unions of connected components must be considered, because it may happen that, for example, $G(Z_i \sqcup Z_j) \hookrightarrow_{fi} G(Y_1)$ without being true that $G(Z_i) \hookrightarrow_{fi} G(Y_1)$ nor $G(Z_j) \hookrightarrow_{fi} G(Y_1)$.)

For each decomposition in the active list, check whether it is possible to obtain the integer k by adding a possible valid index for each $C_{1,i}$, for $i = 1, \dots, p$; and also obtain k by adding a possible valid index for each $C_{2,j}$, for $j = 1, \dots, q$; and delete from the list those not satisfying these two requirements (note that this can algorithmically be done since k is finite; hence, it can be expressed in finitely many ways as a finite sum of positive integers). Again, if the remaining list is empty, the answer must be **No**. Otherwise, we have the list of all possible decompositions $Z = (\bigsqcup_{i=1}^p C_{1,i}) \sqcup (\bigsqcup_{j=1}^q C_{2,j}) \sqcup E_r$, satisfying $G(C_{1,i}) \hookrightarrow_{\lambda_i k_{1,i}} G(Y_1)$, $i = 1, \dots, p$, $G(C_{2,j}) \hookrightarrow_{\mu_j k_{2,j}} G(Y_2)$, $j = 1, \dots, q$, and $\sum_{i=1}^p \lambda_i k_{1,i} = k = \sum_{j=1}^q \mu_j k_{2,j}$. Denote by $L_{1,1}, \dots, L_{1,p}$ and $L_{2,1}, \dots, L_{2,q}$ the images of the corresponding embeddings into $G(Y_1)$ and $G(Y_2)$, respectively.

It remains to check whether at least one of the possibilities on the active list allows us to construct a reduced wedge automaton representing an index k subgroup of $G(Y_1 \sqcup Y_2)$. Try to construct such a wedge automaton in the following way: consider k primary vertices, p 1-secondary vertices, $v_1, \dots, v_p$, labeled by $L_{1,1}, \dots, L_{1,p} \leq_{fi} G(Y_1)$, respectively, and q 2-secondary vertices, $w_1, \dots, w_q$, labeled by $L_{2,1}, \dots, L_{2,q} \leq_{fi} G(Y_2)$, respectively. For each $i = 1, \dots, p$, consider also $\lambda_i k_{1,i}$ edges emanating from v_i , labeled with a set of coset representatives of $G(Y_1)$ modulo $L_{1,i}$ (here is where the condition $L_{1,i} \leq_{\lambda_i k_{1,i}} G(Y_1)$ is crucial); similarly, for each $j = 1, \dots, q$, consider $\mu_j k_{2,j}$ edges emanating from w_j , labeled with a set of coset representatives of $G(Y_2)$ modulo $L_{2,j}$. It remains to connect the $\sum_{i=1}^p \lambda_i k_{1,i} = k$ many edges from the 1-secondary vertices one to one with the k primary vertices, and to connect the $\sum_{j=1}^q \mu_j k_{2,j} = k$ many edges from the 2-secondary vertices one to one with the k primary vertices, in such a way that the whole graph becomes connected and having rank r . If this is impossible for each of the decompositions in the active list, the answer is **No**. Otherwise, we have constructed a reduced wedge automaton $\mathfrak{J}$ satisfying the conditions of Proposition 4.1.1 and so, we have a subgroup of index k in $G(Y_1 \sqcup Y_2)$ such that

$$G(Y_1 \sqcup Y_2) \geq \langle \mathfrak{J} \rangle \cong \left(\bigstar_{i=1}^p L_{1,i} \right) * \left(\bigstar_{j=1}^q L_{2,j} \right) * \mathbb{F}_r \cong \left(\bigstar_{i=1}^p G(C_{1,i}) \right) * \left(\bigstar_{j=1}^q G(C_{2,j}) \right) * \mathbb{F}_r \cong G(Z).$$

In this case, the answer should be **Yes**, with set of indices $\{k\}$.

This completes the proof. $\square$

We finally note that, if we can find such a connected automaton $\mathfrak{J}$, it is not necessary to check whether it has rank r . Indeed, on the one hand we have that

$$\text{rk}(\mathfrak{J}) = 1 - |V(\mathfrak{J})| + |E(\mathfrak{J})| = 1 - p - q + k,$$

where we used that $|E(\mathfrak{J})| = 2k$ by construction. On the other hand, counting the connected components of Z using *Schreier's Formula for Droms groups* on the subgroups $L_{1,i} \leq_{fi} G(Y_1)$ and $L_{2,j} \leq_{fi} G(Y_2)$ we have that

$$\begin{aligned} c_Z &= r + \sum_{i=1}^p ((c_{Y_1} - 1)[G(Y_1) : L_{1,i}] + 1) + \sum_{j=1}^q ((c_{Y_2} - 1)[G(Y_2) : L_{2,j}] + 1) = \\ &= r + p + q + (c_Y - 2)k, \end{aligned}$$

and using the fact that $(c_Y - 1)k = c_Z - 1$ we finally get that $r = 1 - p - q + k$ and, hence, $r = \text{rk}(\mathfrak{J})$ holds.

Chapter 5

Conclusions

In this master's thesis we have focused on *Droms groups*, a subclass of the wider family formed by *Right-angled Artin groups*, groups where the only relations are commutativities between generators. Since these algebraic objects can be defined through a simple undirected graph, we first reviewed a well-known theorem by Droms in [17] that characterized finite Droms graphs as those that do not contain induced copies of C_4 or P_4 , the cycle and path graphs on 4 vertices, respectively.

This characterization allowed us to obtain an alternative definition for the elements of $\mathcal{D}_f$, this one in a recursive way: finite Droms graphs are those that can be formed (starting from an isolated vertex) through cones and disjoint unions from previously constructed elements. This last result, which naturally translate to finitely generated Droms groups, was one of the main tools used in Chapter 4 to study algorithmic problems in $G(\mathcal{D})$. The final result of the chapter established that the family of Droms groups is closed under taking subgroups, a result that is easily seen to be false if we restrict ourselves to the narrower class of finitely generated Droms groups.

In Chapter 4 we studied and solved several algorithmic problems in the family of finitely generated Droms groups. We first addressed subgroup problems on $\mathcal{D}_f$ where, using the recursive characterization of $\mathcal{D}_f$ and *Ivanov automata* theory for describing subgroups of free products of groups, we proved solvability of the *Finite Index Problem* and the *Subgroup Recognition Problem*. These were remarkable results as, for instance, *FIP* is seen to be unsolvable in the wider family of *Right-angled Artin groups*: indeed, as shown in [18], the groups $\mathbb{F}_n \times \mathbb{F}_n = G(E_n \wedge E_n) \in \mathcal{R}$ have unsolvable *FIP*. The reasoning behind this result is based on a reduction of *FIP*($\mathbb{F}_n \times \mathbb{F}_n$) to the finiteness problem in the class of all finitely presented groups, a classical *Markov property* (see [32]) and, therefore, undecidable.

Moreover, a surprising result arose when we proved a graphical generalization for Droms groups of the well-known *Schreier's Formula* for finite index subgroups of free groups.

In the final part of Chapter 4 we addressed embeddability problems. We were able to prove solvability of the *Embeddability Problem* in $\mathcal{D}_f$ and, after characterizing the existence of finite index embeddings between Droms groups using automata theory and *Schreier's Formula for Droms groups*, we were also able to prove solvability of the *Finite Index Embeddability Problem* in $\mathcal{D}_f$.

This last proof suggests a strategy to count the number of subgroups of finite index. However, the related combinatorics are not trivial and is left as possible future work. This would be a nice generalization of a series of results in this direction, detailed below.

We now propose possible future lines of research on the topic: firstly, and as one may

know, there exists a well-known recursion counting the number of subgroups of index n of the finitely generated free abelian group $\mathbb{Z}^m$, namely $T(n, m)$:

$$\begin{cases} T(n, 1) = 1; \\ T(n, m) = \sum_{d|n} d \cdot T(d, m-1), \text{ for } m > 1. \end{cases} \quad (1)$$

An analogous result exists for the free case: if $\mathbb{F}_m$ is a finitely generated free group, M. Hall proved in [19] that the number of subgroups of index n of $\mathbb{F}_m$, namely $R(n, m)$, satisfies the following recursion:

$$\begin{cases} R(1, m) = 1; \\ R(n, m) = n(n!)^{m-1} - \sum_{i=1}^{n-1} ((n-i)!)^{m-1} \cdot R(i, m), \text{ for } n > 1. \end{cases} \quad (2)$$

With the introduction of *Stallings automata* in [37] describing subgroups of free groups, this last formula can be easily recovered: finite index subgroups of free groups are those represented by a finite and saturated Stallings automaton, and their index on the ambient group is exactly the number of vertices of the automaton. Hence, the formula in (2) can also be obtained by counting the number of saturated *Stallings automata* with a fixed number of vertices.

Finally, a recent result by P. López in [27] extended the previous results to the family of free times free-abelian groups; these groups are those of the form $\mathbb{F}_n \times \mathbb{Z}^m$, with $n, m \in \mathbb{Z}^+ \cup \{0\}$, and are easily seen to be Droms groups (in fact, $\mathbb{F}_n \times \mathbb{Z}^m \cong G(E_n \wedge K_m)$). With the help of *enriched automata*, a variation of *Stallings automata* describing subgroups of free times free-abelian groups (see [11]), she was able to prove that the number of subgroups of index k in $\mathbb{F}_n \times \mathbb{Z}^m$, namely $S(k, n, m)$, satisfies the following recursion:

$$S(k, n, m) = \sum_{k_1|k} R(k_1, n) \cdot T\left(\frac{k}{k_1}, m\right) \cdot \left(\frac{k}{k_1}\right)^{k_1(n-1)+1}, \text{ for } k \geq 1. \quad (3)$$

The fact that finitely generated Droms groups can be recursively defined (see Proposition 3.2.5), along with the fact that *FIP* is solvable in $\mathcal{D}_f$ and that we have a graphical characterization for finite index subgroups of a free product of groups in terms of *Ivanov automata* (see Proposition 4.1.1) strongly suggests that a recursive formula further extending (1), (2) and (3) can also be found for Droms groups.

Another possible area of investigation could be trying to describe all possible (isomorphism classes of) subgroups of a fixed finitely generated Droms group in a non-recursive way. Even though $\mathcal{D}_f$ is a quite rigid class, the problem here lies in the description of subgroups of a free product of groups, given by Kurosh Subgroup Theorem, which may be very intricate.

A final line of research could involve studying other algorithmic problems in the family of Droms groups. Although *WP*, *CP*, *IP*, *MP*, *SIP* and *FIP* have already been solved, there are more open questions to be tackled. The *Subgroup Conjugacy Problem*, *SCP* for short, could be an interesting example to start with, even though there are other intriguing possibilities as well.

Bibliography

- [1] A. Baudisch, “Kommutationsgleichungen in semifreien gruppen,” *Acta Mathematica Hungarica*, vol. 29, no. 3-4, pp. 235–249, 1977.
- [2] A. Baudisch, “Subgroups of semifree groups,” *Acta Mathematica Hungarica*, vol. 38, no. 1-4, pp. 19–28, 1981.
- [3] M. Bestvina, “Geometric group theory and 3-manifolds hand in hand: The fulfillment of thurston’s vision,” *Bulletin of the American Mathematical Society*, vol. 51, no. 1, pp. 53–70, 2014.
- [4] R. Charney, “An introduction to right-angled artin groups,” *Geometriae Dedicata*, vol. 125, no. 1, pp. 141–158, 2007.
- [5] M. Clay and D. Margalit, *Office hours with a geometric group theorist*. Princeton University Press, 2017.
- [6] B. Coderch, “Pro-V topologies in groups: The closure problem on the free group and V_p ,” 2025.
- [7] J. Crisp, E. Godelle, and B. Wiest, “The conjugacy problem in right-angled artin groups and their subgroups,” *Journal of topology*, vol. 2, no. 3, pp. 442–460, 2009.
- [8] J. Delgado, “Extensions of free groups: Algebraic, geometric, and algorithmic aspects,” Ph.D. dissertation, Universitat Politècnica de Catalunya, 2017.
- [9] J. Delgado and E. Ventura, “Autòmats de stallings, un camí d’anada i tornada,” *Butlletí de la Societat Catalana de Matemàtiques*, vol. 37, no. 1, pp. 5–59, 2022.
- [10] J. Delgado and E. Ventura, “Stallings automata,” *Languages and Automata, Chapter 3*, edited by B. Steinberg, 2024.
- [11] J. Delgado and E. Ventura, “Stallings automata for free-times-abelian groups: Intersections and index,” *Publicacions matemàtiques*, vol. 66, no. 2, pp. 789–830, 2022.
- [12] J. Delgado, E. Ventura, and A. Zakharov, “Intersection problem for droms raags,” *International Journal of Algebra and Computation*, vol. 28, no. 07, pp. 1129–1162, 2018.
- [13] W. Dicks, “An exact sequence for rings of polynomials in partly commuting indeterminates,” *Journal of Pure and Applied Algebra*, vol. 22, no. 3, pp. 215–228, 1981.
- [14] R. Diestel, “Graph theory, volume 173 of,” *Graduate texts in mathematics*, vol. 7, 2012.
- [15] C. Droms, “Graph groups, coherence, and three-manifolds,” *Journal of Algebra*, vol. 106, no. 2, pp. 484–489, 1987.
- [16] C. Droms, “Isomorphisms of graph groups,” *Proceedings of the American Mathematical Society*, vol. 100, no. 3, pp. 407–408, 1987.

- [17] C. Droms, “Subgroups of graph groups,” *Journal of Algebra*, vol. 110, no. 2, pp. 519–522, 1987.
- [18] A. Goryushkin, “Two algorithmic problems in group theory,” *Res. Rep Math*, vol. 2, p. 4, 2018.
- [19] M. Hall Jr, “Subgroups of finite index in free groups,” *Canadian Journal of Mathematics*, vol. 1, no. 2, pp. 187–190, 1949.
- [20] S. V. Ivanov, “Intersecting free subgroups in free products of groups,” *International Journal of Algebra and Computation*, vol. 11, no. 03, pp. 281–290, 2001.
- [21] S. Ivanov, “On the intersection of finitely generated subgroups in free products of groups,” *International Journal of Algebra and Computation*, vol. 9, no. 05, pp. 521–528, 1999.
- [22] I. Kapovich, R. Weidmann, and A. Myasnikov, “Foldings, graphs of groups and the membership problem,” *International Journal of Algebra and Computation*, vol. 15, no. 01, pp. 95–128, 2005.
- [23] A. Kurosch, “Die untergruppen der freien produkte von beliebigen gruppen,” *Mathematische Annalen*, vol. 109, no. 1, pp. 647–660, 1934.
- [24] G. T. Lee, *Abstract algebra: An introductory course*. Springer, 2018.
- [25] H.-N. Liu, C. Wrathall, and K. Zeger, “Efficient solution of some problems in free partially commutative monoids,” *Information and Computation*, vol. 89, no. 2, pp. 180–198, 1990.
- [26] M. Lohrey and B. Steinberg, “The submonoid and rational subset membership problems for graph groups,” *Journal of Algebra*, vol. 320, no. 2, pp. 728–755, 2008.
- [27] P. López Larios, “Counting subgroups using Stallings automata and generalizations,” 2023.
- [28] L. Lovász, *Combinatorial problems and exercises*, 1979.
- [29] R. C. Lyndon, P. E. Schupp, R. Lyndon, and P. Schupp, *Combinatorial group theory*. Springer, 1977, vol. 89.
- [30] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations*. Dover Publications, 2004.
- [31] J. Nielsen, “Om regning med ikke-kommutative faktorer og dens anvendelse i gruppeteorien,” *Matematisk Tidsskrift. B*, pp. 77–94, 1921.
- [32] M. O. Rabin, *Recursive unsolvability of group theoretic problems*. Princeton University, 1956.
- [33] J. J. Rotman, *An introduction to the theory of groups*. Springer Science & Business Media, 2012, vol. 148.
- [34] O. Schreier, “Die untergruppen der freien gruppen,” in *Abhandlungen aus dem Mathematischen Seminar der universität Hamburg*, Springer, vol. 5, 1927, pp. 161–183.
- [35] J.-P. Serre, *Trees*. Springer Science & Business Media, 2002.
- [36] J. Stallings, “Coherence of 3-manifold fundamental groups,” in *Séminaire Bourbaki vol. 1975/76 Exposés 471–488*, Springer, 2006, pp. 167–173.
- [37] J. R. Stallings, “Topology of finite graphs,” *Inventiones mathematicae*, vol. 71, no. 3, pp. 551–565, 1983.

- [38] D. B. West et al., *Introduction to graph theory*. Prentice hall Upper Saddle River, 2001, vol. 2.