

THE TWISTED CONJUGACY PROBLEM FOR ENDOMORPHISMS OF METABELIAN GROUPS

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Let M be a finitely generated metabelian group explicitly presented in a variety \mathcal{A}^2 of all metabelian groups. An algorithm is constructed which, for every endomorphism $\varphi \in \text{End}(M)$ identical modulo an Abelian normal subgroup N containing the derived subgroup M' and for any pair of elements $u, v \in M$, decides if an equation of the form $(x\varphi)u = vx$ has a solution in M . Thus, it is shown that the title problem under the assumptions made is algorithmically decidable. Moreover, the twisted conjugacy problem in any polycyclic metabelian group M is decidable for an arbitrary endomorphism $\varphi \in \text{End}(M)$.

INTRODUCTION

Let G be a group and $\varphi \in \text{End}(G)$ an endomorphism. We say that elements $u, v \in G$ are φ -twisted conjugate, or, merely, are φ -conjugate, if and only if there exists an element $x \in G$ such that

$$(x\varphi)u = vx.$$

It is easy to see that the property of being φ -conjugate is an equivalence relation $u \sim_{\varphi} v$. For $\varphi = \text{id}$, the relation turns into the usual conjugacy $u \sim v$. Defining a left action of G on its basic set by

$$x \cdot u = (x\varphi)ux^{-1},$$

we can conclude that the orbits coincide with the equivalence classes defined immediately above. The number $R(\varphi)$ of φ -conjugacy classes is called the *Reidemeister number*.

Research on the φ -conjugacy relation is motivated by the topological theory of fixed points of maps, also known as Nielsen theory. For a given self-map

$$f : X \rightarrow X$$

of a compact connected manifold X , a homotopy invariant $N(f)$ (the *Nielsen number*) is defined—this is a lower bound on the number of fixed points of f . On the classical theorem of Wecken, for $\dim X \geq 3$, the

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bound $N(f)$ obtains on a set of maps homotopically equivalent to f . Thus, computing $N(f)$ is a central issue in the topological fixed point theory.

In the paper we study algorithmic aspects of the twisted conjugacy problem for any finitely generated metabelian group M . Namely, we ask the following: Does there exist an algorithm which, for any given endomorphism $\varphi \in \text{End}(M)$ of an effectively presented metabelian group M and for each pair of elements $u, v \in M$, decides whether or not the elements are φ -conjugate in M ? In other words, is there an algorithm determining solvability in M of any equation of the form

$$(x\varphi)u = vx, \tag{1}$$

where $\varphi \in \text{End}(M)$ and $u, v \in M$?

The φ -conjugacy problem for $u, v \in M$ (for any group, not only for a finitely generated metabelian one) can be reduced to the case where one of these elements is trivial. To do this, we replace φ by $\psi = \varphi \circ \sigma_h$, where $\sigma_h : u \mapsto h^{-1}uh$, $u \in M$, is an inner automorphism. Hence,

$$\begin{aligned} (x\varphi)u &= vx \\ \Leftrightarrow x\psi &= wx, \end{aligned} \tag{2}$$

where $w = u^{-1}v$.

Finitely generated metabelian groups have been studied intensively since the publication of P. Hall's fundamental papers in the 1950s. These papers were first to draw attention to the importance of commutative algebra in the theory of finitely generated soluble groups and contained a number of basic results laying groundwork for this theory. It follows from [1] that a finitely generated metabelian group satisfies $\text{max-}n$, the maximal condition on normal subgroups, and is therefore finitely presented in the variety \mathcal{A}^2 of all metabelian groups. In [2], it was stated that any finitely generated metabelian group M is residually finite. This, together with the fact that M is finitely presented in \mathcal{A}^2 , implies that the word problem in M is decidable.

In general, the class of finitely generated metabelian groups has a satisfactory algorithmic theory. A number of results of algorithmic nature were obtained in [3] (see also survey in [4]). We underline the significance of a remarkable result by Noskov [5] who proved that any finitely generated metabelian group has a decidable conjugacy problem. Basically, Noskov's proof and the algorithm he constructed are based on extremely technically sophisticated methods of commutative algebra. In [3], Noskov's algorithm was somewhat simplified. There, use is made of just the following assertion.

NOSKOV'S LEMMA. There is an algorithm which, for any finitely generated (effectively presented) commutative ring R and for a finite subset X of a group $U(R)$ of invertible elements, finds a finite presentation of a subgroup $\text{gp}(X)$.

It is worth observing that the proof of this assertion is also quite complicated.

The main result of this paper is the following:

THEOREM 1. Let M be any finitely generated metabelian group. Then there is an algorithm which decides the twisted conjugacy problem in M for every endomorphism $\psi \in \text{End}(M)$ identical modulo an Abelian normal subgroup N containing the derived subgroup M' .

Also we prove

THEOREM 2. Let M be any polycyclic metabelian group. Then there is an algorithm which decides the twisted conjugacy problem in M for every endomorphism $\varphi \in \text{End}(M)$.

The proof of Theorem 1 relies on some algorithmic results in [3, 5]. In the algorithms used, it suffices to handle a particular case, that of applying Noskov's algorithm, where conjugate elements are contained in an Abelian normal subgroup of M . Theoretically, as noted, this would not lead to an essential simplification of the algorithm: even under such restrictions, the proof and the algorithm remain complicated. At the same time, simplification can well be accomplished for a rather wide class of metabelian groups. This remark is important for a number of applications of the twisted conjugacy problem which call for practical realization of the algorithm as a program (see, e.g., [6]).

The study of the twisted conjugacy problem for a class of solvable groups was initiated in [7]. There, we dealt with a more general problem, that of solvability of any equation of the form $(x\varphi)u = v(x\psi)$, where $\varphi, \psi \in \text{End}(N)$ are arbitrary endomorphisms, in any finitely generated nilpotent group N . In [8], we will work to settle the twisted conjugacy problem for an arbitrary endomorphism of any polycyclic group P . In the polycyclic case, note, the decidability of the problem for any automorphism $\varphi \in \text{Aut}(P)$ follows readily (see [9]) from the fact that the conjugacy problem is decidable in any polycyclic group, stated in [10, 11]. Since there is a finitely presented solvable group of derived length 3 having an undecidable word (and, hence, conjugacy [12]) problem, these results cannot generally be extended to the class of finitely generated solvable groups of derived length at least 3.

In Sec. 1, we give preliminary results. Sec. 2 contains proofs for the theorems.

1. PRELIMINARY RESULTS

We say that a group M in a variety \mathcal{A}^2 of metabelian groups is generated by elements f_1, \dots, f_n and is defined by relations $r_1 = r_1(f) = 1, \dots, r_m = r_m(f) = 1$ if M is the quotient of a free metabelian group M_n with basis f_1, \dots, f_n w.r.t. a normal subgroup generated by r_1, \dots, r_m .

The derived subgroup M' of M can be treated as a finitely generated module over a finitely generated commutative ring $\mathbf{Z}A$, where $A = M/M'$ is the Abelianization of M . The ring $\mathbf{Z}A$ is Noetherian, and M' is finitely presented as a $\mathbf{Z}A$ -module. Therefore, there exists a finite description of M' , even though M' is not finitely generated as a group. It is important that there exists an effective procedure which, given a finitely generated metabelian group M , finds a finite presentation of the $\mathbf{Z}A$ -module M' (see [3]). Moreover, every Abelian normal subgroup $N \geq M'$ can also be treated as a module over a finitely generated commutative ring $\mathbf{Z}B$, where $B = M/N$, and its finite presentation, too, can be found effectively (see [3]).

In the paper we will use some other results from [3] relating to algorithms in finitely generated metabelian groups. We will keep in mind that the word problem is decidable in any finitely generated metabelian group M , as follows by P. Hall's well-known theorem on residual finiteness of M [2]. If necessary, we can effectively determine whether or not elements are equal one to another, and answer some related questions. In addition, we will apply some standard algorithms to finitely generated Abelian groups and group rings over these. Such algorithms are referred to as standard procedures (see [13 or 14]).

Let $M_n = F_n/F_n''$ be a free metabelian group of rank n represented as a quotient of a free group F_n of rank n with basis f_1, \dots, f_n . We will keep the notation for the images of elements $f_i, i = 1, \dots, n$, in M_n . Thus, M_n has basis f_1, \dots, f_n . Denote by $\partial/\partial f_i : F_n \rightarrow \mathbf{Z}F_n, i = 1, \dots, n$, free partial Fox derivatives (see [15-19] for definitions). Then induced free derivatives $\partial/\partial f_i : M_n \rightarrow \mathbf{Z}A_n, i = 1, \dots, n$, are well defined. Here $A_n = M_n/M_n'$ is a free Abelian group of rank n with basis a_1, \dots, a_n corresponding to f_1, \dots, f_n . Denote by \bar{g} the image of an element $g \in M_n$ in A_n under the standard epimorphism $M_n \rightarrow A_n$. More generally, $\bar{\alpha}$ stands for the image of $\alpha \in \mathbf{Z}M_n$ in $\mathbf{Z}A_n$ under the linear extension of the epimorphism

mentioned. For every element $g \in M_n$, the following property holds:

$$\sum_{i=1}^n \partial g / \partial f_i (a_i - 1) = \bar{g} - 1. \quad (3)$$

We consider M'_n as a module over the ring $\mathbf{Z}M_n$ the action on which is defined by conjugations $u \mapsto u^g = gug^{-1}$ by elements of M_n extended by linearity to $\mathbf{Z}M_n$. We may also treat M'_n as a module over the ring $\mathbf{Z}A_n$ because the derived subgroup M'_n itself acts trivially in this instance. For any $u \in M'_n$ and any $\alpha \in \mathbf{Z}M_n$, we then have

$$\partial u^\alpha / \partial f_i = \bar{\alpha} \partial u / \partial f_i, \quad i = 1, \dots, n. \quad (4)$$

Let $\psi \in \text{End}(M_n)$ be an endomorphism identical mod(M'_n). Then ψ is uniquely defined as follows:

$$f_i \psi = u_i f_i, \quad (5)$$

where $u_i \in M'_n$, $i = 1, \dots, n$.

For every element $w \in M_n$, its image under ψ can be written in the form

$$w\psi = u_1^{\partial w / \partial f_1} \dots u_n^{\partial w / \partial f_n} w. \quad (6)$$

Remark 1. Let $M = \text{gp}(f_1, \dots, f_n)$ be an n -generated metabelian group treated as a natural image of M_n . Suppose also that $N \geq M'$ is its Abelian normal subgroup. Then $B = M/N$ is a finitely generated Abelian group with a generating set of elements a_1, \dots, a_n corresponding to the set f_1, \dots, f_n of generators for M . For every word $w = w(f_1, \dots, f_n)$, we define formal derivatives $dw/df_i \in \mathbf{Z}B$, $i = 1, \dots, n$, computed via $\partial w / \partial f_i$ for $w \in M_n$ with values in $\mathbf{Z}B$ (for details, see below).

Clearly, formulas (3), (4), and (6) have the following analogs.

Let $M = \text{gp}(f_1, \dots, f_n) = M_n/R$ be a finitely generated metabelian group represented as a quotient of M_n . Let $N \geq M'$ be an Abelian normal subgroup of M containing the derived subgroup M' . Then there exists a standard epimorphism $\alpha : A_n = M_n/M'_n \rightarrow B = M/N$. Denote by $d/df_i = (\partial/\partial f_i) \circ \alpha$, $i = 1, \dots, n$, induced partial derivatives with values in $\mathbf{Z}B$. These derivatives are applicable to every word $w = w(f)$ treated as an element of M_n . Thus, such are viewed as formal derivatives. In any case the following analog of (3) is valid:

$$\sum_{i=1}^n dg/df_i (a_i - 1) = \tilde{g} - 1 \quad (7)$$

for every word $g = g(f)$, where $\tilde{g} = (\bar{g})\alpha$.

We consider the subgroup N as a module over $\mathbf{Z}M$, or over $\mathbf{Z}B$. For every $u \in N$ and every $\beta \in \mathbf{Z}M$, we then have

$$du^\beta / df_i = \tilde{\beta} du / df_i, \quad i = 1, \dots, n, \quad (8)$$

where $\tilde{\beta}$ is defined naturally.

Let $\psi \in \text{End}(M)$ be an endomorphism identical mod(N). Then ψ is uniquely defined as follows:

$$f_i \psi = u_i f_i, \quad (9)$$

where $u_i \in N$, $i = 1, \dots, n$.

For any word $w = w(f)$ representing an element of M , its image under ψ can be written in the form

$$w\psi = u_1^{dw/df_1} \dots u_n^{dw/df_n} w. \quad (10)$$

LEMMA 2. Let M be a finitely generated metabelian group and $\psi \in \text{End}(M)$ an endomorphism identical mod(N), where $N \geq M'$ is an Abelian normal subgroup of M . Let $M' \leq \text{Fix}_\psi(M)$, where $\text{Fix}_\psi(M) = \{g \in M \mid g\psi = g\}$ is a subgroup consisting of all ψ -invariant elements of M . Also suppose that the center $C = C(M)$ is trivial. Then there is an algorithm constructing a finite set g_1, \dots, g_k of elements which together with M' generate the subgroup $\text{Fix}_\psi(M)$, i.e., the following equality holds:

$$\text{Fix}_\psi(M) = \text{gp}(g_1, \dots, g_k, M'). \quad (11)$$

Proof. Let $M = \text{gp}(f_1, \dots, f_n)$. Since ψ acts identically on M' , the image of every commutator $[f_i, f_j]$, $i, j = 1, \dots, n$, coincides with the commutator itself. Assume that ψ is defined by (9), with $u_1, \dots, u_n \in N$. Using (10), we obtain

$$[f_i, f_j]\psi = [f_i, f_j] \Leftrightarrow u_i^{f_j-1} = u_j^{f_i-1}, \quad i, j = 1, \dots, n. \quad (12)$$

Applying (10) again, we conclude that

$$\begin{aligned} w\psi &= w \\ \Leftrightarrow u_1^{dw/df_1} \dots u_n^{dw/df_n} &= 1. \end{aligned} \quad (13)$$

Both sides of (13) are viewed as elements of the module N . Multiply these by $f_i - 1$, $i = 1, \dots, n$. From (12) and (7), it follows that (13) is equivalent to

$$u_i^{\sum_{i=1}^n dw/df_i(f_i-1)} = u_i^{w-1} = 1 \Leftrightarrow u_i^w = u_i, \quad i = 1, \dots, n. \quad (14)$$

Hence, w belongs to $C_{u_1, \dots, u_n}(M)$, the centralizer of elements u_1, \dots, u_n in M .

Conversely, assume that $w \in C_{u_1, \dots, u_n}(M)$. Let

$$u_1^{dw/df_1} \dots u_n^{dw/df_n} = v.$$

Using the trick described above, we see that

$$v^{f_i-1} = u_i^{w-1} = 1, \quad i = 1, \dots, n.$$

Hence $v \in C$, and $v = 1$ by assumption.

By the above, $\text{Fix}_\psi(M)$ will coincide with the centralizer $C_{u_1, \dots, u_n}(M)$. This centralizer contains the entire derived subgroup M' ; hence it is automatically a nearly normal subgroup of M . Recall that a subgroup H of M is *nearly normal* in the sense of definitions in [3] if $H \cap M'$ is a normal subgroup of M . By [3, Thm. 6.1], there is an algorithm which, given a finitely generated metabelian group M and its finite subset u_1, \dots, u_n , finds the centralizer $C_{u_1, \dots, u_n}(M)$ as a nearly normal subgroup of M . In our case it finds a finite set of elements g_1, \dots, g_k of this centralizer, which together with M' generate the centralizer as a whole. Hence, $\text{Fix}_\psi(M) = \text{gp}(g_1, \dots, g_k, M')$. \square

LEMMA 3. Let M be a finitely generated metabelian group and $\psi \in \text{End}(M)$ its endomorphism identical mod(N), where N is an Abelian normal subgroup of M . Then every term $C_i = C_i(M)$, $i \in \mathbf{N}$, of the upper central series of M is ψ -admissible, and so ψ can be induced to every quotient M/C_i , $i \in \mathbf{N}$.

Proof. It suffices to prove the statement for the center $C = C_1$. Since ψ is identical mod(N) and N is Abelian, the image $c\psi$ of any element $c \in C$ acts trivially on any element $u \in N$.

For every element $g \in M$, we have $g\psi = u(g)g$, $u(g) \in N$. It follows that

$$[g, c] = 1 \Rightarrow [g\psi, c\psi] = [u(g)g, c\psi] = [g, c\psi] = 1.$$

Hence $c\psi \in C$. \square

LEMMA 4. Let M be a finitely generated metabelian group and $\psi \in \text{End}(M)$ an endomorphism identical mod(N), where $N \geq M'$ is an Abelian normal subgroup of M . Also suppose that $M' \leq \text{Fix}_\psi(M)$. Assume that there is an algorithm which finds a finite set of elements generating mod(M') a subgroup $\text{Fix}_{C,\psi}(M)$, the complete preimage of $\text{Fix}_\psi(M/C)$, where C is the center of M (ψ can be induced by Lemma 3). For simplicity, we keep the notation ψ for the induced endomorphism of M/C . Then there is an algorithm which finds a finite set of elements generating mod(M') the subgroup $\text{Fix}_\psi(M)$.

Proof. Let $\text{Fix}_{C,\psi}(M) = \text{gp}(g_1, \dots, g_k, M')$. Note that $\text{Fix}_\psi(M) \leq \text{Fix}_{C,\psi}(M)$. By assumption, we can find elements g_1, \dots, g_k effectively. For every element $g \in \text{Fix}_{C,\psi}(M)$, we have

$$g\psi = c(g)g, \quad c(g) \in C. \quad (15)$$

Define a homomorphism such as

$$\mu : \text{Fix}_{C,\psi}(M) \rightarrow C, \quad \mu : g \mapsto c(g). \quad (16)$$

Note that $\mu|_{M'} = \text{id}$ and $(\text{Fix}_{C,\psi}(M))\mu = (\text{gp}(g_1, \dots, g_k))\mu = \text{gp}(c_i \mid c_i = c(g_i), i = 1, \dots, k)$. Hence, $\text{Fix}_\psi(M) = \ker(\mu)$, and there exists a standard procedure which finds a finite set of elements generating this subgroup mod(M'). \square

COROLLARY 5. Let M be a finitely generated metabelian group. Then there is an algorithm which, for any endomorphism $\psi \in \text{End}(M)$ identical mod(N) with $M' \leq \text{Fix}_\psi(M)$, where $N \geq M'$ is an Abelian normal subgroup of M , finds a finite set of elements generating mod(M') a subgroup $\text{Fix}_\psi(M) \leq M$.

Proof. Thus, we have a finitely generated metabelian group M and an endomorphism $\psi \in \text{End}(M)$ identical mod(N), where $N \geq M'$ is an Abelian normal subgroup of M , which acts trivially on any element $u \in M'$. The upper central series

$$1 \leq C_1 \leq C_2 \leq \dots \leq C_t = C_{t+1}$$

stabilizes in view of the property max- n , valid on M . Therefore, $C(M/C_t)$ is trivial. We know from [3] that there is an algorithm which finds the center and all other terms of the upper series for any finitely generated metabelian group, and also finite presentations of these subgroups. By Lemma 2, there is an algorithm that finds a finite set of elements generating mod($(M/C_t)'$) a subgroup $\text{Fix}_\psi(M/C_t)$. Again we use Lemma 3 for inducing ψ . Appealing to Lemma 4, we find a similar set for the subgroup $\text{Fix}_\psi(M/C_{t-1})$. Continuing this process, we obtain a finite set of elements generating mod(M') the subgroup $\text{Fix}_\psi(M) \leq M$. \square

LEMMA 6. Let M be a finitely generated metabelian group and $\psi \in \text{End}(M)$ an endomorphism identical mod(N), where $N \geq M'$ is an Abelian normal subgroup of M . Also suppose that $M' \leq \text{Fix}_\psi(M)$. Assume $C = C(M) = 1$. Then there is an algorithm deciding if an equation of the form

$$x\psi = ax, \quad a \in N, \quad (17)$$

has a solution in M .

Proof. Let ψ be defined by a map such as in (9). Suppose that $x = x(f)$ is a word presenting a solution for (17). By (10), we then have

$$u_1^{dx/df_1} \dots u_n^{dx/df_n} = a, \quad (18)$$

and vice versa.

Multiplying both sides of (18) by $f_i - 1$, $i = 1, \dots, n$, and using (12) and (7), we derive that

$$u_i^{x-1} = a^{f_i-1} \Leftrightarrow u_i^x = a^{f_i-1} u_i, \quad i = 1, \dots, n. \quad (19)$$

In [3], it was shown that for any finitely generated metabelian group, there exists an algorithm which decides for any two tuples of elements whether or not they are conjugate. This algorithm is a simple generalization of the algorithm in [15] dealing with pairs of elements and deciding their conjugacy. This allows us to verify whether an equation of the form (19) has a solution. If the answer is ‘no,’ then (17) has no solution either. Suppose that the answer is ‘yes’ and x is an appropriate solution. Compute the value of $x\psi$ in this instance. Using (10), we obtain

$$u_1^{dx/df_1} \dots u_n^{dx/df_n} = r. \quad (20)$$

Similarly to the above, we derive

$$u_i^{x-1} = r^{f_i-1}, \quad i = 1, \dots, n.$$

Hence,

$$a^{f_i-1} = r^{f_i-1}, \quad i = 1, \dots, n.$$

By assumption, this entails $a = r$. Therefore, x is a solution for (17). \square

LEMMA 7. Let M be a finitely generated metabelian group and $\psi \in \text{End}(M)$ an endomorphism identical mod(N), where $N \geq M'$ is an Abelian normal subgroup of M . Also suppose that $M' \leq \text{Fix}_\psi(M)$. Let $C = C(M)$ denote the center of M . If there is an algorithm deciding if an equation of the form

$$x\psi = ax \text{ mod}(C), \quad a \in N, \quad (21)$$

has a solution, then there is an algorithm determining whether or not (17) is solvable in M .

Proof. Since C is ψ -admissible by Lemma 3, it follows that if (21) has no solution then (17) has no solution either. Note that Lemma 3 is applied to induce ψ to M/C ; so (21) is induced by (17).

Suppose $x_1 \in M$ is a solution for (21), i.e.,

$$x_1\psi = cax_1, \quad c \in C. \quad (22)$$

If $x \in M$ is a solution for (17) then $z = x^{-1}x_1 \in \text{Fix}_{C,\psi}(M)$, and

$$z\psi = cz. \quad (23)$$

By Corollary 5, the following presentation can be constructed effectively:

$$\text{Fix}_{C,\psi}(M) = \text{gp}(g_1, \dots, g_k, M'). \quad (24)$$

For every $g \in \text{Fix}_{C,\psi}(M)$, $g\psi = c(g)g$, where $c(g) \in C$. Define a homomorphism such as

$$\mu : \text{Fix}_{C,\psi}(M) \rightarrow C, \quad \mu : g \mapsto c(g). \quad (25)$$

Clearly, (23), as well as (17), will have a solution iff $c \in (\text{Fix}_{C,\psi}(M))\mu = \text{gp}(c(g_i) \mid i = 1, \dots, k)$. This can be checked via a standard procedure. \square

COROLLARY 8. Let M be a finitely generated metabelian group and $\psi \in \text{End}(M)$ an endomorphism identical mod(N), where $N \geq M'$ is an Abelian normal subgroup of M . Also suppose that $M' \leq \text{Fix}_\psi(M)$. Then there is an algorithm deciding if an equation of the form (17) is solvable in M .

Proof. The upper central series

$$1 \leq C_1 \leq C_2 \leq \dots \leq C_t = C_{t+1}$$

stabilizes in view of the property max- n on M . Therefore, $C(M/C_t)$ is trivial. By [3], there is an algorithm that finds the center and all other terms of the upper central series in any finitely generated metabelian group M . Again, it supplies us with their finite presentations. By Lemma 6, there is an algorithm that decides if an analog of (17) has a solution in M/C_t . If such a solution exists (otherwise (17) has no solution), then we can reduce (by Lemma 4) the question to the group M/C_{t-1} , and so on. The last step will give the answer for the group M . \square

2. PROOFS OF THE THEOREMS

Proof of Theorem 1. Thus, M is a finitely generated metabelian group and $\psi \in \text{End}(M)$ is an endomorphism identical modulo an Abelian normal subgroup $N \geq M'$. We have to determine whether or not an equation of the form (2) has a solution. We may assume that the condition $w \in N$, which is necessary for solvability, is met.

At a first step, we reduce the problem to the case where ψ is an endomorphism defined as above and acting trivially on every element of the derived subgroup M' . For every element $y \in N$, an element $a(y) \in N$ is defined via

$$y\psi = a(y)y. \tag{26}$$

It is easy to verify that for any elements $y, y_1, y_2 \in N$ and any $g \in M$, the following equalities hold:

$$a(y^{-1}) = a(y)^{-1}, \quad a(y_1 y_2) = a(y_1) a(y_2), \quad a(y^g) = a(y)^g, \quad a(y\psi) = a(y)\psi. \tag{27}$$

Hence, the set $H = H_\psi = \{a(y) \mid y \in N\}$ is an Abelian ψ -admissible normal subgroup of M . We induce an endomorphism, denoted also by ψ , to the quotient $\bar{M} = M/H$. Note that ψ acts trivially on every element of an Abelian normal subgroup $\bar{N} = N/H \leq \bar{M}$.

Equation (2) has a solution in M iff its natural analog has a solution in \bar{M} . Indeed, if this analog is unsolvable, then (2) is clearly also unsolvable in M . Conversely, assume that our analog has a solution in \bar{M} . Then there is an element $z_1 \in M$ for which $z_1\psi = h w z_1$, $h \in H$. Therefore, there exists an element $y \in N$ such that $y\psi = h y$, which implies that $z = y^{-1} z_1$ is a solution for (2) in M .

An Abelian normal subgroup N is finitely generated as a module over $\mathbf{Z}A$, where $A = M/N$, by elements r_i , $i = 1, \dots, m$, which can be constructed effectively. Since ψ is identical mod(N), a normal subgroup $H = H_\psi$ of M is generated as a module by elements $a(r_i)$, $i = 1, \dots, m$. Adding these elements to a set of defining relations for M , we derive a finite description of \bar{M} .

Thus, all conditions of Corollary 8 are satisfied. Hence, there exists an effective algorithm that decides if an equation of the form (2) has a solution. \square

Proof of Theorem 2. Let M be a polycyclic metabelian group and ψ an arbitrary endomorphism of M .

First, we reduce the problem to the case where ψ is an endomorphism identical modulo an Abelian normal subgroup containing the derived subgroup of the entire group. Consider an equation of the form

$$x\psi = wx \text{ mod}(M') \tag{28}$$

induced by (2). Suppose that (28) has a solution $x_1 \in M$ (otherwise (2) has no solution in M). This can be effectively verified via a standard procedure. Then we write

$$x_1\psi = wx_1a, \quad a \in M'. \quad (29)$$

If $x \in M$ is a solution for (2), then $z = x^{-1}x_1 \in M$ is one for

$$z\psi = za. \quad (30)$$

Clearly, z belongs to the subgroup $K = \text{Fix}_{M',\psi}(M) = \{g \in Mg\psi = u(g)g, u(g) \in M'\}$, and $K \geq M'$. Conversely, if (30) is solved by z , which clearly belongs to K , then (2) is solved by $x = x_1z^{-1} \in M$.

Since K is obviously ψ -admissible, we consider $\psi \in \text{End}(K)$. Also K is polycyclic and metabelian. Using a standard procedure, we can find a finite set of elements of K such that

$$K = \text{gp}(g_1, \dots, g_k). \quad (31)$$

Consequently, $N = M'$ is an Abelian ψ -admissible normal subgroup of K containing the derived subgroup K' . Moreover, ψ is identical modulo N . Now we see that all conditions of Theorem 1 regarding K are satisfied. Hence, the twisted ψ -conjugacy problem is decidable in K , and in M too. \square

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