The generic Hanna Neumann Conjecture and Post Correspondence Problem

L. Ciobanu, A. Martino, E. Ventura

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Abstract

Let F be a finitely generated free group, and $K \leqslant F$ be a finitely generated, infinite index subgroup of F. We show that generically many finitely generated subgroups $H \leqslant F$ have trivial intersection with all conjugates of K, thus proving a stronger, generic form of the Hanna Neumann Conjecture. As an application, we show that the equalizer of two free group homomorphisms is generically trivial, which implies that the Post Correspondence Problem is generically solvable in free groups.

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1 Introduction

Let F be a free group, and H and K finitely generated subgroups of F. A classical result of Howson (1954) shows that the intersection of H and K is finitely generated, see [6]. In 1956, Hanna Neumann [11] formulated the following question which is still open, and known as the Hanna Neumann conjecture:

$$\tilde{r}(H \cap K) \leqslant \tilde{r}(h)\tilde{r}(K),$$

where the *reduced rank* of a free group H of rank $rk(H)$ is defined as $\tilde{r}(H) = \max\{r(H) - 1, 0\}$.

Then, in 1990, W. Neumann [12] formulated a stronger version of the question as follows. For H and K as above let $H\backslash F/K = \{HxK \mid x \in F\}$ be the set of double cosets, and for $x \in F$ let $H^x = x^{-1}Hx$. If $HxK = HyK$ it follows that $\tilde{r}(H^x \cap K) = \tilde{r}(H^y \cap K)$. Then the Strengthened Hanna Neumann Conjecture consists of the inequality

$$\sum_{HgK \in H\backslash F/K} \tilde{r}(H^g \cap K) \leqslant \tilde{r}(H)\tilde{r}(K).$$

In 1992, G. Tardos [15] proved this strong version of the conjecture when one of the involved subgroups has rank two. In 1994, W. Dicks [2] translated the problem into a graph-theoretic conjecture, and in 1996 G.Tardos [16] resolved the inequality when both subgroups have rank three. W. Dicks and E. Formanek [3] improved Tardos' result proving the inequality when one of the subgroups has rank three.

In the present paper we prove that, for every finitely generated, infinite index subgroup $K \leqslant F$, generically many subgroups $H \leqslant F$ (that is, "most" subgroups of F in a certain precise sense) satisfy the following identity:

$$\sum_{g \in F} r(H^g \cap K) = 0, \tag{1}$$

which clearly implies (a stronger form of) the Strengthened Hanna Neumann Conjecture in a generic version. The notion of genericity will be made precise in the following section.

We would like to mention the results of [10] and [8], which state that for any positively generated subgroup H the Strengthened Hanna Neumann Conjecture holds for any K. However, for a subgroup H, the property of being positively generated is not a generic property.

After the completion of this work, we have become aware of the result [1] of G. Arzhantseva, which also imply the genericity of the Strengthened Hanna Neumann Conjecture, although this implication is not stated in her paper. More specifically, if K is a finitely generated subgroup of infinite index in F, she proves that generically many tuples of elements of F, say $h_1, \ldots, h_r \in F$, satisfy that the normal closure $\ll h_1, \ldots, h_r \gg$ has trivial intersection with K. Hence, for a generic finitely generated $H \leqslant F$, one can deduce $H^g \cap K = 1$ for all $g \in F$. This implies equation (1) for generically many subgroups H and K.

In Section 4 we present an application of the main result. We show that the equalizer of two free group homomorphisms is generically trivial. Let F_1 and F_2 be two subgroups of arbitrary finite rank. The *equalizer* of two homomorphisms α and β from F_1 to F_2 is the set of elements (understood as reduced words) in F_1 for which $\alpha(x) = \beta(x)$. This is closely related to the Post Correspondence Problem in free groups.

2 Definitions and notation

Let F_k be a free group of rank $k \geqslant 2$ with generating set $A = \{a_1, \ldots, a_k\}$, viewed as the fundamental group of the wedge of k circles. This naturally leads to working with graphs; all graphs considered here are going to be oriented and finite (unless otherwise stated).

Definition. Let H be a finitely generated subgroup of rank r of the free group F_k, and let X_H be the corresponding covering space of the wedge of k circles (infinite except when H has finite index in F_k). That is, vertices of X_H are cosets, $V(X_H) = \{Hx \mid x \in F_k\}$, and edges are of the form (Hx, a) going from Hx to Hxa, for all $x \in F_k$ and $a \in A$. Note that X_H is an A-labeled oriented graph, with a distinguished basepoint $* = H1$, and with every vertex being the initial vertex (and the terminal vertex as well) of exactly k edges, labeled by the k symbols in A (see [14] for more details).

The *core* of H, denoted C_H, is the smallest subgraph of X_H containing the basepoint $*$, and having fundamental group H. Since H is finitely generated, C_H is a finite graph with all vertices of degree at least two, except possibly $*$. Like X_H, the graph C_H is an A-labeled oriented graph, with every vertex being the initial vertex (and the terminal vertex) of at most k edges, labeled by pairwise different letters in A. A vertex is *saturated* if it is the initial vertex of exactly k edges, as well as the terminal vertex of k edges too. And it happens that the subgroup H has finite index in F_k if and only if all vertices of C_H are saturated, see [14] for details.

Clearly, every path p in C_H spells a word w_p on $A^{\pm 1}$, i.e. an element $w_p \in F_k$, which is reduced if and only if p has no backtracking (here we understand that crossing backwards an edge labeled a, reads a^{-1}). It is also clear that p is closed (resp. closed at $*$) if and only if some conjugate of w_p (resp. w_p) belongs to H. Dually, we say that a word w is *readable in* C_H if there exists a path p in C_H (with whatever initial and terminal vertices) such that $w_p = w$. Finally, a *segment* of a word w is a subword, i.e. a word $w^{\#}$ such that $w = w' \cdot w^{\#} \cdot w''$ for some words w' and w'', and with no cancelation in the two products; the segment $w^{\#}$ is called an *initial segment* (resp. a *terminal segment*) when w' (resp. w'') is trivial. Analogously, we can talk about *segments* of paths.

Definition. Let C_H and C_K be the cores of two subgroups $H, K \leqslant F_k$, respectively. Then we define the *pullback* of H and K, denoted $C_{H,K}$, in the following way: the set of vertices is the cartesian product $V(C_H) \times V(C_K)$ and, for every $a \in A$, the set of oriented a-labeled edges is the cartesian product of the sets of a-labeled edges in C_H and C_K, where the edge $((Hx, a), (Ky, a))$ (simply denoted $((Hx, Ky), a)$) starts at vertex (Hx, Ky) and ends at vertex (Hxa, Kya). It is easy to see that the connected component of $C_{H,K}$ containing the basepoint $(*, *) = (H, K)$, after iteratively removing several possible vertices of degree one, becomes isomorphic to the core of $H \cap K$; in a similar way, the other possible components of $C_{H,K}$ correspond to intersections of the form $H \cap K^x$, see [14] for details.

We say that H and K have *trivial* pullback when $C_{H,K}$ is a forest (i.e. all its components are trees). Algebraically, this corresponds to saying that $H \cap K^x = 1$ for every $x \in F_k$.

Definition. The random selection of a finitely generated subgroup $H \leqslant F_k$ will consist of choosing a random tuple of words $\{h_1, \ldots, h_r\}$ in F_k of length bounded by n, consider the subgroup $H = \langle h_1, \ldots, h_r \rangle$, and then let n tend to infinity. Meanwhile, r is a fixed parameter.

If $\mathcal{P}$ is a property, we say that *generically many finitely generated subgroups of F_k satisfy $\mathcal{P}$* if, for every $r \geqslant 1$, the proportion of r-tuples of words of length less than or equal n in F_k which generate a subgroup satisfying $\mathcal{P}$ (among all possible r-tuples) tends to 1 when n tends to infinity. Furthermore, we say that this genericity is *exponential* when the mentioned limit tends to 1 exponentially fast, for every r.

We shall prove a generic version of (in fact, a stronger form of) the Strengthened Hanna Neumann Conjecture by showing that, for every given finitely generated, infinite index subgroup $K \leqslant F_k$, and every given $r \geqslant 1$, generically many finitely generated subgroups $H \leqslant F_k$ have trivial pullback with K (i.e. $C_{H,K}$ is a forest). Additionally, this genericity will be proven to be exponential. To this goal, we shall argue by induction on r.

Let us first set some useful notation. Fix an ambient free group F_k with $k \geqslant 2$, and a finitely generated subgroup $K \leqslant F_k$. For any given positive integers n, r, we define the following sets of tuples of words in F_k (in the definitions, H will denote the subgroup $\langle h_1, \ldots, h_r \rangle$):

- $B(n) = B_1(n) = \{h \in F_k \mid |h| \leqslant n\}$,
- $B_r(n) = \{(h_1, \ldots, h_r) \mid h_i \in F_k, |h_i| \leqslant n\} = B(n) \times \overset{r)}{\cdots} \times B(n)$,
- $TP_r(K) = \{(h_1, \ldots, h_r) \mid h_i \in F_k,\ C_{H,K} \text{ is trivial }\}$,
- $NTP_r(K) = \{(h_1, \ldots, h_r) \mid h_i \in F_k,\ C_{H,K} \text{ is non-trivial }\}$,
- $FG_r = \{(h_1, \ldots, h_r) \mid h_i \in F_k,\ r(H) = r\}$,
- $NFG_r = \{(h_1, \ldots, h_r) \mid h_i \in F_k,\ r(H) < r\}$,
- $II_r = \{(h_1, \ldots, h_r) \mid h_i \in F_k,\ [F_k : H] = \infty\}$.
- $NII_r = \{(h_1, \ldots, h_r) \mid h_i \in F_k,\ [F_k : H] < \infty\}$,

We remark that the initials used to denote each of these sets mean "ball", "trivial pullback", "non-trivial pullback", "free generating", "non-free generating", "infinite index", and "finite index", respectively.

Note that the first two sets are finite. An easy computation shows that $|B(n)| = \frac{2k(2k-1)^n - 2}{2k-2}$ and so, $(2k-1)^n \leqslant |B(n)| \leqslant 2(2k-1)^n$. Hence, $|B_r(n)| = \left(\frac{2k(2k-1)^n - 2}{2k-2}\right)^r$. The rest of sets are infinite (except the last one) and we shall be interested in estimating the cardinal of their intersection with $B_r(n)$ for any given n.

We shall need several results from [9]. In that preprint (see Theorem 2 there) it is shown that, generically, any tuple of bounded length words is a basis of the subgroup it generates (this result is also proven in [7], in a different way). In our terminology,

Proposition 1 ([9]). *For every positive integer r, the following limit exists and equals 0,*

$$\lim_{n\to\infty} \frac{|NFG_r \cap B_r(n)|}{|B_r(n)|} = 0.$$

Furthermore, the convergence is exponentially fast.

Among tuples in FG_r (they are called *viable* in [9]), Claim 4 of [9] proves two assertions that, in our terminology, can be stated in the following way.

Proposition 2 ([9]). *For every positive integer r, the following limit exists and equals 0,*

$$\lim_{n\to\infty} \frac{|FG_r \cap NII_r \cap B_r(n)|}{|B_r(n)|} = 0.$$

Furthermore, the convergence is exponentially fast.

Proposition 3 ([9]). *For every positive integer r, there exist constants M_r and $1 < \gamma_r < 2k-1$ depending only on r (and the ambient rank k) such that, for every infinite index subgroup $H \leqslant F_k$ of rank r, the total number of reduced paths in C_H and with length n, is at most $M_r\gamma_r^n$.*

3 Main Result

The main result of this note is the following.

Theorem 1. *Let F_k be the free group of rank k, and let $K \leqslant F_k$ be an infinite index subgroup of rank s. Generically many subgroups of F_k have trivial pullback with K i.e., for every positive integer r, the following limit exists and equals 1,*

$$\lim_{n\to\infty} \frac{|TP_r(K) \cap B_r(n)|}{|B_r(n)|} = 1.$$

Furthermore, the convergence is exponentially fast.

Proof. The proof goes by induction on r (including the exponential behavior).

For the case $r = 1$, let $h \in NTP_1(K) \cap B_1(n)$. Write $h = h_1 h_2 h_1^{-1}$ with h_2 cyclically reduced, and denote the lengths by $n_1 = |h_1|$ and $n_2 = |h_2|$. Note that $2n_1 + n_2 \leqslant n$, and that $C_{\langle h\rangle}$ is a circle labeled h_2 with a (possibly empty) tail labeled h_1, from the basepoint $*$ to a vertex in the circle. Consider now one of the shortest non-trivial, reduced, and closed paths p in $C_{\langle h\rangle,K}$, which exist by the hypothesis that this pullback is nontrivial. The path p projects to a nontrivial, reduced, and closed path both in $C_{\langle h\rangle}$ and C_K so, the first projection must cross the circle labeled h_2. Thus, a subpath of p, and so a subpath of its projection to C_K, reads h_2. This means that h_2 is readable in C_K and hence, by Proposition 3, it has at most $M_s\gamma_s^{n_2}$ possibilities, for some constants M_s and $\gamma_s < 2k-1$. Thus, separating first the case where $2n_1+n_2 \leqslant n/2$, we have

$$\frac{|NTP_1(K)\cap B_1(n)|}{|B_1(n)|} \leqslant \frac{|B(\lfloor n/2\rfloor)| + \sum\limits_{\substack{n_1,\, n_2 \geqslant 0 \\ n/2 < 2n_1+n_2 \leqslant n}} (2k-1)^{n_1}\cdot M_s\gamma_s^{n_2}}{(2k-1)^n} \leqslant$$

$$\frac{2(2k-1)^{n/2}}{(2k-1)^n} + \sum_{\substack{n_1,\, n_2 \geqslant 0 \\ n/2 < 2n_1+n_2 \leqslant n}} \frac{M_s\gamma_s^{n_2}}{(2k-1)^{n_1}(2k-1)^{n_2}} \leqslant$$

$$\sum_{\substack{n_1 > n/8,\ n_2 \geqslant 0 \\ n/2 < 2n_1+n_2 \leqslant n}} \frac{M_s}{(2k-1)^{n/8}} + \sum_{\substack{n_1 \geqslant 0,\ n_2 > n/4 \\ n/2 < 2n_1+n_2 \leqslant n}} M_s\left(\frac{\gamma_s}{2k-1}\right)^{n/4} \leqslant$$

$$n^2 M_s\left((\frac{1}{2k-1})^{n/8} + (\frac{\gamma_s}{2k-1})^{n/4}\right),$$

which tends to zero exponentially fast, when $n\to\infty$.

Now, for given $r\geqslant 1$, assume that the theorem holds for r and let us prove it for $r+1$. This will require us to find an estimate for $|TP_{r+1}(K)\cap B_{r+1}(n)|$ when n is big enough, using the fact that $|TP_r(K)\cap B_r(n)|/|B_r(n)|$ is as close to 1 as we wish. Or better, passing to the complements, we shall see that $|NTP_{r+1}(K)\cap B_{r+1}(n)|/|B_{r+1}(n)|$ tends to zero when n tends to infinity, using the same fact for $|NTP_r(K)\cap B_r(n)|/|B_r(n)|$.

Let $(h_1,\ldots,h_r,h_{r+1})\in NTP_{r+1}(K)\cap B_{r+1}(n)$ and write $H=\langle h_1,\ldots,h_r\rangle$ and $H'=\langle h_1,\ldots,h_r,h_{r+1}\rangle$. Then, one of the following four situations must hold:

(i) $(h_1,\ldots,h_r)\in NTP_r(K)\cap B_r(n)$ (and no conditions on h_{r+1}), or

(ii) $(h_1,\ldots,h_r)\in TP_r(K)\cap NFG_r\cap B_r(n)$ (and no conditions on h_{r+1}), or

(iii) $(h_1,\ldots,h_r)\in TP_r(K)\cap FG_r\cap NII_r\cap B_r(n)$ (and no conditions on h_{r+1}), or

(iv) $(h_1,\ldots,h_r)\in TP_r(K)\cap FG_r\cap II_r\cap B_r(n)$, and either an initial segment and a terminal segment of h_{r+1} whose lengths add up at least $\frac{1}{2}|h_{r+1}|$ are both readable in C_H, or a segment of h_{r+1} of length at least $\frac{1}{2}|h_{r+1}|$ is readable in C_K (we shall refer to this condition by saying that h_{r+1} is *half readable (h.r.) in* (C_H,C_K)).

In fact, this is obvious except the very last condition on h_{r+1}. Assume that $(h_1,\ldots,h_r)\in TP_r(K)\cap FG_r\cap II_r\cap B_r(n)$; in other words, assume that the core graphs C_H and C_K have trivial pullback $C_{H,K}$, have ranks r and s respectively, and have at least one vertex each, which is not saturated.

In this situation, let h'_{r+1} be the longest initial segment of h_{r+1} which is readable in C_H starting at $*$, and let h''_{r+1} be the longest terminal segment of h_{r+1} which is readable in C_H ending at $*$ (in principle, these segments can be empty, or can even overlap each other). If $|h'_{r+1}| + |h''_{r+1}| \geqslant \frac{1}{2}|h_{r+1}|$ then we are done. Otherwise, $h_{r+1} = h'_{r+1} \cdot h^{\#}_{r+1} \cdot h''_{r+1}$, where there is no cancelation in either product, and $|h^{\#}_{r+1}| \geqslant \frac{1}{2}|h_{r+1}|$. In this case, $C_{H'}$ looks exactly like C_H with an attached handle labeled $h^{\#}_{r+1}$. Consider now one of the shortest non-trivial, reduced, and closed paths p in $C_{H',K}$ (which exist by the hypothesis that this pullback is nontrivial). The path p projects to a nontrivial, reduced, and closed path both in $C_{H'}$ and C_K so, the first projection must cross the handle labeled $h^{\#}_{r+1}$ (because we are also assuming that $C_{H,K}$ is a forest). Thus, a subpath of p, and so a subpath of its projection to C_K, reads $h^{\#}_{r+1}$. This means that $h^{\#}_{r+1}$ is readable in C_K.

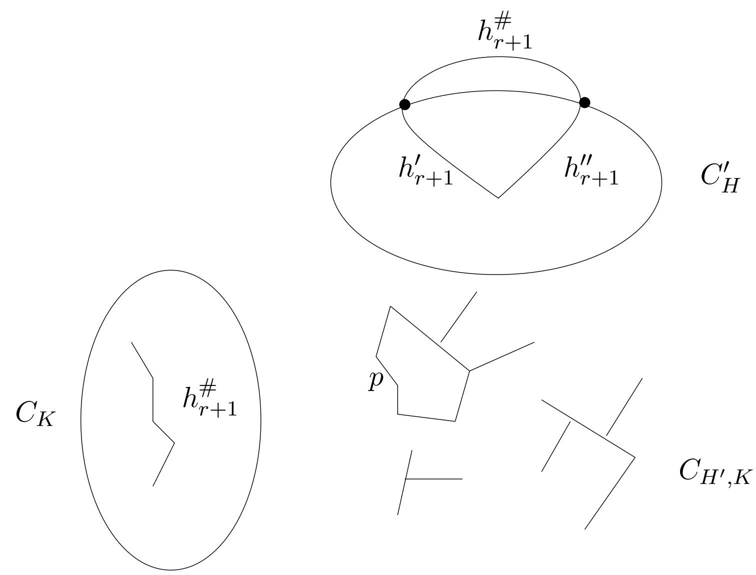

Figure 1: Nontrivial pullback, case (iv)

Once we have this tetrachotomy, the corresponding estimate follows easily:

$$\frac{|NTP_{r+1}(K) \cap B_{r+1}(n)|}{|B_{r+1}(n)|} \leqslant \frac{|NTP_r(K) \cap B_r(n)|}{|B_r(n)|} \cdot \frac{2(2k-1)^n}{|B(n)|} +$$

$$+ \frac{|TP_r(K) \cap NFG_r \cap B_r(n)|}{|B_r(n)|} \cdot \frac{2(2k-1)^n}{|B(n)|} +$$

$$+ \frac{|TP_r(K) \cap FG_r \cap NII_r \cap B_r(n)|}{|B_r(n)|} \cdot \frac{2(2k-1)^n}{|B(n)|} +$$

$$+\frac{|TP_r(K)\cap FG_r\cap II_r\cap B_r(n)|}{|B_r(n)|}\cdot\frac{|\{w\in B(n)\mid w \text{ is h.r. in } (C_H,C_K)\}|}{|B(n)|}.$$

The first summand is bounded above by $2|NTP_r(K)\cap B_r(n)|/|B_r(n)|$, which tends to zero exponentially fast when $n\to\infty$, by the inductive hypothesis. The second summand is bounded above by $2|NFG_r\cap B_r(n)|/|B_r(n)|$, which tends to zero exponentially fast by Proposition 1. The third summand is bounded above by $2|FG_r\cap NII_r\cap B_r(n)|/|B_r(n)|$, which tends to zero exponentially fast by Proposition 2. Finally, for the last summand let us ignore the first factor (it is less than one), let us separate those w's with $|w|\leqslant n/2$, and let us count how many half readable words w do exist with $|w|\geqslant n/2:$ either their initial and terminal segments w' and w'', have length adding up $n/4$ or more; or their middle segment $w^{\#}$ has length $n/4$ or more. By Proposition 3 (and using here that both H and K are infinite index subgroups of F_k) the number of w's that fall into the first and second cases are, at most,

$$\sum_{\substack{i,j\geqslant 0\\ n/4\leqslant i+j\leqslant n}} M_r\gamma_r^i\cdot|B(|w|-i-j)|\cdot M_r\gamma_r^j\ \leqslant \sum_{\substack{i,j\geqslant 0\\ n/4\leqslant i+j\leqslant n}} M_r^2\gamma_r^{i+j}\cdot 2(2k-1)^{n-i-j}$$

and

$$\sum_{n/4\leqslant i\leqslant n} M_s\gamma_s^i\cdot|B(|w|-i)|\ \leqslant \sum_{n/4\leqslant i\leqslant n} M_s\gamma_s^i\cdot 2(2k-1)^{n-i},$$

respectively. Hence, the last summand in the equation above can be bounded by

$$\frac{1}{(2k-1)^n}\left(2(2k-1)^{n/2}+\sum_{\substack{i,j\geqslant 0\\ n/4\leqslant i+j\leqslant n}} 2M_r^2\gamma_r^{i+j}(2k-1)^{n-i-j}+\right.$$

$$\left.+\sum_{n/4\leqslant i\leqslant n} 2M_s\gamma_s^i(2k-1)^{n-i}\right)\leqslant$$

$$\frac{2}{(2k-1)^{n/2}}+\sum_{\substack{i,j\geqslant 0\\ n/4\leqslant i+j\leqslant n}} 2M_r^2\Big(\frac{\gamma_r}{2k-1}\Big)^{i+j}+\sum_{n/4\leqslant i\leqslant n} 2M_s\Big(\frac{\gamma_s}{2k-1}\Big)^{i}\leqslant$$

$$\frac{2}{(2k-1)^{n/2}}+2M_r^2n^2\Big(\frac{\gamma_r}{2k-1}\Big)^{n/4}+2M_sn\Big(\frac{\gamma_s}{2k-1}\Big)^{n/4},$$

which again tends to zero, exponentially fast when $n\to\infty$, because both γ_r and γ_s are strictly less than $2k-1$. This completes the proof.

□

4 The Post Correspondence Problem

The main result of this paper has applications with respect to the Post Correspondence Problem in free groups. The Post Correspondence Problem is one of the most famous undecidable problems in theoretical computer science, and in a more algebraic language it can be stated as follows: given two morphisms α and β of a free semigroup, decide whether there are any elements x in the semigroup such that $\alpha(x) = \beta(x)$. This problem is unsolvable for a semigroup with at least 7 generators, solvable for a semigroup with 2 generators, and it is not known whether it is solvable or not for a semigroup with $3, 4, 5$ or 6 generators [13].

Very little is known about the Post Correspondence Problem in free groups. The only result in this direction is the following, due to Goldstein and Turner [**?**]. Let F_1 and F_2 be two free groups of finite ranks n and m, with $m, n \geq 2$. The *equalizer* of two homomorphisms α and β from F_1 to F_2 is the set of elements (understood as reduced words) in F_1 for which $\alpha(x) = \beta(x)$. Goldstein and Turner have proved that the equalizer of two homomorphisms is a finitely generated subgroup in case one of the two maps is injective. Is it not known whether deciding if the equalizer is trivial or not is a solvable problem.

However, we have the following corollary to Theorem 1. Genericity in the result below follows the approach taken in the previous sections of this paper and refers to choosing tuples of elements of bounded length which represent the images of the homomorphisms in question. One then lets the length of the words go to infinity.

Corollary 1. *Let F_1 and F_2 be two free groups of ranks n and m, $m, n \geq 2$. Let α and β be two homomorphisms from F_1 to F_2. Then the equalizer of α and β is generically trivial, that is, equal to the identity element of F_1.*

Proof. Let us suppose that F_1 has generators $x_1, \ldots, x_n$. Let $\alpha(x_i) = a_i$ and $\beta(x_i) = b_i$, where $a_i, b_i \in F_2$, for all $i \in \{1, \ldots, n\}$. Suppose that there exists a reduced word $w \in F_1$ such that $\alpha(w) = \beta(w) = v$. Since $\alpha(w) \in \langle a_1, \ldots, a_n \rangle$, and $\beta(w) \in \langle b_1, \ldots, b_n \rangle$, we get that v is in the intersection of the subgroups $H = \langle a_1, \ldots, a_n \rangle$ and $K = \langle b_1, \ldots, b_n \rangle$.

By the main result of this paper, the intersection of the subgroups H and K is generically trivial (since subgroups of finite index form a negligible set, the index does not play a role), thus the word v is with probability 1 going to be the identity element in F_2. In order to prove that the equalizer is indeed trivial we need to prove that the kernels of the two homomorphisms do not have a significant intersection. However, by Theorem 1 of [9], α and β are generically injective. This implies that generically the equalizer of the two homomorphisms is equal to the identity element of F_1. $\square$

This result implies that the Post Correspondence Problem is generically solvable in free groups, since for two randomly chosen homomorphisms their equalizer is trivial.

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Laura Ciobanu, University of Fribourg, Dept. of Mathematics, 1700 Fribourg, Switzerland. *E-mail*: laura.ciobanu@unifr.ch

Armando Martino, Universitat Politècnica de Catalunya, Dept. of Mathematics, 08860 Castelldefels, Spain. *E-mail*: Armando.Martino@upc.edu

Enric Ventura, Universitat Politècnica de Catalunya, Dept. of Applied Mathematics III, 08242 Barcelona, Catalonia. *E-mail*: enric.ventura@upc.edu