TWISTED CONJUGACY IN SOLVABLE BAUMSLAG–SOLITAR GROUPS

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Abstract. In this article, we solve the twisted conjugacy problem for solvable Baumslag–Solitar groups $BS(1,n)$, i.e., we propose an algorithm which, given two elements $u, v \in BS(1,n)$ and an automorphism $\varphi \in \mathrm{Aut}(BS(1,n))$, decides whether $v = (x\varphi)^{-1}ux$ for some $x \in BS(1,n)$. Also, we prove Orbit Decidability for the full automorphism group $\mathrm{Aut}(BS(1,n))$ — given two elements $u, v \in BS(n,1)$, decides whether u, v can be mapped to each other by some automorphism in $\mathrm{Aut}(BS(1,n))$ — as well as for cyclic subgroups of $\mathrm{Out}(BS(1,n))$. As a direct consequence, we obtain a solution to the conjugacy problem for certain semidirect products of $BS(1,n)$ by torsion-free hyperbolic groups.

1. Introduction

The study of Algorithmic Group Theory started with the famous three *Dehn Problems* (see [10]), namely the *Word Problem*, the *Conjugacy Problem* and the *Isomorphism Problem*:

Word Problem, $WP(G)$: For a finite presentation $G = \langle X \mid R\rangle$, *given a word on the generators $w \in F(X)$, decide whether w represents the trivial element in G, $w =_G 1$.*

Conjugacy Problem, $CP(G)$: For a finite presentation $G = \langle X \mid R\rangle$, *given two words on the generators $u, v \in F(X)$, decide whether u and v represent conjugate elements in G, $u \sim_G v$.*

Isomorphism Problem, IP: *Given two finite presentations, $\langle X \mid R\rangle$ and $\langle Y \mid S\rangle$, decide whether they present isomorphic groups, $\langle X \mid R\rangle \simeq \langle Y \mid S\rangle$.*

As it is well known, these three problems are unsolvable in their full generality, but there are lots of interesting results in the literature solving them on certain families of groups, or analyzing the solvability boundary on some others. In the present article we will focus on the conjugacy problem in close relation with another two algorithmic problems: the *Twisted Conjugacy Problem* and the *Orbit Decidability Problem*.

Let G be a group and $\varphi \in \mathrm{Aut}(G)$, $g \mapsto g\varphi$, be an automorphism. Two elements $u, v \in G$ are *φ-twisted conjugated*, denoted $u \sim_\varphi v$, if there exists $x \in G$ such that $v = (x\varphi)^{-1}ux$. It is easy to see that $\sim_\varphi$ is an equivalence relation on G, which coincides with the usual conjugation when $\varphi = \mathrm{Id}$. This variation of standard conjugacy was first introduced by Reidemeister with clear topological motivations related to Nielsen Fixed Point Theory; see [17]. The corresponding algorithmic problem is the following generalization of the Conjugacy Problem:

Twisted Conjugacy Problem, $TCP(G)$: For a finite presentation $G = \langle X \mid R\rangle$, *given an automorphism $\varphi\colon G \to G$ (given by the images of the generators), and two words on the generators, $u, v \in F(X)$, decide whether u and v represent φ-twisted conjugate elements in G, i.e., $u \sim_\varphi v$.*

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Of course, CP coincides with the particular case of TCP with $\varphi = \mathrm{Id}$; so, a positive solution to $TCP(G)$ immediately gives a positive solution to $CP(G)$ (which automatically gives a positive solution to $WP(G)$ as well). However, as one might expect, in an arbitrary group G, twisted conjugacy classes may be much more complicated to understand than standard conjugacy classes. For instance, in the case of free groups, $CP(F_n)$ is very easy both conceptually and computationally, while $TCP(F_n)$ is known to be solvable but much harder in both senses; see [3, Theorem 1.5]. Also, similarly to the classical result stating the existence of a finitely presented group G with solvable $WP(G)$ but unsolvable $CP(G)$ (see, for example, [15]), there is a more recent result stating the existence of a finitely presented group G with solvable $CP(G)$ but unsolvable $TCP(G)$; see [4, Corollary 4.9].

In [3], Bogopolski–Martino–Maslakova–Ventura solved the conjugacy problem for free-by-cyclic groups, $CP(F_n \rtimes_\varphi \mathbb{Z})$. The proof had two clearly separate ingredients: firstly a solution to the Twisted Conjugacy Problem for free groups, $TCP(F_n)$, and, secondly, a result from Brinkmann [5] providing an algorithm to decide, given an automorphism $\varphi \in \mathrm{Aut}(F_n)$, and two elements $u, v \in F_n$, whether v is conjugate to some iterated image of u, i.e., $v \sim u\varphi^k$, for some $k \in \mathbb{Z}$. On the other hand, more than 30 years earlier, Miller [15] already studied the much bigger family of free-by-free groups, and showed that the Conjugacy Problem is unsolvable within that framework.

Few years after the publication of [3], Bogopolski–Martino–Ventura [4] realized that, in their solution to $CP(F_n \rtimes_\varphi \mathbb{Z})$, the case where the two inputs belonged to outside the normal subgroup F_n ($u, v \notin F_n$) boils down to $TCP(F_n)$, while the opposite case ($u, v \in F_n$) corresponded directly to Brinkmann's result (note that if $u \in F_n$ and $v \notin F_n$, or viceversa, then they automatically cannot be conjugate to each other). This pattern happened to be much more general: in the free-by-free group $G = F_n \rtimes_{\varphi_1,\ldots,\varphi_m} F_m$, the decision on whether two elements $u, v \notin F_n$ are conjugated to each other boils down to $TCP(F_n)$ (so, for *every* free-by-free group, the CP *is solvable when restricted* to inputs outside F_n), while deciding whether two elements $u, v \in F_n$ are conjugated to each other *in* G, becomes a much harder problem than Brinkmann's one (and mandatorily sometimes unsolvable, according to Miller's result [15]). Much beyond this family, the main result in [4] is the following one, exactly in the same spirit, but applying to *any* (non necessarily split) short exact sequence of groups:

Theorem 1.1 (Bogopolski–Martino–Ventura, [4]). *Let* $1 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 1$ *be an algorithmic short exact sequence of groups such that*

(i) *F has solvable twisted conjugacy problem;*
(ii) *H has solvable conjugacy problem; and*
(iii) *for every $1 \neq h \in H$, the subgroup $\langle h \rangle$ has finite index on its centralizer $C_H(h)$, and there is an algorithm which computes a finite set of coset representatives, $z_{h,1}, \ldots, z_{h,t_h} \in H$, $C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}$.*

Then, the following are equivalent:

(a) *the conjugacy problem for G is solvable;*
(b) *the conjugacy problem for G restricted to F is solvable;*
(c) *the action subgroup $A_G = \{\varphi_g \mid g \in G\} \leqslant \mathrm{Aut}(F)$ is orbit decidable.*

The term *algorithmic* referred to a short exact sequence as above, just means the assumption of algorithms for computing images of elements by α and β, pre-images by β, and pre-images by

α of elements in $\ker\beta = \operatorname{Im}\alpha$; typically, this is the case when F, G and H are given by finite presentations, and α and β are given by images of the generators; see [4] for details. As for the *action subgroup* associated to a short exact sequence, A_G, this is just the set of restrictions to $F \trianglelefteq G$, namely $\varphi_g \colon F \to F$, of the inner automorphisms of G, namely $\gamma_g \colon G \to G$, $x \mapsto g^{-1}xg$ (note that φ_g is not, in general, inner as automorphism of F; we can only say that $A_G \leqslant \operatorname{Aut}(F)$). Finally, the notion of *Orbit Decidability* is crucial:

Orbit Decidability Problem, $OD(A)$: For a finite presentation $G = \langle X \mid R\rangle$ and a group of automorphisms $A \leqslant \operatorname{Aut}(G)$, *given two words on the generators $u, v \in F(X)$, decide whether the corresponding elements $u, v \in G$ can be mapped to each other by some automorphism in A, i.e., whether there exists $\alpha \in A$ such that $u\alpha = v$.*

Remark 1.2. Sometimes the notion of Orbit Decidability for $A \leqslant \operatorname{Aut}(G)$ asks to decide whether there exists $\alpha \in A$ such that $u\alpha \sim v$. In general this is not exactly the same, but the two notions obviously coincide when $\operatorname{Inn}(F) \leqslant A$, as is always the case for the action subgroups coming from short exact sequences. Therefore, Theorem 1.1 is perfectly valid with the two (coinciding) notions of orbit decidabillity.

A subgroup $A \leqslant \operatorname{Aut}(G)$ is said to be *orbit decidable* if $OD(A)$ is solvable. Particularizing to the case where both F and H are free groups (hypotheses (i), (ii) and (iii) do hold), the group G siting in the middle of the short exact sequence is free-by-free, and Theorem 1.1 states that $CP(G)$ is solvable if and only if the corresponding action subgroup is orbit decidable: sometimes this is the case (for example, when $H = \mathbb{Z}$, in which case $A_G/(A_G \cap \operatorname{Inn}(F))$ is cyclic and $OD(A_G)$ is, precisely, Brinkmann's problem), and sometimes it is not (like in the classical Miller's examples of free-by-free groups with unsolvable Conjugacy Problem); see [4] for more details.

After the publication of [4], several research papers have appeared following the same strategy but in other situations: anytime we have a family of groups (to put in place of F) for which we can solve the Twisted Conjugacy Problem, it becomes interesting to study Orbit Decidability for (all, certain) subgroups of automorphisms $A \leqslant \operatorname{Aut}(F)$, and deduce from it the solvability or unsolvability of the Conjugacy Problem for extensions of F by families of groups H satisfying (ii) and (iii) (for example, torsion-free hyperbolic groups; see [4]). This strategy was followed in [4] itself to study the Conjugacy Problem within the family of free-by-free-abelian groups, where the authors positively solved it for all groups of the form $\mathbb{Z}^2 \rtimes F_m$, and $\mathbb{Z}^n \rtimes_A F_m$ with virtually solvable action subgroup $A \leqslant \mathrm{GL}_n(\mathbb{Z})$; and provided the first known examples of groups of the form $\mathbb{Z}^4 \rtimes F_n$ with unsolvable Conjugacy Problem. Similarly, Šunić–Ventura [18] constructed the first known examples of automaton groups with unsolvable Conjugacy Problem. González-Meneses–Ventura [14] followed the same project for Braid groups, showing that all Braid-by-hyperbolic groups have solvable Conjugacy Problem. Burillo–Matucci–Ventura [6] did the same for Thompson's group F, proving in this case the existence of Thomson-by-hyperbolic groups with unsolvable Conjugacy Problem. Recently, Blufstein–Valiunas [2] and Crowe [8, 9] have followed similar projects for certain large-type Artin groups and for dihedral Artin groups, respectively.

In the present paper we follow the above project for the family of solvable Baumslag–Solitar groups $BS(1,n)$: we first solve the Twisted Conjugacy Problem, and then investigate Orbit Decidability of certain subgroups of $\operatorname{Aut}(BS(1,n))$, connecting it with the Conjugacy Problem for some extensions of $BS(1,n)$.

At the beginning of Section 2 we consider the group $BS(1,n)$ and briefly survey some elementary properties, a normal form for its elements, and its semi-direct product structure. Then we move on to the automorphism group $\mathrm{Aut}(BS(1,n))$: in Subsection 2.2, following the results from Collins–Levin [7] and O'Neil [16], we give a more direct proof of the explicit form for the automorphisms of $BS(1,n)$. In Subsection 2.3, we give an easy solution to the Conjugacy Problem for $BS(1,n)$.

Then, in Section 3, we focus on and solve the Twisted Conjugacy Problem (see Theorem 3.2); in Section 4, we prove orbit decidability for the full automorphism group $\mathrm{Aut}(BS(1,n))$ (see Theorem 4.2); and, in Section 5, we prove orbit decidability for cyclic subgroups of $\mathrm{Out}(BS(1,n))$ (see Theorem 5.2). According to the above project, each of these last two results has the corresponding corollary about solvability of the Conjugacy Problem in the corresponding extensions of $BS(1,n)$, namely Corollaries 4.3 and 5.3. We conclude with a couple of natural Open Questions in Section 6.

General notation and conventions. For a group G, $\mathrm{Aut}(G)$ denotes the group of automorphisms of G. We write them all with the argument on the left, that is, we denote by $(x)\varphi$ (or simply $x\varphi$) the image of the element x by the homomorphism φ; accordingly, we denote by $\varphi\psi$ the composition $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$. Specifically, we will reserve the letter γ for right conjugations, $\gamma_g \colon G \to G, x \mapsto g^{-1}xg$. Following the same convention, when thinking of a matrix A as a map, it will always act on the right of horizontal vectors, $\mathbf{v} \mapsto \mathbf{v}A$. We denote by $\mathrm{M}_{n\times m}(\mathbb{Z})$ the $n \times m$ (additive) group of matrices over $\mathbb{Z}$, and by $\mathrm{GL}_m(\mathbb{Z})$ the linear group over the integers. We use the function $|\cdot|_a$ to count the number of a-occurrences in a word w.

2. Solvable Baumslag–Solitar groups

The so-called Baumslag–Solitar groups $BS(m,n)$ are the class of two-generated one-related groups presented by

$$BS(m,n) = \langle a, t \mid a^m = t^{-1}a^n t \rangle ,$$

for $m, n \in \mathbb{Z}$. They all are HNN-extensions of $\mathbb{Z}$ (with associated subgroups $m\mathbb{Z}$ and $n\mathbb{Z}$) sitting in the middle of the well known short exact sequence

$$\begin{array}{ccccccccc} 1 & \to & \ll a \gg & \to & BS(m,n) & \xrightarrow{\pi} & \mathbb{Z} = \langle t \rangle & \to & 1. \\ & & & & a & \mapsto & 1 & & \\ & & & & t & \mapsto & t & & \end{array} \tag{1}$$

Note that the homomorphism $\pi \colon BS(m,n) \twoheadrightarrow \mathbb{Z}$, $w \mapsto t^{|w|_t}$, is well defined because the defining relation is t-balanced (not being the case for a, unless $m = n$). Of course, $\ker \pi = \ll a \gg = \{w(a,t) \mid |w|_t = 0\} \trianglelefteq BS(m,n)$.

In this paper we will concentrate in the solvable groups within this family, namely those with $m = 1$, $BS(1,n)$. As the first elementary (and special) examples, we have $BS(0,1) \simeq \mathbb{Z}$, $BS(1,1) \simeq \mathbb{Z}^2$, and $BS(1,-1) = \langle a, t \mid a = t^{-1}a^{-1}t \rangle$, which is the fundamental group of the Klein bottle. Note that $BS(1,n)^{\mathrm{ab}} = \mathbb{Z} \oplus \mathbb{Z}/|n-1|\mathbb{Z}$, with $\ll a \gg$ being, for $n \neq 1$, the only normal subgroup whose quotient is isomorphic to $\mathbb{Z}$; in particular, $\ll a \gg$ is invariant under any automorphism of $BS(1,n)$, a crucial property for the arguments below.

For the rest of the paper, we assume $n \neq \pm 1$.

2.1. **Normal Form.** Looking at the defining relation under the forms $a^n t = ta$ and $a^{-n}t = ta^{-1}$, one observes that, in an arbitrary word $w(a,t)$, each positive occurrence of t can always be moved to the right, at the price of multiplying by n the exponents of the jumped a's. Similarly, the defining relation written in the forms $t^{-1}a^n = at^{-1}$ and $t^{-1}a^{-n} = a^{-1}t^{-1}$ tells us that each negative occurrence of t's can always be moved to the left, at the price of multiplying by n the exponents of the jumped a's. Summarizing, and using induction on p, for every $p, k \in \mathbb{Z}$, $p \geq 0$, we have

$$t^p a^k = a^{kn^p} t^p \quad \text{and} \quad a^k t^{-p} = t^{-p} a^{kn^p}. \tag{2}$$

Repeating sufficiently many of these jumps, any element $w(a,t) \in BS(1,n)$ can be expressed in the form $g = t^{-p}a^k t^q$, where $p, k, q \in \mathbb{Z}$, $p, q \geq 0$.

To understand the structure of $BS(1,n)$, we consider the following monomorphism, which proves linearity for this family of groups.

Lemma 2.1. *The map*

$$\begin{array}{rcl} \varphi\colon BS(1,n) & \to & \mathrm{GL}_2(\mathbb{Z}[1/n]) \leqslant \mathrm{GL}_2(\mathbb{Q}) \\ a & \mapsto & A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\ t & \mapsto & T = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) \end{array} \tag{3}$$

defines a monomorphism of groups. In particular, $BS(1,n)$ is a linear group.

Proof. It is straightforward to check that the map (3) preserves the relation $a = t^{-1}a^n t$ and, hence, it is well defined:

$$T^{-1}A^nT = \begin{pmatrix} 1/n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A.$$

Take an arbitrary element $g = t^{-p}a^k t^q \in BS(1,n)$. If $(t^{-p}a^k t^q)\varphi = T^{-p}A^kT^q$ equals the identity,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n^{-p} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n^q & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n^{q-p} & k\,n^{-p} \\ 0 & 1 \end{pmatrix},$$

then $k = 0$ and $p = q$ and, therefore, $g = 1$. This shows that φ is injective. □

As mentioned above, every element of $BS(1,n)$ can be written in the form $g = t^{-p}a^k t^q$, with $p, k, q \in \mathbb{Z}$, $p, q \geq 0$. From now on, for technical reasons, we will prefer to write $g = t^{-p}a^k t^q = t^{-p}a^k t^p t^c$, where $p, k, c \in \mathbb{Z}$, $p \geq 0$ — close to being a normal form, as proven in the following Corollary.

Corollary 2.2. *Let $p_i, k_i, c_i \in \mathbb{Z}$ with $p_i \geq 0$, for $i = 1, 2$. In the group $BS(1,n)$, $t^{-p_1}a^{k_1}t^{p_1+c_1} = t^{-p_2}a^{k_2}t^{p_2+c_2}$ if and only if $c_1 = c_2 \in \mathbb{Z}$ and $k_1/n^{p_1} = k_2/n^{p_2} \in \mathbb{Z}[1/n]$.*

Proof. Applying π from (1), it is clear that $c_1 = c_2$ is a necessary condition. Then, the assumed equality can be expressed as

$$1 = t^{-p_1}a^{k_1}t^{p_1-p_2}a^{-k_2}t^{p_2} = t^{-p_1-p_2}a^{k_1n^{p_2}}a^{-k_2n^{p_1}}t^{p_1+p_2},$$

which is equivalent to $a^{k_1n^{p_2}-k_2n^{p_1}} = 1$ and so, to $k_1n^{p_2} - k_2n^{p_1} = 0 \in \mathbb{Z}$, and to $k_1/n^{p_1} = k_2/n^{p_2} \in \mathbb{Z}[1/n]$ (since, by Lemma 2.1, a has infinite order in $BS(1,n)$). □

This suggests to use the following notation: for $\alpha = k/n^p \in \mathbb{Z}[1/n]$ (here, $k, p \in \mathbb{Z}$, $p \geq 0$) write

$$a^\alpha := t^{-p}a^k t^p \in BS(1,n).$$

Note that this is coherent because the equality among rational numbers $kn/n^{p+1} = k/n^p$ corresponds to the equality $t^{-p-1}a^{kn}t^{p+1} = t^{-p}(t^{-1}a^n t)^k t^p = t^{-p}a^k t^p$ among the corresponding elements in $BS(1,n)$. This is specific for powers of a so, we only accept rational exponents for the letter a.

Moreover, with this notation, the two rules (2) stated above and saying that t jumps to the right (and t^{-1} to the left) of any integral power of a at the price of multiplying its exponent by n, can be unified and extended to the more homogeneous rule saying that t^c (for $c \in \mathbb{Z}$ with no signum distinction !), jumps to the right of rational powers of a at the price of multiplying its exponent by n^c. Furthermore, this new exponential notation is compatible with the standard rules of computation:

Lemma 2.3. *For any $\alpha = k/n^p$, $\beta = \ell/n^q \in \mathbb{Z}[1/n]$ ($k, \ell, p, q \in \mathbb{Z}$, $p, q \geq 0$) and $c, r \in \mathbb{Z}$, we have*

- (i) $t^c a^\alpha = a^{\alpha n^c} t^c$ *(equivalently, $a^\alpha t^c = t^c a^{\alpha n^{-c}}$);*
- (ii) $a^\alpha a^\beta = a^{\alpha+\beta}$*;*
- (iii) $(a^\alpha)^r = a^{r\alpha}$*;*
- (iv) $t^{-c}a^\alpha t^c = a^{\alpha/n^c}$*;*
- (v) $\big(a^\alpha t^c\big)^{-1} = a^{-\alpha n^{-c}} t^{-c}$*;*
- (vi) $\big(a^\alpha t^c\big)^r = a^{\frac{n^{rc}-1}{n^c-1}\alpha} t^{rc}$.

Proof. For $c \geq 0$ we have $t^c a^\alpha = t^c t^{-p} a^k t^p = t^{-p} a^{kn^c} t^c t^p = a^{\alpha n^c} t^c$. And, for $c \leq 0$, we have $t^c a^\alpha = t^c t^{-p} a^k t^p = t^{c-p} a^k t^{p-c} t^c = a^{k/n^{p-c}} t^c = a^{\alpha n^c} t^c$. In a similar fashion, we get the equivalent version $a^\alpha t^c = t^c a^{\alpha n^{-c}}$. This proves (i).

For (ii), $a^\alpha a^\beta = t^{-p}a^k t^p t^{-q} a^\ell t^q = t^{-p-q} a^{kn^q} a^{\ell n^p} t^{p+q} = a^{(kn^q+\ell n^p)/n^{p+q}} = a^{\alpha+\beta}$.

For (iii), $(a^\alpha)^r = \big(t^{-p}a^k t^p\big)^r = t^{-p}a^{rk}t^p = a^{rk/n^p} = a^{r\alpha}$.

For (iv), $t^{-c}a^\alpha t^c = a^{\alpha n^{-c}} t^{-c} t^c = a^{\alpha/n^c}$, using (i).

For (v), $\big(a^\alpha t^c\big)^{-1} = t^{-c}a^{-\alpha} = a^{-\alpha n^{-c}} t^{-c}$, again using (i).

Finally, (vi) is obvious for $r = 0, 1$. For $r \geq 2$ we use induction on r:

$$\big(a^\alpha t^c\big)^{r+1} = a^\alpha t^c a^{\frac{n^{rc}-1}{n^c-1}\alpha} t^{rc} = a^{\alpha + n^c \frac{n^{rc}-1}{n^c-1}\alpha} t^{(r+1)c} = a^{(1+n^c\frac{n^{rc}-1}{n^c-1})\alpha} t^{(r+1)c} =$$

$$= a^{\frac{n^c-1+n^{(r+1)c}-n^c}{n^c-1}\alpha} t^{(r+1)c} = a^{\frac{n^{(r+1)c}-1}{n^c-1}\alpha} t^{(r+1)c}.$$

And, for $r \leq 0$, we apply the just proved formula to $-r \geq 0$ and get

$$\big(a^\alpha t^c\big)^r = \Big(\big(a^\alpha t^c\big)^{-1}\Big)^{-r} = \big(a^{-\alpha n^{-c}} t^{-c}\big)^{-r} = a^{\frac{n^{rc}-1}{n^{-c}-1}(-\alpha n^{-c})} t^{rc} = a^{-\frac{n^{rc}-1}{1-n^c}\alpha} t^{rc} = a^{\frac{n^{rc}-1}{n^c-1}\alpha} t^{rc}.$$

This completes the proof. □

Another straightforward consequence is the structure of $\ker\pi = \ll a \gg$.

Corollary 2.4. *We have $\ker\pi = \ll a \gg \simeq \mathbb{Z}[1/n]$, an additive subgroup of $\mathbb{Q}$. In particular, it is abelian and not finitely generated.*

Proof. Clearly,

$$\ker \pi = \ll a \gg = \{a^{\alpha}t^{c} \in BS(1,n) \mid c=0\} = \{a^{\alpha} \mid \alpha \in \mathbb{Z}[1/n]\} \simeq \mathbb{Z}[1/n],$$

since, by Lemma 2.3(ii), $a^{\alpha}a^{\beta} = a^{\alpha+\beta}$. This is an additive subgroup of $\mathbb{Q}$, abelian and not finitely generated. (Alternatively, this can also be seen by observing that the image of $\ker \pi = \ll a \gg$ under the monomorphism φ of (3) is $\langle \left(\begin{smallmatrix} 1 & 1/n^p \\ 0 & 1 \end{smallmatrix}\right),\, p \geq 0 \rangle = \{ \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) \mid \alpha \in \mathbb{Z}[1/n]\} \leqslant \mathrm{GL}_2(\mathbb{Z}[1/n])$). □

Using this notation and the above arguments, we can conclude the following easy and practical normal form for elements in $BS(1,n)$, reflecting the semidirect product structure of $BS(1,n)$ inherited from (1):

Proposition 2.5. *Every element $g \in BS(1,n)$ can be written, in a unique way, as $g = a^{\alpha}t^{c}$, for some $\alpha \in \mathbb{Z}[1/n]$ and some $c \in \mathbb{Z}$. Moreover, $\big(a^{\alpha_1}t^{c_1}\big)\big(a^{\alpha_2}t^{c_2}\big) = a^{\alpha_1+n^{c_1}\alpha_2}t^{c_1+c_2}$.*

Proof. Moving negative powers of t to the left and positive powers to the right, such a form always exists, for any element $g \in BS(1,n)$. Unicity is given by Corollary 2.2. Finally, the product rule follows easily,

$$\big(a^{\alpha_1}t^{c_1}\big)\big(a^{\alpha_2}t^{c_2}\big) = a^{\alpha_1}a^{n^{c_1}\alpha_2}t^{c_1+c_2} = a^{\alpha_1+n^{c_1}\alpha_2}t^{c_1+c_2}, \tag{4}$$

from Lemma 2.3. □

2.2. Automorphisms of $BS(1,n)$.

Following a previous work by Collins–Levin [7], J. O'Neill [16] gave an explicit description of the automorphisms of $BS(1,n)$. For completeness, we give here a more direct and compact proof.

Proposition 2.6 (J. O'Neill, [16])**.** *For $n \neq \pm 1$, the automorphism group of $BS(1,n)$ is*

$$\mathrm{Aut}(BS(1,n)) = \{\varphi_{\alpha,\beta} \mid \alpha \in \mathbb{Z}[1/n]^{*},\, \beta \in \mathbb{Z}[1/n]\}$$

where, for $\alpha = k/n^p$ and $\beta = \ell/n^q$ with $k, \ell, p, q \in \mathbb{Z}$, $p, q \geq 0$, $\varphi_{\alpha,\beta}$ is defined as

$$\begin{array}{rcl} \varphi_{\alpha,\beta}\colon BS(1,n) & \to & BS(1,n) \\ a & \mapsto & a^{\alpha} = t^{-p}a^{k}t^{p} \\ t & \mapsto & a^{\beta}t = t^{-q}a^{\ell}t^{q+1}. \end{array} \tag{5}$$

Moreover, $\varphi_{\alpha_1,\beta_1} \circ \varphi_{\alpha_2,\beta_2} = \varphi_{\alpha_1\alpha_2,\beta_1\alpha_2+\beta_2}$, and $(\varphi_{\alpha,\beta})^{-1} = \varphi_{\alpha^{-1},-\beta\alpha^{-1}}$.

Proof. It is clear that any automorphism $\varphi \in \mathrm{Aut}(BS(1,n))$ must be of the above form: it must leave $\ll a \gg$ invariant (because it is the only normal subgroup in $BS(1,n)$ with quotient isomorphic to $\mathbb{Z}$, since $n \neq 1$) and so, it must send a to an element with $|a\varphi|_t = 0$, i.e., of the form a^{α}, for some $\alpha \in \mathbb{Z}[1/n]$; the necessity of the invertibility condition for $\alpha = k/n^p \in \mathbb{Z}[1/n]^{*}$ (just meaning that k divides some power of n) will be seen later. On the other hand, φ induces an automorphism of $\mathbb{Z}$ and so, it must send t to an element with $|t\varphi|_t = \pm 1$, i.e., $t\varphi = a^{\beta}t^{\pm 1}$; however, $t\varphi = a^{\beta}t^{-1}$ is not compatible with the defining relation from $BS(1,n)$,

$$(a^{\beta}t^{-1})^{-1}(a^{\alpha})^{n}(a^{\beta}t^{-1}) = ta^{-\beta}a^{n\alpha}a^{\beta}t^{-1} = ta^{n\alpha}t^{-1} = a^{n^2\alpha} \neq a^{\alpha},$$

since a is of infinite order, and $n \neq \pm 1$. Therefore, all automorphisms of $BS(1,n)$ have the form (5).

Let us see that every such $\varphi_{\alpha,\beta}$ is, in fact, an endomorphism. It is straightforward to check it is well defined, by proving the preservation of the defining relation:

$$\begin{aligned} a &\overset{\varphi_{\alpha,\beta}}{\mapsto} a^{\alpha}, \\ t^{-1}a^n t &\overset{\varphi_{\alpha,\beta}}{\mapsto} \big(a^{\beta}t\big)^{-1}\big(a^{\alpha}\big)^{n}\big(a^{\beta}t\big) = t^{-1}a^{-\beta}a^{n\alpha}a^{\beta}t = t^{-1}a^{n\alpha}t = a^{\alpha}. \end{aligned}$$

Now let us check that the composition $\varphi_{\alpha_1,\beta_1}\circ\varphi_{\alpha_2,\beta_2}$ equals $\varphi_{\alpha_1\alpha_2,\beta_1\alpha_2+\beta_2}$. In fact, for $\alpha_1 = k_1/n^{p_1}$, $\beta_1 = \ell_1/n^{q_1}$ and $\alpha_2 = k_2/n^{p_2}$, $\beta_2 = \ell_2/n^{q_2}$, with $k_i, \ell_i, p_i, q_i \in \mathbb{Z}$, $p_i, q_i \geq 0$, $i = 1, 2$, we have

$$\begin{aligned} a \overset{\varphi_{\alpha_1,\beta_1}}{\mapsto} t^{-p_1}a^{k_1}t^{p_1} \overset{\varphi_{\alpha_2,\beta_2}}{\mapsto} & \big(a^{\beta_2}t\big)^{-p_1}\big(a^{\alpha_2}\big)^{k_1}\big(a^{\beta_2}t\big)^{p_1} = \\ &= a^{\frac{n^{-p_1}-1}{n-1}\beta_2}t^{-p_1}a^{k_1\alpha_2}a^{\frac{n^{p_1}-1}{n-1}\beta_2}t^{p_1} = \\ &= a^{\frac{n^{-p_1}-1}{n-1}\beta_2}a^{n^{-p_1}k_1\alpha_2}a^{n^{-p_1}\frac{n^{p_1}-1}{n-1}\beta_2} = \\ &= a^{\frac{n^{-p_1}-1}{n-1}\beta_2+\alpha_1\alpha_2+\frac{1-n^{-p_1}}{n-1}\beta_2} = \\ &= a^{\alpha_1\alpha_2}, \end{aligned}$$

and

$$\begin{aligned} t \overset{\varphi_{\alpha_1,\beta_1}}{\mapsto} t^{-q_1}a^{\ell_1}t^{q_1+1} \overset{\varphi_{\alpha_2,\beta_2}}{\mapsto} & \big(a^{\beta_2}t\big)^{-q_1}\big(a^{\alpha_2}\big)^{\ell_1}\big(a^{\beta_2}t\big)^{q_1+1} = \\ &= a^{\frac{n^{-q_1}-1}{n-1}\beta_2}t^{-q_1}a^{\ell_1\alpha_2}a^{\frac{n^{q_1+1}-1}{n-1}\beta_2}t^{q_1+1} = \\ &= a^{\frac{n^{-q_1}-1}{n-1}\beta_2}a^{n^{-q_1}\ell_1\alpha_2}a^{n^{-q_1}\frac{n^{q_1+1}-1}{n-1}\beta_2}t = \\ &= a^{\frac{n^{-q_1}-1}{n-1}\beta_2+\beta_1\alpha_2+\frac{n-n^{-q_1}}{n-1}\beta_2}t = \\ &= a^{\beta_1\alpha_2+\beta_2}t. \end{aligned}$$

Therefore, $\varphi_{\alpha_1,\beta_1}\circ\varphi_{\alpha_2,\beta_2} = \varphi_{\alpha_1\alpha_2,\beta_1\alpha_2+\beta_2}$, as we wanted to see. In particular, such an endomorphism $\varphi_{\alpha,\beta}$ is an automorphism if and only if there exists $\varphi_{\alpha',\beta'}$ such that $\varphi_{\alpha,\beta}\circ\varphi_{\alpha',\beta'} = \varphi_{\alpha\alpha',\beta\alpha'+\beta'} = \mathrm{Id}$, which in turn implies that $\alpha\alpha' = 1$ in $\mathbb{Z}[1/n]$, and so α is invertible. Furthermore, whenever $\alpha \in \mathbb{Z}[1/n]^*$, we have

$$\varphi_{\alpha,\beta}\circ\varphi_{\alpha^{-1},-\beta\alpha^{-1}} = \varphi_{\alpha^{-1},-\beta\alpha^{-1}}\circ\varphi_{\alpha,\beta} = \varphi_{1,0} = \mathrm{Id}\,.$$

This completes the proof. □

Finally, note that $\varphi_{\alpha,\beta}$ maps an arbitrary element $a^{\nu}t^{c} \in BS(1,n)$ (with $\nu = m/n^r \in \mathbb{Z}[1/n]$, $m, r, c \in \mathbb{Z}$, $r \geq 0$) to

$$\begin{aligned} \big(a^{\nu}t^{c}\big)\varphi_{\alpha,\beta} = \big(t^{-r}a^{m}t^{r+c}\big)\varphi_{\alpha,\beta} &= \big(a^{\beta}t\big)^{-r}a^{m\alpha}\big(a^{\beta}t\big)^{r+c} = \\ &= a^{\frac{n^{-r}-1}{n-1}\beta}t^{-r}a^{m\alpha}a^{\frac{n^{r+c}-1}{n-1}\beta}t^{r+c} = \\ &= a^{\frac{n^{-r}-1}{n-1}\beta}a^{mn^{-r}\alpha}a^{n^{-r}\frac{n^{r+c}-1}{n-1}\beta}t^{c} = \\ &= a^{\frac{n^{-r}-1}{n-1}\beta+mn^{-r}\alpha+\frac{n^{c}-n^{-r}}{n-1}\beta}t^{c} = \\ &= a^{\nu\alpha+\frac{n^{c}-1}{n-1}\beta}t^{c}. \end{aligned} \tag{6}$$

2.3. **The Conjugacy Problem in $BS(1,n)$.** Let us observe the role of inner automorphisms. Left conjugation by an arbitrary element $a^\beta t^r \in BS(1,n)$, where $\beta \in \mathbb{Z}[1/n]$, $r \in \mathbb{Z}$, works as

$$\begin{array}{rcl} a & \mapsto & \left(a^\beta t^r\right) a \left(a^\beta t^r\right)^{-1} = a^\beta t^r a t^{-r} a^{-\beta} = a^\beta a^{n^r} a^{-\beta} = a^{n^r}, \\ t & \mapsto & \left(a^\beta t^r\right) t \left(a^\beta t^r\right)^{-1} = a^\beta t a^{-\beta} = a^\beta a^{-n\beta} t = a^{(1-n)\beta} t, \end{array}$$

That is, left conjugation by $a^\beta t^r$ equals $\varphi_{n^r,(1-n)\beta}$. In particular, left conjugation by a equals $\varphi_{1,1-n}$, and left conjugation by t equals $\varphi_{n,0}$. Therefore,

$$\operatorname{Inn}(BS(1,n)) = \left\{ \varphi_{\alpha,\beta} \mid \alpha = n^r \ (r \in \mathbb{Z}),\ \beta \in (1-n)\mathbb{Z}[\frac{1}{n}] \right\} = \langle \varphi_{1,1-n},\ \varphi_{n,0} \rangle \trianglelefteq \operatorname{Aut}(BS(1,n)).$$

As a straightforward consequence, we can deduce the following solution to the Conjugacy Problem for $BS(1,n)$, a well-know classical result (see [1]):

Proposition 2.7. *The Conjugacy Problem is solvable in $BS(1,n)$.*

Proof. Let $a^{\nu_1} t^{c_1}$, $a^{\nu_2} t^{c_2} \in BS(1,n)$ be two arbitrary elements, where $\nu_i = k_i / n^{p_i} \in \mathbb{Z}[1/n]$ with $k_i, p_i, c_i \in \mathbb{Z}$, $p_i \geq 0$, $i = 1,2$. By the previous analysis, $c_1 = c_2$ is a necessary condition for ν_1, ν_2 to be conjugated to each other; assume this and call it just c. Inverting the inputs, if necessary, we can further assume $c \geq 0$.

Since $a^{\nu_i} t^c = t^{-p_i} a^{k_i} t^{p_i + c} \sim a^{k_i} t^c$, $i = 1,2$, we have that $a^{\nu_1} t^c \sim a^{\nu_2} t^c$ if and only if $a^{k_1} t^c \sim a^{k_2} t^c$. And, by equation (6), this happens if and only if there exists $\beta \in \mathbb{Z}[1/n]$ and $r \in \mathbb{Z}$ such that

$$k_2 = k_1 n^r - (n^c - 1)\beta. \tag{7}$$

For $c = 0$ this will happen if and only if the integers k_1 and k_2 differ, precisely, in a multiplicative power of n; this is easily decidable by comparing the prime decompositions of k_1, k_2 and n.

Assume $c \geq 1$. Since $n^c - 1$ is coprime to n, equation (7) can only admit solutions with $\beta \in \mathbb{Z}$. In this case, the two inputs are conjugate to each other if and only if

$$k_2 \equiv k_1 n^r \pmod{n^c - 1}$$

for some $r \in \mathbb{Z}$. And checking this for $r = 0, 1, \ldots, c-1$ is enough; this completes the proof. □

Remark 2.8. For later use, let us be more explicit here. Given the two elements above $a^{\nu_1} t^{c_1}$, $a^{\nu_2} t^{c_2} \in BS(1,n)$, a necessary condition for them to be conjugated to each other is $c_1 = c_2$ (call it c). In this case, $a^{\nu_1} t^c \sim a^{\nu_2} t^c$ if and only if $a^{k_1} t^c \sim a^{k_2} t^c$, which happens if and only if, for some $r \in \mathbb{Z}$, we have $k_2 \equiv k_1 n^r \pmod{n^c - 1}$, *understanding equality in $\mathbb{Z}$, $k_2 = k_1 n^r$, in the case* $c = 0$. This condition is easy to check, in practice.

3. The twisted conjugacy problem in $BS(1,n)$

Our goal in this section is to make a further step solving the Twisted Conjugacy Problem for the groups $BS(1,n)$. First, we record the following general and elementary fact, which is true in an arbitrary group G. The proof is direct from the definitions.

Lemma 3.1. *Let G be a group, $u, v, w \in G$, $\varphi, \psi \in \operatorname{Aut}(G)$ and $\gamma_w \in \operatorname{Inn}(G)$. Then, the following are equivalent:*

(a) $u \sim_\varphi v$,
(b) $u\psi \sim_{\psi^{-1}\varphi\psi} v\psi$,

(c) $wu \sim_{\varphi\gamma_{w^{-1}}} wv$,
(d) $uw \sim_{\gamma_{w^{-1}}\varphi} vw$. □

Theorem 3.2. *The Twisted Conjugacy Problem is solvable in $BS(1,n)$.*

Proof. Suppose we are given a fixed automorphism $\varphi_{\alpha,\beta} \in \mathrm{Aut}(BS(1,n))$, where $\alpha = k/n^p \in \mathbb{Z}[1/n]^*$, $\beta = \ell/n^q \in \mathbb{Z}[1/n]$ with $k, \ell, p, q \in \mathbb{Z}$, $p, q \geq 0$. Suppose we are also given two elements $u = a^{\nu_1}t^{c_1}$, $v = a^{\nu_2}t^{c_2} \in BS(1,n)$, where $\nu_i = m_i/n^{r_i} \in \mathbb{Z}[1/n]$ with $m_i, r_i, c_i \in \mathbb{Z}$, $r_i \geq 0$, $i = 1, 2$. We have to decide whether $a^{\nu_1}t^{c_1}$ and $a^{\nu_2}t^{c_2}$ are $\varphi_{\alpha,\beta}$-twisted conjugated to each other, i.e., whether $u \sim_{\varphi_{\alpha,\beta}} v$. Observe that, again, $c_1 = c_2$ (call it just c) is an obvious necessary condition.

Applying Lemma 3.1 (a)⇔(d) with $w = t^{-c}$, and (a)⇔(b) with $\psi = \gamma_{t^{-r_1}}$, we have that

$$u = a^{\nu_1}t^c \sim_{\varphi_{\alpha,\beta}} a^{\nu_2}t^c = v \quad \Leftrightarrow \quad a^{\nu_1} \sim_{\gamma_{t^c}\varphi_{\alpha,\beta}} a^{\nu_2} \quad \Leftrightarrow \quad a^{m_1} \sim_{\gamma_{t^{r_1}}\gamma_{t^c}\varphi_{\alpha,\beta}\gamma_{t^{-r_1}}} t^{-(r_2-r_1)}a^{m_2}t^{r_2-r_1}.$$

Thus, adjusting the twisting automorphism appropriately, we are reduced to check twisted conjugation for inputs of the form $u = a^{m_1}$, $v = t^{-r}a^{m_2}t^r$, with $m_1, m_2, r \in \mathbb{Z}$, $r \geq 0$.

Writing the possible $\varphi_{\alpha,\beta}$-twisted conjugator as $g = a^{\nu}t^d$, $\nu \in \mathbb{Z}[1/n]$, $d \in \mathbb{Z}$, and using equation (6), we have

$$g\varphi_{\alpha,\beta} = a^{\nu\alpha + \frac{n^d-1}{n-1}\beta}t^d;$$

now, using Lemma 2.3(iv),

$$\big(g\varphi_{\alpha,\beta}\big)^{-1}ug = t^{-d}a^{-\nu\alpha - \frac{n^d-1}{n-1}\beta}a^{m_1}a^{\nu}t^d = a^{n^{-d}\left(-\nu\alpha - \frac{n^d-1}{n-1}\beta + m_1 + \nu\right)}.$$

Hence, $u = a^{m_1} \sim_{\varphi_{\alpha,\beta}} t^{-r}a^{m_2}t^r = v$ if and only if

$$\frac{m_2}{n^r}n^d = -\nu\alpha - \frac{n^d-1}{n-1}\beta + m_1 + \nu,$$

for some $\nu \in \mathbb{Z}[1/n]$ and $d \in \mathbb{Z}$; rearranging and simplifying, we get

$$(n-1)m_2n^d + (n-1)n^r\nu(\alpha-1) + (n^d-1)n^r\beta = (n-1)n^rm_1.$$

Write $\nu = y/n^x$, $x, y \in \mathbb{Z}$, $x \geq 0$, and apply the change of variable $\mathbb{Z} \ni z = d + x$; our equation becomes equivalent to the integral equation

$$(n-1)m_2n^d + (n-1)n^r\frac{y}{n^x}\frac{k-n^p}{n^p} + (n^d-1)n^r\frac{\ell}{n^q} = (n-1)n^rm_1,$$

$$(n-1)m_2n^{d+x+p+q} + (n-1)y(k-n^p)n^{r+q} + (n^d-1)\ell n^{r+p+x} = (n-1)m_1n^{r+p+q+x},$$

$$\Big(\ell n^{r+p} + (n-1)m_1n^{r+p+q}\Big)n^x + \Big((n-1)(n^p-k)n^{r+q}\Big)y = \Big((n-1)m_2n^{p+q} + \ell n^{r+p}\Big)n^z.$$

From the data (namely $k, p, \ell, q, m_1, m_2, r, n \in \mathbb{Z}$, $p, q, r \geq 0$, $n \neq \pm 1$) compute the three integers

$$A = \ell n^{r+p} + (n-1)m_1n^{r+p+q}, \qquad B = (n-1)(n^p-k)n^{r+q}, \qquad C = (n-1)m_2n^{p+q} + \ell n^{r+p}.$$

Summarizing $A, B, C \in \mathbb{Z}$, and $u = a^{m_1}$ and $v = t^{-r}a^{m_2}t^r$ are $\varphi_{\alpha,\beta}$-twisted conjugated to each other if and only if the equation

$$An^x + By = Cn^z \tag{8}$$

has an integral solution, for the unknowns $x, y, z \in \mathbb{Z}$, with $x \geq 0$. If $B = \pm 1$ this is always the case; so, assume $B \neq \pm 1$.

If $C = 0$ it is easy to check: if further $B = 0$ then equation (8) has no solution unless $A = 0$; and if $B \neq 0$ compute the finite set $\mathcal{N} = \{1, n, n^2, \ldots\} \subseteq \mathbb{Z}/B\mathbb{Z}$ (by computing the successive powers of n until obtaining the first repetition modulo B) and check whether the set $A\mathcal{N}$ contains $0 \pmod B$.

So, assume $C \neq 0$, write $C = C'n^s$ with $s \geq 0$ and $C' \neq 0$ not being multiple of n, and note that all possible solutions to equation (8) have $z \geq -s$. So, with the change of variable $\mathbb{Z} \ni z' = z + s$, (8) is equivalent to

$$An^x + By = C'n^{z'}, \tag{9}$$

for the unknowns $x, y, z' \in \mathbb{Z}$, with $x, z' \geq 0$. Let us distinguish two more cases.

Case 1: $B = 0$. Our equation becomes $An^x = C'n^{z'}$, which has the desired solution if and only if A/C' is a rational number being a (positive or negative) power of n. This can be checked by looking at the prime factorizations of A, C', and n.

Case 2: $B \neq 0$. Now, equation (9) is equivalent to

$$An^x \equiv C'n^{z'} \pmod B,$$

for the unknowns $x, z' \in \mathbb{Z}$, with $x, z' \geq 0$. Compute the finite set $\mathcal{N} = \{1, n, n^2, \ldots\} \subseteq \mathbb{Z}/B\mathbb{Z}$ (note that there are infinitely many powers to compute, but they take only finitely many values modulo B, so we only have to compute until getting the first repetition modulo B). Clearly, our equation admits a solution if and only if the subsets $A\mathcal{N}$ and $C'\mathcal{N}$ from $\mathbb{Z}/B\mathbb{Z}$ intersect non-trivially, a fact which is easily decidable. This completes the proof. □

4. Orbit Decidability for the full $\mathrm{Aut}(BS(1, n))$

In this section we are going to solve the Orbit Decidability Problem for the whole automorphism group $\mathrm{Aut}(BS(1, n))$, i.e., given two elements $u, v \in BS(1, n)$ we will decide whether there exists $\varphi_{\alpha,\beta} \in \mathrm{Aut}(BS(1, n))$ such that $u\varphi_{\alpha,\beta} = v$. Before, we need to remind a folklore algorithmic result about the ring $\big(\mathbb{Z}[1/n], +, \cdot\big)$. Note that, till now, we have considered $\mathbb{Z}[1/n]$ just as an additive group, and $\mathbb{Z}[1/n]^*$ as the (multiplicative) group of units of the ring $\big(\mathbb{Z}[1/n], +, \cdot\big)$.

It is well known that, as a ring, $\mathbb{Z}[1/n]$ is a subring of the field of rational numbers $\mathbb{Q}$, being itself an Euclidean domain; in particular: (i) we have an easy algorithm to compute greatest common divisors of elements in $\mathbb{Z}[1/n]$; and (ii) $\mathbb{Z}[1/n]$ is a principal ideal domain; see [13] for a general basic reference for these kind of commutative rings.

Proposition 4.1. *There is an algorithm which, given $\alpha, \delta \in \mathbb{Z}[1/n]$ decides whether the coset $\alpha + \delta\mathbb{Z}[1/n]$ (of the principal ideal generated by δ) contains an invertible element.*

Proof. Note that generators of (principal) ideals in $\mathbb{Z}[1/n]$ work up to products by units of the ambient ring. So, without loss of generality, we can assume $\delta \in \mathbb{Z}$. If $\delta = 0$ or $\delta \in \mathbb{Z}[1/n]^*$ the result is immediate. So, assume $\delta \neq 0$ is multiple of some prime not appearing in the prime factorization of $n = p_1^{e_1} \cdots p_m^{e_m}$. The coset of interest is the subset

$$\alpha + \delta\mathbb{Z}[1/n] = \{\alpha + \delta\beta \mid \beta \in \mathbb{Z}[1/n]\} \subseteq \mathbb{Z}[1/n].$$

So, given $\alpha = k/n^r \in \mathbb{Z}[1/n]$, $k, r \in \mathbb{Z}$, $r \geq 0$, and given $\delta \in \mathbb{Z}$ with the above conditions, we have to decide whether there exists $x, z \in \mathbb{Z}$, $z \geq 0$, such that

$$\frac{k}{n^r} + \delta\frac{x}{n^z} = \frac{kn^z + \delta x n^r}{n^{r+z}} \in \mathbb{Z}[1/n]^*$$

or, equivalently, $kn^z + \delta n^r x$ belongs to $\mathbb{Z}[1/n]^*$. In other words, we have to decide whether there exists $x, z \in \mathbb{Z}$, $z \geq 0$, such that the integer $kn^z + \delta n^r x$ factorizes involving only the primes $\{p_1, \ldots, p_m\}$. In order to decide this, compute the finite sets

$$\mathcal{N} = \{1, n, n^2, \ldots\} \subseteq \mathbb{Z}/\delta\mathbb{Z},$$
$$\mathcal{P}_i = \{1, p_i, p_i^2, \ldots\} \subseteq \mathbb{Z}/\delta\mathbb{Z},$$

for $i = 1, \ldots, m$ (note that, as in the proof of Theorem 3.2 above, both are finite and computable). Compute also the finite set

$$\mathcal{P} = \{y_1 \cdots y_m \mid y_1 \in \mathcal{P}_1, \ldots, y_m \in \mathcal{P}_m\} \subseteq \mathbb{Z}/\delta\mathbb{Z},$$

and let us distinguish two cases: $z \geq r$ and $0 \leq z \leq r$.

- There exists $x, z \in \mathbb{Z}$, $z \geq r$, such that $kn^z + \delta n^r x$ is invertible in $\mathbb{Z}[1/n]$ if and only if there exists $x, z' \in \mathbb{Z}$, $z' \geq 0$, such that $kn^{z'} + \delta x \in \mathbb{Z}[1/n]^*$, and this happens if and only if $k\mathcal{N}$ and $\mathcal{P}$ intersect non-trivially as subsets of $\mathbb{Z}/\delta\mathbb{Z}$.
- There exists $x, z \in \mathbb{Z}$, $0 \leq z \leq r$, such that $kn^z + \delta n^r x$ is invertible in $\mathbb{Z}[1/n]$ if and only if there exists $x \in \mathbb{Z}$ such that at least one of $k + \delta n^r x, \ldots, k + \delta n^0 x$ belongs to $\mathbb{Z}[1/n]^*$; which is equivalent to the existence of $x \in \mathbb{Z}$ such that $k + \delta x \in \mathbb{Z}[1/n]^*$. And this happens if and only if $k \in \mathcal{P} \pmod{\delta}$.

Clearly, these two conditions are decidable. This completes the proof. □

Theorem 4.2. *The full automorphism group $A = \mathrm{Aut}(BS(1,n))$ is orbit decidable.*

Proof. We first recall from (6) that $\mathrm{Aut}(BS(1,n)) = \{\varphi_{\alpha,\beta} \mid \alpha \in \mathbb{Z}[1/n]^*, \beta \in \mathbb{Z}[1/n]\}$ and the action on $BS(1,n)$ is described by the equation $(a^\nu t^c)\varphi_{\alpha,\beta} = a^{\nu\alpha + \frac{n^c-1}{n-1}\beta} t^c$.

Let $u = a^{\nu_1} t^{c_1}$, $v = a^{\nu_2} t^{c_2} \in BS(1,n)$ be two given elements, where $\nu_i = k_i/n^{p_i} \in \mathbb{Z}[1/n]$, $k_i, p_i, c_i \in \mathbb{Z}$, $p_i \geq 0$, $i = 1, 2$. We have to decide whether u and v belong to the same orbit, i.e., whether $u\varphi_{\alpha,\beta} = v$, for some $\alpha \in \mathbb{Z}[1/n]^*$ and some $\beta \in \mathbb{Z}[1/n]$. As observed above, $c_1 = c_2$ is a necessary condition; let us assume this and just denote it by c. Replacing u and v by their inverses, if necessary, we can assume $c \geq 0$.

Consider now the equation

$$\nu_2 = \nu_1\alpha + \mu\beta,$$

with unknowns $\alpha \in \mathbb{Z}[1/n]^*$ and $\beta \in \mathbb{Z}[1/n]$, and where $\mu = \frac{n^c-1}{n-1} \in \mathbb{Z} \leqslant \mathbb{Z}[1/n]$. Using the fact that $\mathbb{Z}[1/n]$ is an Euclidean domain, compute $\gcd(\nu_1, \mu)$; if ν_2 is not multiple of it then, clearly, the above equation has no solution and we are done. So assume it is and, simplifying $\gcd(\nu_1, \mu)$, we are reduced to consider the equation

$$\nu_2' = \nu_1'\alpha + \mu'\beta \tag{10}$$

where, now, $\gcd(\nu_1', \mu') = 1$. Using Bezout's identity, we can effectively compute a particular solution to it, say $\alpha_0, \beta_0 \in \mathbb{Z}[1/n]$ satisfying

$$\nu_2' = \nu_1'\alpha_0 + \mu'\beta_0.$$

If, by chance, α_0 were invertible, α_0, β_0 would be a valid solution to our problem and we would conclude that u and v belong to the same orbit. Otherwise, we need to run over *all* possible alternative solutions to (10) and check whether at least one of them has α invertible. Note that, for any other possible solution to (10), say $\nu_2' = \nu_1'\alpha + \mu'\beta$, we have $\nu_1'(\alpha - \alpha_0) + \mu'(\beta - \beta_0) = 0$

and so (since $\gcd(\nu_1', \mu') = 1$), $\alpha - \alpha_0 = \lambda\mu'$ and $\beta - \beta_0 = -\lambda\nu_1'$, for some $\lambda \in \mathbb{Z}[1/n]$. This means that *all* solutions to (10) are of the form

$$\left.\begin{array}{rcl} \alpha & = & \alpha_0 + \lambda\mu' \\ \beta & = & \beta_0 - \lambda\nu_1' \end{array}\right\},$$

for $\lambda \in \mathbb{Z}[1/n]$. It remains to decide whether $\alpha_0 + \lambda\mu'$ is invertible for some $\lambda \in \mathbb{Z}[1/n]$. This can be effectively done by Proposition 4.1. □

Theorem 4.2 is the analog for $BS(1,n)$ of the classical Whitehead's result for free groups. As explained in the introduction, the following is an immediate consequence via Theorem 1.1.

Corollary 4.3. *Let H be a torsion-free hyperbolic group (with m generators) and let $\varphi_1, \ldots, \varphi_m \in \mathrm{Aut}(BS(1,n))$ be automorphisms such that $\langle \varphi_1, \ldots, \varphi_m \rangle = \mathrm{Aut}(BS(1,n))$. Then, the Conjugacy Problem is solvable in any group of the form $BS(1,n) \rtimes_{\varphi_1,\ldots,\varphi_m} H$.* □

5. Orbit Decidability for cyclic subgroups

In this final section we will prove orbit decidability for subgroups $\mathrm{Inn}(BS(1,n)) \leqslant A \leqslant \mathrm{Aut}(BS(1,n))$ such that $A/\mathrm{Inn}(BS(1,n))$ is cyclic. This is the analog for $BS(1,n)$ of Brinkman's result for free groups; see [5]. We remind that

$$\mathrm{Inn}(BS(1,n)) = \left\{ \varphi_{\alpha,\beta} \mid \alpha = n^r\ (r \in \mathbb{Z}),\ \beta \in (1-n)\mathbb{Z}[\frac{1}{n}] \right\} = \langle \varphi_{1,1-n},\ \varphi_{n,0} \rangle \trianglelefteq \mathrm{Aut}(BS(1,n)).$$

So, our target subgroups are those of the form $A = \langle \varphi_{1,1-n},\ \varphi_{n,0}, \varphi_{\alpha_0,\beta_0} \rangle \leqslant \mathrm{Aut}(BS(1,n))$, for a given $\alpha_0 \in \mathbb{Z}[1/n]^*$ and $\beta_0 \in \mathbb{Z}[1/n]$.

Lemma 5.1. *For any $1 \neq \alpha \in \mathbb{Z}[1/n]^*$, $\beta \in \mathbb{Z}[1/n]$, and $r \in \mathbb{Z}$, we have*

$$(\varphi_{\alpha,\beta})^r = \varphi_{\alpha^r, \frac{\alpha^r - 1}{\alpha - 1}\beta}.$$

For the special case $\alpha = 1$, we have $(\varphi_{1,\beta})^r = \varphi_{1,r\beta}$.

Proof. The case $\alpha = 1$ works by a straightforward induction.

Suppose $\alpha \neq 1$. For $r = 0, 1$ the equality is clear. For $r \geq 2$, using induction, we have

$$(\varphi_{\alpha,\beta})^r = \varphi_{\alpha,\beta} \circ (\varphi_{\alpha,\beta})^{r-1} = \varphi_{\alpha,\beta} \circ \varphi_{\alpha^{r-1}, \frac{\alpha^{r-1}-1}{\alpha-1}\beta} = \varphi_{\alpha^r, \beta\alpha^{r-1} + \frac{\alpha^{r-1}-1}{\alpha-1}\beta} =$$

$$= \varphi_{\alpha^r, (\alpha^{r-1} + \frac{\alpha^{r-1}-1}{\alpha-1})\beta} = \varphi_{\alpha^r, \frac{\alpha^r-1}{\alpha-1}\beta}.$$

Finally, for $r \leq -1$ we apply Proposition 2.6 and the already proven formula for $-r \geq 1$, to get

$$(\varphi_{\alpha,\beta})^r = (\varphi_{\alpha,\beta}^{-1})^{-r} = (\varphi_{\alpha^{-1}, -\beta\alpha^{-1}})^{-r} = \varphi_{(\alpha^{-1})^{-r}, \frac{(\alpha^{-1})^{-r}-1}{\alpha^{-1}-1}(-\beta\alpha^{-1})} = \varphi_{\alpha^r, -\frac{\alpha^r-1}{1-\alpha}\beta} = \varphi_{\alpha^r, \frac{\alpha^r-1}{\alpha-1}\beta}.$$

This concludes the proof. □

Theorem 5.2. *For any $\alpha_0 \in \mathbb{Z}[1/n]^*$ and $\beta_0 \in \mathbb{Z}[1/n]$, the subgroup $A = \langle \varphi_{1,1-n},\ \varphi_{n,0}, \varphi_{\alpha_0,\beta_0} \rangle \leqslant \mathrm{Aut}(BS(1,n))$ is orbit decidable.*

Proof. The values of $\alpha_0 \in \mathbb{Z}[1/n]^*$ and $\beta_0 \in \mathbb{Z}[1/n]$ (i.e., the automorphism $\varphi_{\alpha_0,\beta_0} \in \mathrm{Aut}(BS(1,n))$) are fixed along all the argument. Let now $u = a^{\nu_1}t^{c_1}$, $v = a^{\nu_2}t^{c_2} \in BS(1,n)$ be two given elements, where $\nu_i = k_i/n^{p_i} \in \mathbb{Z}[1/n]$, $k_i, p_i, c_i \in \mathbb{Z}$, $p_i \geq 0$, $i = 1,2$, and we have to decide whether $u\varphi = v$ for some $\varphi \in A = \langle \varphi_{1,1-n}, \varphi_{n,0}, \varphi_{\alpha_0,\beta_0}\rangle \leqslant \mathrm{Aut}(BS(1,n))$. Equivalently, we have to decide whether $u(\varphi_{\alpha_0,\beta_0})^s \sim v$, for some $s \in \mathbb{Z}$. Since $a^{\nu_i}t^{c_i} \sim a^{k_i}t^{c_i}$, $i = 1,2$, this is equivalent to checking whether $(a^{k_1}t^{c_1})(\varphi_{\alpha_0,\beta_0})^s = (a^{k_2}t^{c_2})\varphi_{n^r,(1-n)\beta}$, for some $s \in \mathbb{Z}$ and some $\varphi_{n^r,(1-n)\beta} \in \mathrm{Inn}(BS(1,n))$. As in the previous section, $c_1 = c_2$ is a clear necessary condition for this to happen; so, let us assume this equality, and denote it just by $c \in \mathbb{Z}$; without loss of generality, we can further assume $c \geq 0$.

To analyze this fact, Lemma 5.1 gives us control on the powers $(\varphi_{\alpha_0,\beta_0})^s$ which, by equation (6), act over $a^{k_1}t^c$ as

$$(a^{k_1}t^c)(\varphi_{\alpha_0,\beta_0})^s = \begin{cases} (a^{k_1}t^c)\varphi_{1,s\beta_0} = a^{k_1+\frac{n^c-1}{n-1}s\beta_0}t^c & \text{if } \alpha_0 = 1, \\ (a^{k_1}t^c)\varphi_{\alpha_0^s,\frac{\alpha_0^s-1}{\alpha_0-1}\beta_0} = a^{k_1\alpha_0^s+\frac{n^c-1}{n-1}\frac{\alpha_0^s-1}{\alpha_0-1}\beta_0}t^c & \text{if } \alpha_0 \neq 1. \end{cases}$$

On the other hand, an arbitrary conjugation $\varphi_{n^r,(1-n)\beta}$, where $r \in \mathbb{Z}$, $\beta \in \mathbb{Z}[\frac{1}{n}]$, acts on $a^{k_2}t^c$ as

$$(a^{k_2}t^c)\varphi_{n^r,(1-n)\beta} = a^{k_2n^r+\frac{n^c-1}{n-1}(1-n)\beta}t^c = a^{k_2n^r-(n^c-1)\beta}t^c.$$

Hence (given $\pm 1 \neq n \in \mathbb{Z}$, $k_1, k_2, c \in \mathbb{Z}$, $\beta_0 \in \mathbb{Z}[1/n]$ and $\alpha_0 \in \mathbb{Z}[1/n]^*$), we have to decide whether there exist $s, r \in \mathbb{Z}$ and $\beta \in \mathbb{Z}[1/n]$ such that $(a^{k_1}t^c)(\varphi_{\alpha_0,\beta_0})^s = a^{k_2n^r-(n^c-1)\beta}t^c$. Let us distinguish two cases $\alpha_0 = 1$ and $\alpha_0 \neq 1$.

If $\alpha_0 = 1$ we have to decide whether there exist $s, r \in \mathbb{Z}$ and $\beta \in \mathbb{Z}[1/n]$ such that

$$k_1 + \frac{n^c-1}{n-1}s\beta_0 = k_2n^r - (n^c-1)\beta,$$

i.e.,

$$(n-1)k_1 + (n^c-1)s\beta_0 = (n-1)k_2n^r - (n-1)(n^c-1)\beta.$$

Rearranging, this is equivalent to

$$An^r + Bs + C = D\beta, \tag{11}$$

where $A = (1-n)k_2$, $B = (n^c-1)\beta_0$, $C = (n-1)k_1$, and $D = (1-n)(n^c-1)$ are fixed (and computable) elements from $\mathbb{Z}[1/n]$. Note that $0 \neq D \in \mathbb{Z}$ and $\gcd(D,n) = 1$. If $D = \pm 1$ the equation has obvious solutions (for $r, s \in \mathbb{Z}$ and $\beta \in \mathbb{Z}[1/n]$); so, we can assume $D \neq -1, 0, 1$. Now, construct the finite set

$$\mathcal{N} = \{1, n, n^2, \ldots\} \subseteq \mathbb{Z}[1/n]/D\mathbb{Z}[1/n] \simeq \mathbb{Z}/D\mathbb{Z},$$

and observe that, since $\gcd(n,D) = 1$, the first repetition will happen against 1; hence, $\mathcal{N} = \{n^r \pmod{D} \mid r \in \mathbb{Z}\}$. We conclude that equation (11) has a solution for $s, r \in \mathbb{Z}$ and $\beta \in \mathbb{Z}[1/n]$ if and only if 0 belongs to the set $A\mathcal{N} + B(\mathbb{Z}[1/n]/D\mathbb{Z}[1/n]) + \{C\} \subseteq \mathbb{Z}[1/n]/D\mathbb{Z}[1/n]$. This is clearly decidable.

Finally, for $\alpha_0 \neq 1$ we have to decide whether there exist $s, r \in \mathbb{Z}$ and $\beta \in \mathbb{Z}[1/n]$ such that

$$k_1\alpha_0^s + \frac{n^c-1}{n-1}\frac{\alpha_0^s-1}{\alpha_0-1}\beta_0 = k_2n^r - (n^c-1)\beta,$$

i.e.,

$$(n-1)(\alpha_0-1)k_1\alpha_0^s + (n^c-1)(\alpha_0^s-1)\beta_0 = (n-1)(\alpha_0-1)k_2n^r - (n-1)(\alpha_0-1)(n^c-1)\beta.$$

Rearranging, this is equivalent to

$$An^r + B\alpha_0^s + C = D\beta, \tag{12}$$

where, as in the previous case, $A = (1-n)(\alpha_0 - 1)k_2$, $B = (n-1)(\alpha_0 - 1)k_1 + (n^c - 1)\beta_0$, $C = -(n^c - 1)\beta_0$, and $D = (1-n)(\alpha_0 - 1)(n^c - 1) \neq 0$ are fixed (and computable) elements from $\mathbb{Z}[1/n]$. With a change of variable of the form $\beta = \gamma\beta'$ for the appropriate unit $\gamma \in \mathbb{Z}[1/n]^*$, we can assume that $D \in \mathbb{Z}$ and $\gcd(D, n) = 1$. If $D = \pm 1$ the equation has obvious solutions (for $s, r \in \mathbb{Z}$ and $\beta \in \mathbb{Z}[1/n]$); so, we can assume $D \neq -1, 0, 1$. Now, similarly to the previous case, construct the finite sets

$$\mathcal{N} = \{1, n, n^2, \ldots\} \subseteq \mathbb{Z}[1/n]/D\mathbb{Z}[1/n] \simeq \mathbb{Z}/D\mathbb{Z},$$
$$\mathcal{A} = \{1, \alpha_0, \alpha_0^2, \ldots\} \subseteq \mathbb{Z}[1/n]/D\mathbb{Z}[1/n] \simeq \mathbb{Z}/D\mathbb{Z},$$

where, again, the first repetition will happen against 1 in both cases, since $\gcd(n, D) = \gcd(\alpha_0, D) = 1$; hence, $\mathcal{N} = \{n^r \pmod{D} \mid r \in \mathbb{Z}\}$ and $\mathcal{A} = \{\alpha_0^s \pmod{D} \mid s \in \mathbb{Z}\}$. We conclude that equation (12) has a solution for $s, r \in \mathbb{Z}$ and $\beta \in \mathbb{Z}[1/n]$ if and only if 0 belongs to the set $A\mathcal{N} + B\mathcal{A} + \{C\} \subseteq \mathbb{Z}[1/n]/D\mathbb{Z}[1/n]$. This is clearly decidable, concluding the proof. □

As in the previous section, the following is an immediate consequence via Theorem 1.1.

Corollary 5.3. *Let H be a torsion-free hyperbolic group (with m generators) and let $\varphi_1, \ldots, \varphi_m \in \operatorname{Aut}(BS(1,n))$ be automorphisms such that $\langle \overline{\varphi_1}, \ldots, \overline{\varphi_m} \rangle \leqslant \operatorname{Out}(BS(1,n))$ is cyclic. Then, the Conjugacy Problem is solvable in any group of the form $BS(1,n) \rtimes_{\varphi_1,\ldots,\varphi_m} H$.* □

6. Questions

At this point, a natural pending question about the group $BS(1,n)$ is the following.

Question 6.1. *What is the behavior of the finitely generated subgroups $\operatorname{Inn}(BS(1,n)) \leqslant A \leqslant \operatorname{Aut}(BS(1,n))$ not covered in Theorems 4.2 or 5.2 ? Are all of them orbit decidable? Or are there some of them being orbit undecidable? In other words, is the Conjugacy Problem solvable for all semidirect products of the form $BS(1,n) \rtimes_{\varphi_1,\ldots,\varphi_m} H$, with H torsion-free hyperbolic and $\varphi_1, \ldots, \varphi_m \in \operatorname{Aut}(BS(1,n))$? Or are there some of them having unsolvable Conjugacy Problem?*

It is known that $\operatorname{Out}(BS(1,n))$ is virtually abelian and, in fact, metabelian as well, see [7]; more concretely, in our notation, it is not difficult to see that the subgroup

$$\langle \overline{\varphi_{p_1,0}}, \ldots, \overline{\varphi_{p_m,0}} \rangle \leqslant \operatorname{Out}(BS(1,n))$$

is abelian and has index $n-1$, where $n = p_1^{e_1} \cdots p_m^{e_m}$ is the prime factorization of n. So, the structure of subgroups of $\operatorname{Out}(BS(1,n))$ is relatively simple and this seems to point into the direction that one should be able to understand their actions on the conjugacy classes of $BS(1,n)$. From this perspective, it may seem more reasonable to think that all subgroups of $\operatorname{Aut}(BS(1,n))$ are orbit decidable. However, we have not yet been able to prove this in full generality. As a possible negative news, several recent papers like [11] and [12] point out some other algorithmic problems, apparently not far from ours, which happen to be undecidable, in general, for metabelian groups, or even for 2×2 matrices over $\mathbb{Q}$.

Question 6.2. *Is the Twisted Conjugacy Problem solvable for an arbitrary Baumslag–Solitar group $BS(m,n)$? In the affirmative case, can one follow a similar program and detect orbit decidable and/or orbit undecidable subgroups of $\operatorname{Aut}(BS(m,n))$ (together with the corresponding families of semidirect products having solvable/unsolvable Conjugacy Problem)?*

This question seems much more ambitious since, in general, $\mathrm{Aut}(BS(m,n))$ is much more complicated and less known than $\mathrm{Aut}(BS(1,n))$. For example, Collins–Levin [7] proved that, for m multiple of $n \neq 0, \pm 1$ but having the same set of prime divisors, the group $\mathrm{Aut}(BS(m,n))$ is not even finitely generated.

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