Deciding endo-fixedness

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Abstract

Let F be a finitely generated free group. We present an algorithm such that, given a subgroup $H \leqslant F$, decides whether H is the fixed subgroup of some family of automorphisms, or family of endomorphisms of F and, in the affirmative case, finds such a family. The algorithm uses both combinatorial and geometric methods.

1 Introduction

For all the paper, let $A = \{a_1, \ldots, a_n\}$ be an alphabet with n different letters, and F be the free group (of rank $r(F) = n$) with basis A.

Let $\mathrm{End}\,(F)$ denote the endomorphism monoid of F, and $\mathrm{Aut}\,(F)$ the automorphism group of F, so $\mathrm{Aut}\,(F)$ is the group of units of $\mathrm{End}\,(F)$. Throughout, we let elements of $\mathrm{End}\,(F)$ act on the right on F, so $x \mapsto (x)\phi$.

In the last decade a lot of literature has appeared studying the fixed subgroup of a single, or a family, of automorphisms, or endomorphisms, of F (see the survey [19] for details). But very few algorithmic results are known in this direction. Only recently, O. Maslakova (see [13]) has found an algorithm to compute a set of generators for the fixed subgroup of an automorphism of F (which is quite complicated, and whose complexity is quite high). One can easily extend this to compute generators for the fixed subgroup of a family of automorphisms, but the same questions using endomorphisms are still open.

In this note, we shall address the dual problem. We present an algorithm such that, given a subgroup $H \leqslant F$, decides whether H is the fixed subgroup of some family of automorphisms, or family of endomorphisms of F and, in the affirmative case, finds such a family.

Recognizing whether H is the fixed subgroup of a family of automorphisms is not difficult, because a classical result by McCool already said that the stabilizer of a finitely generated subgroup of F is finitely generated (as subgroup of $\mathrm{Aut}\,(F)$), see Theorem 2.1 below. However, recognizing whether H is the fixed subgroup of a family of endomorphisms is trickier, because the stabilizer of H, in general, need not be finitely generated as a submonoid of $\mathrm{End}\,(F)$. The main technique used to deal with this problem is the graphical tool called "fringe of a subgroup", which allows to compute the collection of algebraic extensions of a given subgroup H.

In the rest of this introduction we define the main concepts, and state the results to be used, with the correct references. Along the following section we prove the main result and give the two announced deciding algorithms.

Definition 1.1 For any $S \subseteq \mathrm{End}\,(F)$, let $\mathrm{Fix}\,(S)$ denote the subset consisting of those elements of

F which are fixed by every element of S (read $\mathrm{Fix}\,(S) = F$ for the case where S is empty). Then $\mathrm{Fix}\,(S)$ is a subgroup of F, called the *fixed subgroup of* S.

A subgroup H of F is called an *endo-fixed* subgroup of F if $H = \mathrm{Fix}\,(S)$ for some $S \subseteq \mathrm{End}\,(F)$. If S can be chosen to lie in $\mathrm{Aut}\,(F)$ we further say that H is an *auto-fixed* subgroup of F.

A subgroup H of F is called a 1-*endo-fixed* subgroup of F if $H = \mathrm{Fix}(\phi)$ for some $\phi \in \mathrm{End}\,(F)$ (here, and throughout, to simplify notation we write $\mathrm{Fix}(\phi)$ rather than $\mathrm{Fix}(\{\phi\})$). If ϕ can be chosen to lie in $\mathrm{Aut}\,(F)$, we further say that H is a 1-*auto-fixed* subgroup of F. For example, any maximal cyclic subgroup of F is 1-auto-fixed, since it is the subgroup fixed by a suitable inner automorphism. And non-maximal cyclic subgroups of F are not even endo-fixed, because every endomorphism fixing a power of an element must fix the element itself.

Notice that, since $\mathrm{Fix}\,(S) = \cap_{\alpha \in S} \mathrm{Fix}\,(\alpha)$, an auto-fixed (resp. endo-fixed) subgroup is an intersection of 1-auto-fixed (resp. 1-endo-fixed) subgroups, and vice-versa. And, clearly, the families of auto-fixed and endo-fixed subgroups of F are closed under arbitrary intersections. $\Box$

The most important result about 1-auto-fixed subgroups of F were obtained by M. Bestvina and M. Handel in [2], where they showed that every 1-auto-fixed subgroup of F has rank at most $r(F)$, which had previously been conjectured by G. P. Scott. Soon after the announcement of this result, and using it, Imrich and Turner showed, in [5], that any 1-endo-fixed subgroup of F also has rank at most $r(F)$. Later, Dicks-Ventura [4], using the techniques of [2], showed that any auto-fixed subgroup of F has rank at most $r(F)$; in fact, they proved a stronger result, namely that any mono-fixed subgroup of F is F-inert (a subgroup $H \leqslant F$ is F*-inert* if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F$). And after this, G. M. Bergman [1], using the result of [4], showed that any endo-fixed subgroup of F also has rank at most n (it is not known whether endo-fixed subgroups of F are necessarily F-inert; this is known as the inertia conjecture, see [12] or [19]). This brief history is appropriate for our purposes, but is far from complete; for example, it does not mention the ground-breaking work of S. M. Gersten, who first showed that 1-auto-fixed subgroups are finitely generated.

A natural question that arises in this context is what is the relation between the four mentioned families of subgroups of F, namely 1-auto-fixed, 1-endo-fixed, auto-fixed and endo-fixed subgroups. The relation between these families is partially known, though not completely. For example, in [10], Martino-Ventura showed that every auto-fixed (resp. endo-fixed) subgroup of F is a free factor of a 1-auto-fixed (resp. 1-endo-fixed) subgroup of F. But they also gave an example of a free factor of a 1-auto-fixed subgroup which is not even endo-fixed. It is conjectured, but not known in general, that the families of 1-auto-fixed and auto-fixed subgroups (resp. 1-endo-fixed and endo-fixed subgroups) do coincide; in other words, it is not known whether the family of 1-auto-fixed (resp. 1-endo-fixed) subgroups is closed under intersections. This is only known to be true when $n = 2$, and when the involved fixed subgroups have maximal rank. We don't include in this discussion the families of 1-mono-fixed and mono-fixed subgroups, because they are known to coincide with the families of 1-auto-fixed and auto-fixed subgroups, respectively (see Theorem 11 in [11]).

On the other direction, it is known that the families of 1-endo-fixed and 1-auto-fixed subgroups (and the families of endo-fixed and auto-fixed subgroups, as well) do not coincide. In [11], the authors exhibit the first known examples of 1-endo-fixed subgroups which are not 1-auto-fixed (see also [3] for more interesting calculations about this phenomena).

Finally, we mention a computational result that will be used later (it only has theoretical interest because no precise bound on the complexity is known, and one expects it to be quite high).

Theorem 1.2 (Maslakova, [13]) *Let $\varphi\colon F \to F$ be an automorphism of a finitely generated free group F. Then, a basis for $Fix(\varphi)$ is computable.*

Definition 1.3 Let $H \leqslant F$. We denote by $\mathrm{Aut}_H(F)$ the subgroup of $\mathrm{Aut}\,(F)$ consisting of all automorphisms of F which fix every element of H (this is usually called the *stabilizer* of H). Analogously, we denote by $\mathrm{End}_H(F)$ the submonoid of $\mathrm{End}\,(F)$ consisting of all endomorphisms of F which fix every element of H. Obviously, $\mathrm{Aut}_H(F) \leqslant \mathrm{End}_H(F)$.

Now, $\mathrm{Aut}_{(-)}(F)$ is a function from the set of subgroups of F to the set of subsets of $\mathrm{Aut}\,(F)$, and $\mathrm{Fix}\,(-)$ is a function in the reverse direction. This pair of functions form a Galois connection, and their images are called *closed* subsets (in $\mathrm{Aut}\,(F)$ and F, respectively). Clearly, $\mathrm{Aut}\,(F)$-closed subgroups of F are precisely the auto-fixed subgroups. Mimicking the classical Galois notions, we define the *auto-closure* of H in F, denoted $a\text{-}Cl_F(H)$, as $\mathrm{Fix}\,(\mathrm{Aut}_H(F))$, i.e. the smallest auto-fixed subgroup of F containing H.

Replacing Aut to End everywhere in the previous paragraph we obtain another Galois connection, and we similarly define the *endo-closure* of H in F, denoted $e\text{-}Cl_F(H)$, as $\mathrm{Fix}\,(\mathrm{End}_H(F))$, i.e. the smallest endo-fixed subgroup of F containing H. Since $\mathrm{Aut}_H(F) \leqslant \mathrm{End}_H(F)$, an obvious relation between closures is that

$$e\text{-}Cl_F(H) = \mathrm{Fix}\,(\mathrm{End}_H(F)) \leqslant \mathrm{Fix}\,(\mathrm{Aut}_H(F)) = a\text{-}Cl_F(H).$$

However, the equality does not hold in general, because of the existence of 1-endo-fixed subgroups which are not auto-fixed.

Note that, by the results mentioned above, the ranks of $a\text{-}Cl_F(H)$ and $e\text{-}Cl_F(H)$ are always less than or equal $r(F)$, even if that of H is not. □

The main goal of this note is to show that, for any finitely generated $H \leqslant F$ (given by a set of generators), one can algorithmically compute a basis for both $a\text{-}Cl_F(H)$ and $e\text{-}Cl_F(H)$. Using this algorithm, one can immediately decide whether the given H is auto-fixed (resp. endo-fixed) or not: H is auto-fixed (resp. endo-fixed) if and only if $a\text{-}Cl_F(H) = H$ (resp. $e\text{-}Cl_F(H) = H$).

To do this, we need to use the concepts of retract and stable image, and the graphical technique called "fringe of a subgroup" to compute the set of algebraic extensions of H. We briefly review on these two topics, now.

A subgroup $H \leqslant F$ is called a *retract of* F (just *retract* if there is no risk of confusion) if there exists a homomorphism $\rho\colon F \to H$ which fixes the elements of H; such a morphism is called a *retraction*. The obvious examples of retracts are the free factors of F, but there are retracts which are not free factors. Recognizing retracts is algorithmically possible (as showed in Proposition 4.6 of [14] following an argument indicated by Turner), though quite complicated because it uses Makanin's algorithm to solve equations in free groups.

Proposition 1.4 (4.6 in [14]) *Let $H \leqslant F$ be a finitely generated subgroup of F, given by a finite set of generators. It is decidable whether H is a retract of F.*

For a given endomorphism $\varphi\colon F \to F$, define the *stable image* of φ as $F\varphi^\infty = \cap_{m=1}^{\infty} F\varphi^m$. With a simple argument, Imrich and Turner showed in [5] that: 1) $F\varphi^\infty$ is a φ-invariant subgroup of F, 2) the restriction of φ to its stable image is always an automorphism, and 3) $F\varphi^\infty$ is a retract of F. This will be used later in order to reduce a certain computation with endomorphisms, to a similar one with automorphisms.

Let $H \leqslant K \leqslant F$. We say that the extension $H \leqslant K$ is *algebraic*, denoted $H \leqslant_{\text{alg}} K$, if H is not contained in any proper free factor of K. The opposite situation consists of H being a *free factor* of K, denoted $H \leqslant_{\text{ff}} K$. It is not difficult to see that every extension $H \leqslant K$ of finitely generated (free) subgroups of F can be decomposed, in a unique way, as an algebraic extension followed by a free extension, namely $H \leqslant_{\text{alg}} L \leqslant_{\text{ff}} K$ (just take L to be the smallest free factor of K containing H). The uniqueness refers to the fact that L is completely determined by the original extension $H \leqslant K$; again, mimicking the classical Galois theory, L is called the *algebraic closure of H in K*. We refer the reader to [14] for a detailed development of these ideas, including an analysis of the similarities and the significant differences with respect to classical field theory.

The important fact in this story is an old result by M. Takahasi, originally proven by combinatorial methods in [17] (and reproduced in Section 2.4, Exercise 8, of [8]). However, the modern graphical techniques developed by Stalling's in the 1980's (see [16]) lead to a new, clear, concise and natural proof of Takahasi's Theorem, which was discovered independently by Ventura in [18], and by Kapovich-Miasnikov in [6]. Margolis, Sapir and Weil, also independently, considered the same construction in [9] for a slightly different purposes. See [14] for a unification of these three points of view, written in the language of algebraic extensions. In this setting, Takahasi's Theorem says the following.

Theorem 1.5 (Takahasi) *Let $H \leqslant F$ be a subgroup of a free group F. If H is finitely generated then it has finitely many algebraic extensions, i.e.*

$$\mathcal{AE}(H) = \{K \leqslant F \mid H \leqslant_{\text{alg}} K\}$$

is finite. Furthermore, bases of the elements in $\mathcal{AE}(H)$ are computable from a set of generators of H.

Sketch of proof. Think $F = \langle A \mid \ \rangle$ as the fundamental group of a bouquet of n circles, and then H as the corresponding covering $X(H)$, which can be though as a graph with labels from A on the edges (this graph is easily computable from any given set if generators of H). Except when H is of finite index in F, the graph $X(H)$ if infinite but, if H is finitely generated, $X(H)$ consists on a finite core $\Gamma(H)$ with attached infinite trees (each isomorphic to connected parts of the Cayley graph of F with respect to A). Now consider an arbitrary extension $H \leqslant K \leqslant F$. It corresponds to another covering $X(K)$, which is in turn covered by $X(H)$. That is, $X(K)$ is a quotient of $X(H)$ and so, can be obtained from $X(H)$ by performing several identifications of vertices and edges. These identifications may destroy $\Gamma(H)$, but some quotient of $\Gamma(H)$ always remains as a subgraph of $X(K)$ (in fact, of $\Gamma(K)$). If H is finitely generated then $\Gamma(H)$ is finite, and so has finitely many quotients, which are computable from $\Gamma(H)$ (i.e. from any given set of generators of H). This gives a computable finite list of extensions of H, called the *fringe of H*, $\mathcal{O}(H) = \{H_1, \ldots, H_p\}$, $p \geqslant 1$. And, by construction, it is clear that, for every $H \leqslant K$, there exists $i = 1, \ldots, p$ such that $H \leqslant H_i \leqslant_{\text{ff}} K$. This implies that $\mathcal{AE}(H) \subseteq \mathcal{O}(H)$ and so, we already have a proof of the finiteness part of Takahasi's Theorem. Unfortunately, the equality between these two sets is not true in general, as one can find free factor relations between the H_i's. But, after a cleaning process (checking for every pair (i, j) whether $H_i \leqslant_{\text{ff}} H_j$ and, in this case, deleting H_j from the list) one can algorithmically compute $\mathcal{AE}(H) = \{H_1, \ldots, H_q\}$, $1 \leqslant q \leqslant p$. See [15] for a fast algorithm to check free factorness. □

Note that the smallest and the biggest of the H_i's in $\mathcal{O}(H)$ correspond, respectively, to the quotient identifying nothing, which gives H itself, and to the quotient identifying all vertices down to a single one, which gives $\langle A' \rangle \leqslant_{\rm ff} F$, where $A' \subseteq A$ is the set of all letters involved in the generators of H. Note also that the first one belongs to $\mathcal{AE}(H)$ (since $H \leqslant_{\rm alg} H$) and the same happens for either the second one or a free factor of it. In particular, $\mathcal{AE}(H)$ contains at least H, and a free factor of F. This fact will be used later.

Finally, we mention one of the results in [16]. Given two finitely generated subgroups $H, K \leqslant F$ (by sets of generators, say), one can algorithmically compute a basis for $H \cap K$ using the technique of pull-backs of graphs.

Proposition 1.6 (Stallins [16]) *Let $H, K \leqslant F$ be two finitely generated subgroups of a free group F, given by finite sets of generators. Then, a basis for $H \cap K$ is algorithmically computable.*

2 The algorithm

Let $H \leqslant F$ be a finitely generated subgroup of F, given by a set of generators. We shall give two algorithms to compute a basis for $a\text{-}Cl_F(H)$ and $e\text{-}Cl_F(H)$. The basic elementary result that we use is due to McCool (see Proposition 5.7 in Chapter I of [7], and the subsequent paragraph):

Theorem 2.1 (McCool, [7]) *Let $H \leqslant F$ be a finitely generated subgroup of a finitely generated free group, given by a finite set of generators. Then the stabilizer, $Aut_H(F)$, of H is also finitely generated (in fact finitely presented), and a finite set of generators (and relations) is algorithmically computable from H.*

2.1 The automorphism case

By Theorem 2.1, $\mathrm{Aut}_H(F)$ is finitely generated; furthermore, a list of generators, say $\mathrm{Aut}_H(F) = \langle \varphi_1, \ldots, \varphi_m \rangle \leqslant \mathrm{Aut}\,(F)$, can be found algorithmically from a set of generators of H . Now it is clear that

$$a\text{-}Cl_F(H) = \mathrm{Fix}\,(\mathrm{Aut}_H(F)) = \cap_{\varphi \in \mathrm{Aut}_{H(F)}} \mathrm{Fix}\,(\varphi) = \mathrm{Fix}\,(\varphi_1) \cap \cdots \cap \mathrm{Fix}\,(\varphi_m).$$

By Maslakova's Theorem 1.2, we can then compute generators for $\mathrm{Fix}\,(\varphi_1), \ldots, \mathrm{Fix}\,(\varphi_m)$ and, using Proposition 1.6, find a basis for $a\text{-}Cl_F(H)$. Hence we have proven

Theorem 2.2 *Let $H \leqslant F$ be a finitely generated subgroup of a free group, given by a finite set of generators. Then, a basis for the auto-closure $a\text{-}Cl_F(H)$ of H is algorithmically computable, together with a finite set of automorphisms $\varphi_1, \ldots, \varphi_m \in Aut(F)$ such that $a\text{-}Cl_F(H) = Fix(\varphi_1) \cap \cdots \cap Fix(\varphi_m)$.* $\square$

Corollary 2.3 *Let $H \leqslant F$ be a finitely generated subgroup of a free group, given by a finite set of generators. Then, it is algorithmically decidable whether H is auto-fixed and, in the affirmative case, find a finite set of automorphisms $\varphi_1, \ldots, \varphi_m \in Aut(F)$ such that $H = Fix(\varphi_1) \cap \cdots \cap Fix(\varphi_m)$.* $\square$

Proof. Apply Theorem 2.2 to H. If $a\text{-}Cl_F(H)$ is strictly bigger than H, then H is not auto-fixed (there are elements outside H which are fixed by *every* automorphism of F fixing H). Otherwise, $a\text{-}Cl_F(H) = H$ and the algorithm in Theorem 2.2 also outputs a finite list of automorphisms $\varphi_1, \ldots, \varphi_m \in \mathrm{Aut}\,(F)$ such that $\mathrm{Fix}\,(\varphi_1) \cap \cdots \cap \mathrm{Fix}\,(\varphi_m) = a\text{-}Cl_F(H) = H$. $\square$

We don't play attention to the complexity of this algorithm; it only has theoretical interest and it is not effective. McCool's algorithm is a brute force search which is not conceptually complicated, but has strongly exponential complexity. And Maslakova's algorithm is conceptually much more sophisticated, and its complexity also seems to be quite high.

2.2 The endomorphism case

There is no hope that a similar strategy with endomorphisms instead of automorphisms could work; it is known that $\mathrm{End}_H(F)$ is not always finitely generated as submonoid of $\mathrm{End}\,(F)$ and so, there is no hope of having a version of McCool's result for endomorphisms. Moreover, Maslakova's Theorem makes strong use of train tracks, a machinery that works only for monomorphisms and definitely does not work in presence of kernel; in fact, at the time of writing, no algorithm is known to compute the fixed subgroup of an arbitrary endomorphism of F.

Instead, we will use algebraic extensions, Takahasi's Theorem and retractions to reduce the computation of $e\text{-}Cl_F(H)$ to finitely many computations of auto-closures.

Theorem 2.4 *Let $H \leqslant F$ be a finitely generated subgroup of a free group, given by a finite set of generators. Then, a basis for the endo-closure $e\text{-}Cl_F(H)$ of H is algorithmically computable, together with a finite set of endomorphisms $\varphi_1, \ldots, \varphi_m \in End(F)$ such that $e\text{-}Cl_F(H) = Fix(\varphi_1) \cap \cdots \cap Fix(\varphi_m)$.*

Proof. Consider the set of algebraic extensions of H, $\mathcal{AE}(H) = \{H_1, H_2, \ldots, H_q\}$, and the subset of those which are retracts of F, say $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$, $1 \leqslant r \leqslant q$ (note that $\mathcal{AE}_{ret}(H)$ is not empty because, as we noted above, $\mathcal{AE}(H)$ contains at least a free factor (and so a retract) of F). By Theorem 1.5, we can algorithmically compute $q \geqslant 1$, and a basis for each $H_1, \ldots, H_q$. Now, using Theorem 1.4, we can algorithmically decide which ones of these H_i's are retracts of F, and so compute $r \geqslant 1$ and $\mathcal{AE}_{ret}(H) = \{H_1, \ldots, H_r\}$. Then, write the generators of H as words on the generators of each one of these H_i's, and apply r times Theorem 2.2 to compute bases for $a\text{-}Cl_{H_1}(H), \ldots, a\text{-}Cl_{H_r}(H)$. Finally, use Proposition 1.6 to find a basis for $\cap_{i=1}^{r} a\text{-}Cl_{H_i}(H)$.

We claim that $\cap_{i=1}^{r} a\text{-}Cl_{H_i}(H) = e\text{-}Cl_F(H)$, which will conclude the proof. In fact, we shall prove this equality under the form

$$\bigcap_{i=1}^{r} \bigcap_{\substack{\alpha \in \mathrm{Aut}\,(H_i) \\ H \leqslant \mathrm{Fix}\,(\alpha)}} \mathrm{Fix}\,(\alpha) = \bigcap_{\substack{\beta \in \mathrm{End}\,(F) \\ H \leqslant \mathrm{Fix}\,(\beta)}} \mathrm{Fix}\,(\beta), \tag{1}$$

by showing that every intersecting subgroup in one side in also present in the opposite side.

Let $\beta \in \mathrm{End}\,(F)$ be such that $H \leqslant \mathrm{Fix}\,(\beta)$. Consider the stable image of β, which contains H, and the algebraic closure of H in it, $H \leqslant_{\mathrm{alg}} H_i \leqslant_{\mathrm{ff}} F\beta^{\infty} \leqslant F$. The endomorphism β restricts to an automorphism of $F\beta^{\infty}$ which, in turn, restricts to an automorphism $\alpha = \beta_{|H_i}$ of H_i (because images of free factors of $F\beta^{\infty}$ under β, are again free factors of $F\beta^{\infty}$). And, clearly, $\mathrm{Fix}\,(\alpha) = \mathrm{Fix}\,(\beta)$. This shows inclusion "$\leqslant$" in equation (1).

Let $H_i \in \mathcal{AE}_{ret}(H)$, and $\alpha \in \mathrm{Aut}\,(H_i)$ with $H \leqslant \mathrm{Fix}\,(\alpha)$. Let $\rho\colon F \to H_i$ be a retraction, and consider $\beta = \rho\alpha\iota \in \mathrm{End}\,(F)$, where $\iota\colon H_i \to F$ is the inclusion map. It is clear that $H \leqslant \mathrm{Fix}\,(\alpha) = \mathrm{Fix}\,(\beta)$ and so, $\mathrm{Fix}\,(\alpha)$ is also one of the subgroups appearing the intersection on the right hand side. This shows inclusion "$\geqslant$" in equation (1) and completes the proof. □

Corollary 2.5 *Let $H \leqslant F$ be a finitely generated subgroup of a free group, given by a finite set of generators. Then, it is algorithmically decidable whether H is endo-fixed and, in the affirmative case, find a finite set of endomorphisms $\varphi_1, \ldots, \varphi_m \in End(F)$ such that $H = Fix(\varphi_1) \cap \cdots \cap Fix(\varphi_m)$.* $\square$

Since by McCool's Theorem 2.1 stabilizers of finitely generated subgroups are finitely generated, we can immediately deduce that every auto-fixed subgroup is a finite intersection of 1-auto-fixed subgroups. That is, for every set $S \subseteq \mathrm{Aut}(F)$ there exists $S_0 \subseteq \mathrm{Aut}(F)$ such that $|S_0| < \infty$ and $\mathrm{Fix}(S) = \mathrm{Fix}(S_0)$. Even without $\mathrm{End}_H(F)$ being finitely generated in general, the same result is true for endomorphisms, as can immediately be deduced from Corollary 2.5.

Corollary 2.6 *Every endo-fixed subgroup is a finite intersection of* 1*-endo-fixed subgroups. That is, for every $S \subseteq End(F)$ there exists $S_0 \subseteq End(F)$ such that $|S_0| < \infty$ and $Fix(S) = Fix(S_0)$.* $\square$

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