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Volume equivalence of subgroups of free groups

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ABSTRACT

Let F_n be a free group of finite rank $n \geq 2$. Due to Kapovich, Levitt, Schupp and Shpilrain (2007) [1], two finitely generated subgroups H and K of F_n are called volume equivalent, if for every free and discrete isometric action of F_n on an \mathbb{R} -tree T , we have $\text{vol}(T_H/H) = \text{vol}(T_K/K)$. We give a more algebraic and combinatorial characterization of volume equivalence and discuss a counterexample of volume equivalence in order to justify our characterization. We also provide a specific example to answer a question of Kapovich, Levitt, Schupp and Shpilrain in the negative: volume equivalence does not imply equality in rank.

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1. Introduction

Let F_n be a free group of finite rank $n \geq 2$. Two elements g and h in F_n are called *translation equivalent* if for every free and discrete isometric action of F_n on an \mathbb{R} -tree T , the elements g and h have equal translation lengths on T , namely $\inf_{x \in T} d(x, gx) = \inf_{x \in T} d(x, hx)$. This concept was first introduced by Kapovich, Levitt, Schupp and Shpilrain [1] motivated by several notions studied in Teichmüller theory related to spectral geometry of hyperbolic surfaces, particularly the notions of “hyperbolic equivalence” and “simple intersection theory” there (cf. [7]). It is easy to see that if g is conjugate to $h^{\pm 1}$ in F_n , then g and h are translation equivalent. However it was not obvious and not known that more complicated examples of translation equivalence exist. The paper of Kapovich et al. [1] gave several combinatorial characterizations of translation equivalence in free

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groups and also exhibited two unexpected sources of translation equivalence: one associated with traces of $SL(2, \mathbb{C})$ -representations of F_n (where the proof is ultimately analytic) and the other of a combinatorial nature involving some \mathbb{R} -tree techniques. In the same paper, Kapovich et al. proposed a problem about other possible sources of translation equivalence in free groups, which was later solved in the affirmative by the first author [4]. Furthermore, the first author [5] proved that there is an algorithm to decide whether or not two given elements of F_2 are translation equivalent.

Kapovich et al. [1] also introduced two natural generalizations of the notion of translation equivalence: one is the bounded translation equivalence in free groups and the other is the volume equivalence of subgroups of free groups which is the main topic of the present paper.

Definition 1.1. Two elements g and h in F_n are called *boundedly translation equivalent* if there is $C > 0$ such that

$$\frac{1}{C} \leq \frac{\|\phi(g)\|}{\|\phi(h)\|} \leq C$$

for every automorphism ϕ of F_n .

Clearly every pair of translation equivalent elements in F_n are boundedly translation equivalent, but not vice versa. As an easy example of bounded translation equivalent elements which are not translation equivalent, we can take two elements $g = a$ and $h = a[a, b]$ in $F(a, b)$ which is the free group with basis $\{a, b\}$. Indeed, it was shown in [6, Introduction] that, for these two elements g and h ,

$$\|\phi(h)\| = \|\phi(g)\| + 4$$

for every automorphism ϕ of $F(a, b)$ and thus

$$\frac{1}{5} \leq \frac{\|\phi(g)\|}{\|\phi(h)\|} \leq 1$$

for every automorphism ϕ of $F(a, b)$. The first author also proved in [6] that bounded translation equivalence in the free group of rank two is algorithmically decidable. Recently, Kapovich and Lustig [2] showed that any two “sufficiently random” elements in F_n are boundedly translation equivalent.

Definition 1.2. Two finitely generated subgroups H and K of F_n are called *volume equivalent*, if for every free and discrete isometric action of F_n on an \mathbb{R} -tree T , we have $\text{vol}(T_H/H) = \text{vol}(T_K/K)$. Here T_H is the unique minimal H -invariant subtree of T ; thus T_H/H is a finite graph whose edges inherit the same lengths as their lifts to T . The *volume* $\text{vol}(T_H/H)$ is the sum of the lengths of the edges of this graph.

Once again, if H is conjugate to K , then, obviously, H and K are volume equivalent. Also, if H and K are subgroups of index $m < \infty$ in a subgroup L of F_n , then H and K are volume equivalent. If $H = \langle h \rangle$ and $K = \langle g \rangle$ are infinite cyclic subgroups of F_n , then H and K are volume equivalent if and only if g and h are translation equivalent. Furthermore, if H and K are volume equivalent in F_n and if $\phi : F_n \rightarrow F_m$ is an injective homomorphism, then $\phi(H)$ and $\phi(K)$ are volume equivalent in F_m . Apart from these observations, no other sources of volume equivalence have been known.

Let F_n be the free group of finite rank $n \geq 2$ with basis Σ . It is well known that every finitely generated subgroup H of F_n has a unique finite graphical representation Γ_H up to graph isomorphism. The graph Γ_H is a cyclically reduced Σ -labeled directed graph which depends only on the conjugacy class $[H]$. We refer the reader to [3] and [8] for the construction of the graph Γ_H . By $V(\Gamma_H)$, we denote the number of vertices of the graph Γ_H , whereas $E(\Gamma_H)$ denotes the number of edges of Γ_H . As in [8], for $S \subset \Sigma^{\pm 1}$, the *capacity* of S , denoted by $\text{cap}(\Gamma_H; S)$, means the number of vertices v

of Γ_H such that the set of labels of incoming edges of v meets both S and its complement S^c . The terminology “capacity” comes from hipergraph versions of Whitehead graphs for finitely generated subgroups of free groups.

We now introduce our main results of the present paper. Motivated by [1, Theorem A], we give a more algebraic and combinatorial characterization of volume equivalence as follows.

Theorem A.

- (1) For $n \geq 3$, two finitely generated subgroups H and K of F_n are volume equivalent if and only if $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ for every automorphism ϕ of F_n .
- (2) Two finitely generated subgroups H and K of $F_2 = F(a, b)$ are volume equivalent if and only if $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ and $cap(\Gamma_{\phi(H)}; \{a^{\pm 1}\}) = cap(\Gamma_{\phi(K)}; \{a^{\pm 1}\})$ for every automorphism ϕ of F_2 .

The following corollary is immediate.

Corollary 1.3.

- (1) For $n \geq 3$, two finitely generated subgroups H and K of F_n are volume equivalent if and only if $vol(X(F_n, A)_H/H) = vol(X(F_n, A)_K/K)$ for Cayley graphs $X(F_n, A)$ of F_n with respect to all possible free bases A of F_n .
- (2) Two finitely generated subgroups H and K of F_2 are volume equivalent if and only if $vol(X(F_2, A)_H/H) = vol(X(F_2, A)_K/K)$ and $cap(X(F_2, A)_H/H; \{x^{\pm 1}\}) = cap(X(F_2, A)_K/K; \{x^{\pm 1}\})$ for every free basis A of F_2 and every $x \in A$.

The proof of Theorem A will appear in Section 2, and it is only a modification and generalization of the proof of [1, Theorem A]. As stated in Theorem A, our combinatorial characterization of volume equivalence in F_2 requires one additional condition on “capacities”. We justify that this extra condition cannot be dropped by proving the following theorem, whose proof will be given also in Section 2.

Theorem B. *Let*

$$H = \langle [a, b] \rangle \quad \text{and} \quad K = \langle a, b^2, bab^{-1} \rangle$$

be subgroups of $F(a, b)$. Then we have $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ for every automorphism ϕ of $F(a, b)$, but H and K are not volume equivalent.

Introducing the notion of volume equivalence, Kapovich et al. [1] asked whether the volume equivalence of two subgroups H and K of F_n necessarily implies that H and K have the same rank. The following theorem answers this question in the negative.

Theorem C. *Let*

$$H = \langle [b, a^{-1}]b[b, a^{-1}]ba^{-1}ba \rangle,$$

$$K = \langle [b, a^{-1}]^2, ba^{-1}[b, a]b^{-2}a \rangle$$

be subgroups of $F(a, b)$. Then H and K are volume equivalent in $F(a, b)$ but $rank(H) \neq rank(K)$.

The proof of Theorem C will appear in Section 4, and it heavily relies on the technical analysis to be done in Section 3 where we will analyze the effect of automorphisms of F_2 on the subgroup graphs of F_2 with the use of an explicit description of automorphisms of F_2 obtained in [5].

Remark 1.4. (1) Using arguments similar to the proof of Theorem C, we can also prove:

(i) Two subgroups

$$H = \langle [a, b^{-1}]a[a, b^{-1}]^2, a^{-1}b^{-1}[a, b^{-1}] \rangle,$$

$$K = \langle [a, b^{-1}]a[a, b^{-1}]^2, a^{-1}b^{-1}[a, b^{-1}]a[a, b^{-1}] \rangle$$

are volume equivalent in $F(a, b)$.

(ii) Two subgroups

$$H = \langle a^5, (ab^{-1})^3 a^{-2}, ab^{-1}a^2b^{-1} \rangle,$$

$$K = \langle a^5, (ab^{-1})^3 a^{-1}, ab^{-1}a^2b^{-1} \rangle$$

satisfy $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ for every automorphism ϕ of $F(a, b)$, but H and K are not volume equivalent in $F(a, b)$.

(2) If $H = \langle h \rangle$ and $K = \langle g_1, g_2 \rangle$ are as in Theorem C and if K_1 is a subgroup of K of index $m \geq 1$, then K_1 is volume equivalent to $\langle h^m \rangle$. The rank of K_1 is equal to $m + 1$. Thus Theorem C yields an example of a subgroup of $F(a, b)$ of arbitrary rank $k \geq 2$ which is volume equivalent to a cyclic subgroup.

2. A combinatorial characterization of volume equivalence

Throughout this section, let F_n be the free group of finite rank $n \geq 2$ with basis Σ . We begin this section by recalling that a *Whitehead automorphism* α of F_n is defined to be an automorphism of one of the following two types (cf. [5]):

- (W1) α permutes elements in $\Sigma^{\pm 1}$.
- (W2) α is defined by a letter $x \in \Sigma^{\pm 1}$ and a set $S \subset \Sigma^{\pm 1} \setminus \{x, x^{-1}\}$ in such a way that if $c \in \Sigma^{\pm 1}$ then
 - (a) $\alpha(c) = cx$ provided $c \in S$ and $c^{-1} \notin S$; (b) $\alpha(c) = x^{-1}cx$ provided both $c, c^{-1} \in S$; (c) $\alpha(c) = c$ provided both $c, c^{-1} \notin S$.

If α is of type (W2), we write $\alpha = (S, x)$. Note that in the expression of $\alpha = (S, x)$ it is conventional to include the defining letter x in the defining set S , but for the sake of brevity of notation we will omit x from S as defined above.

We establish two lemmas that are used in the proof of Theorem A from the Introduction. If Γ is a finite cyclically reduced Σ -labeled directed graph and $x \in \Sigma^{\pm 1}$, then $n(\Gamma; x)$ denotes the number of edges of Γ labeled by x . Clearly $n(\Gamma; x) = n(\Gamma; x^{-1})$.

Lemma 2.1. *For two finitely generated subgroups H and K of F_n , if $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ for every automorphism ϕ of F_n , then $n(\Gamma_H; x) = n(\Gamma_K; x)$ for every $x \in \Sigma^{\pm 1}$.*

Proof. Let $\Sigma = \{a_1, a_2, \dots, a_n\}$. For each $i, j \in \{1, \dots, n\}$ with $i \neq j$, let $\phi_{ij} = (\{a_i\}, a_j)$ be a Whitehead automorphism of F_n of type (W2). By [8], we have

$$V(\Gamma_{\phi_{ij}(H)}) - V(\Gamma_H) = \text{cap}(\Gamma_H; \{a_i, a_j\}) - n(\Gamma_H; a_j). \tag{1}$$

Since ϕ_{ij} is an automorphism of F_n , $\Gamma_{\phi_{ij}(H)}$ has the same number of faces as Γ_H does. So, by (1), we have

$$E(\Gamma_{\phi_{ij}(H)}) - E(\Gamma_H) = \text{cap}(\Gamma_H; \{a_i, a_j\}) - n(\Gamma_H; a_j). \tag{2}$$

For a similar reason, we get

$$E(\Gamma_{\phi_{ji}(H)}) - E(\Gamma_H) = \text{cap}(\Gamma_H; \{a_i, a_j\}) - n(\Gamma_H; a_i). \tag{3}$$

Subtract (3) from (2) to obtain

$$E(\Gamma_{\phi_{ij}(H)}) - E(\Gamma_{\phi_{ji}(H)}) = n(\Gamma_H; a_i) - n(\Gamma_H; a_j).$$

So, for each $i = 1, \dots, n$, we have

$$\sum_{j \neq i} \{E(\Gamma_{\phi_{ij}(H)}) - E(\Gamma_{\phi_{ji}(H)})\} = (n - 1)n(\Gamma_H; a_i) - \sum_{j \neq i} n(\Gamma_H; a_j).$$

Here, since $\sum_{j \neq i} n(\Gamma_H; a_j) = E(\Gamma_H) - n(\Gamma_H; a_i)$, we obtain

$$n(\Gamma_H; a_i) = \frac{1}{n} \left[E(\Gamma_H) + \sum_{j \neq i} \{E(\Gamma_{\phi_{ij}(H)}) - E(\Gamma_{\phi_{ji}(H)})\} \right].$$

The same argument is applied to K to get

$$n(\Gamma_K; a_i) = \frac{1}{n} \left[E(\Gamma_K) + \sum_{j \neq i} \{E(\Gamma_{\phi_{ij}(K)}) - E(\Gamma_{\phi_{ji}(K)})\} \right].$$

By the hypothesis of the lemma, we finally have

$$n(\Gamma_H; a_i^{-1}) = n(\Gamma_H; a_i) = n(\Gamma_K; a_i) = n(\Gamma_K; a_i^{-1})$$

for every $i = 1, \dots, n$, which proves the lemma. \square

Lemma 2.2. For two finitely generated subgroups H and K of F_n with $n \geq 3$, if $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ for every automorphism ϕ of F_n , then $\text{cap}(\Gamma_H; S) = \text{cap}(\Gamma_K; S)$ for every subset $S \subset \Sigma^{\pm 1}$ such that $S^{\pm 1} = S$.

Proof. Let $\Sigma = \{a_1, a_2, \dots, a_n\}$, and let S be a subset of $\Sigma^{\pm 1}$ such that $S^{\pm 1} = S$. Without loss of generality, we may assume that $S = \{a_1, \dots, a_k\}^{\pm 1}$ with $k < n$. Put $N = E(\Gamma_H) = E(\Gamma_K)$. There are two cases to consider.

Case 1. $1 < k < n - 1$.

For a finite cyclically reduced Σ -labeled directed graph Γ , by $L(\Gamma)$ we denote the number of vertices v of Γ such that v has only two incident edges labeled by a_1 and a_n into the vertex v . Now let ϕ be an automorphism of F_n defined by

$$\begin{aligned} \phi(a_1) &= (a_1^{-N} a_2 a_1^N)^{-N} a_1 (a_1^{-N} a_2 a_1^N)^N; \\ \phi(a_i) &= a_1^{-N} a_i a_1^N \quad \text{for } 2 \leq i \leq k; \\ \phi(a_j) &= a_n^{-N} a_j a_n^N \quad \text{for } k + 1 \leq j \leq n - 1; \\ \phi(a_n) &= (a_n^{-N} a_{n-1} a_n^N)^{-N} a_n (a_n^{-N} a_{n-1} a_n^N)^N. \end{aligned}$$

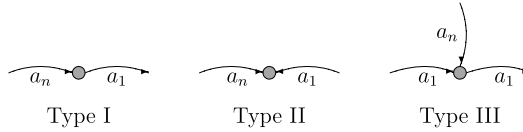


Fig. 1.

It is not hard to observe that

$$\begin{aligned} \text{cap}(\Gamma_H; S) &= L(\Gamma_{\phi(H)}), \\ \text{cap}(\Gamma_K; S) &= L(\Gamma_{\phi(K)}), \end{aligned} \tag{4}$$

and that in the graphs $\Gamma_{\phi(H)}$ and $\Gamma_{\phi(K)}$ there is no vertex v' such that v' has more than two incident edges two labeled by a_1 and a_n into the vertex v' . This observation yields that

$$\begin{aligned} E(\Gamma_{\psi\phi(H)}) - E(\Gamma_{\phi(H)}) &= n(\Gamma_{\phi(H)}; a_1) - 2L(\Gamma_{\phi(H)}), \\ E(\Gamma_{\psi\phi(K)}) - E(\Gamma_{\phi(K)}) &= n(\Gamma_{\phi(K)}; a_1) - 2L(\Gamma_{\phi(K)}), \end{aligned} \tag{5}$$

where ψ is the Whitehead automorphism $(\{a_1\}, a_n)$ of F_n . Here, by the hypothesis of the lemma, $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ and $E(\Gamma_{\psi\phi(H)}) = E(\Gamma_{\psi\phi(K)})$, which implies

$$E(\Gamma_{\psi\phi(H)}) - E(\Gamma_{\phi(H)}) = E(\Gamma_{\psi\phi(K)}) - E(\Gamma_{\phi(K)}).$$

Also by Lemma 2.1, we have $n(\Gamma_{\phi(H)}; a_1) = n(\Gamma_{\phi(K)}; a_1)$. It then follows from (5) that

$$L(\Gamma_{\phi(H)}) = L(\Gamma_{\phi(K)}),$$

so that, by (4), we finally have

$$\text{cap}(\Gamma_H; S) = \text{cap}(\Gamma_K; S).$$

Case 2. $k = 1$ or $k = n - 1$.

For every finite cyclically reduced Σ -labeled directed graph Γ , since

$$\text{cap}(\Gamma; \{a_1^{\pm 1}, \dots, a_{n-1}^{\pm 1}\}) = \text{cap}(\Gamma; \{a_n^{\pm 1}\}),$$

we may assume that $S = \{a_1^{\pm 1}\}$. Let $L_1(\Gamma)$, $L_2(\Gamma)$ and $L_3(\Gamma)$ denote the number of vertices of Γ of type I, II and III, respectively, as depicted in Fig. 1.

Let ϕ be an automorphism of F_n defined by

$$\begin{aligned} \phi(a_1) &= a_1; \\ \phi(a_i) &= a_n^{-N} a_i a_n^N \quad \text{for } 2 \leq i \leq n - 1; \\ \phi(a_n) &= (a_n^{-N} a_{n-1} a_n^N)^{-N} a_n (a_n^{-N} a_{n-1} a_n^N)^N. \end{aligned}$$

It is not hard to observe that

$$\begin{aligned} \text{cap}(\Gamma_H; S) &= \text{cap}(\Gamma_{\phi(H)}; S) = L_1(\Gamma_{\phi(H)}) + L_2(\Gamma_{\phi(H)}) + L_3(\Gamma_{\phi(H)}), \\ \text{cap}(\Gamma_K; S) &= \text{cap}(\Gamma_{\phi(K)}; S) = L_1(\Gamma_{\phi(K)}) + L_2(\Gamma_{\phi(K)}) + L_3(\Gamma_{\phi(K)}). \end{aligned} \tag{6}$$

Let ψ_1 and ψ_2 be the Whitehead automorphisms $(\{a_n\}, a_1)$ and $(\{a_n\}, a_1^{-1})$ of F_n , respectively. Then, by the hypothesis of the lemma,

$$E(\Gamma_{\psi_j\phi(H)}) - E(\Gamma_{\phi(H)}) = E(\Gamma_{\psi_j\phi(K)}) - E(\Gamma_{\phi(K)}),$$

for $j = 1, 2$. Note that

$$\begin{aligned} E(\Gamma_{\psi_1\phi(H)}) - E(\Gamma_{\phi(H)}) &= n(\Gamma_{\phi(H)}; a_n) - 2L_2(\Gamma_{\phi(H)}) - L_3(\Gamma_{\phi(H)}), \\ E(\Gamma_{\psi_1\phi(K)}) - E(\Gamma_{\phi(K)}) &= n(\Gamma_{\phi(K)}; a_n) - 2L_2(\Gamma_{\phi(K)}) - L_3(\Gamma_{\phi(K)}), \end{aligned}$$

and that

$$\begin{aligned} E(\Gamma_{\psi_2\phi(H)}) - E(\Gamma_{\phi(H)}) &= n(\Gamma_{\phi(H)}; a_n) - 2L_1(\Gamma_{\phi(H)}) - L_3(\Gamma_{\phi(H)}), \\ E(\Gamma_{\psi_2\phi(K)}) - E(\Gamma_{\phi(K)}) &= n(\Gamma_{\phi(K)}; a_n) - 2L_1(\Gamma_{\phi(K)}) - L_3(\Gamma_{\phi(K)}). \end{aligned}$$

Here, since $n(\Gamma_{\phi(H)}; a_n) = n(\Gamma_{\phi(K)}; a_n)$ by Lemma 2.1, we have

$$\begin{aligned} 2L_2(\Gamma_{\phi(H)}) + L_3(\Gamma_{\phi(H)}) &= 2L_2(\Gamma_{\phi(K)}) + L_3(\Gamma_{\phi(K)}), \\ 2L_1(\Gamma_{\phi(H)}) + L_3(\Gamma_{\phi(H)}) &= 2L_1(\Gamma_{\phi(K)}) + L_3(\Gamma_{\phi(K)}). \end{aligned}$$

Adding these two equalities yields

$$2L_1(\Gamma_{\phi(H)}) + 2L_2(\Gamma_{\phi(H)}) + 2L_3(\Gamma_{\phi(H)}) = 2L_1(\Gamma_{\phi(K)}) + 2L_2(\Gamma_{\phi(K)}) + 2L_3(\Gamma_{\phi(K)}),$$

that is,

$$L_1(\Gamma_{\phi(H)}) + L_2(\Gamma_{\phi(H)}) + L_3(\Gamma_{\phi(H)}) = L_1(\Gamma_{\phi(K)}) + L_2(\Gamma_{\phi(K)}) + L_3(\Gamma_{\phi(K)}).$$

It then follows from (6) that

$$\text{cap}(\Gamma_H; S) = \text{cap}(\Gamma_K; S),$$

as required. \square

Now we are in a position to prove Theorem A from the Introduction.

Theorem A.

- (1) For $n \geq 3$, two finitely generated subgroups H and K of F_n are volume equivalent if and only if $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ for every automorphism ϕ of F_n .
- (2) Two finitely generated subgroups H and K of $F_2 = F(a, b)$ are volume equivalent if and only if $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ and $\text{cap}(\Gamma_{\phi(H)}; \{a^{\pm 1}\}) = \text{cap}(\Gamma_{\phi(K)}; \{a^{\pm 1}\})$ for every automorphism ϕ of F_2 .

Proof. We first prove the “only if” part of (1) and (2). Suppose that H and K are volume equivalent in F_n ($n \geq 2$). So, for every free and discrete isometric action of F_n on an \mathbb{R} -tree X , $vol(X_H/H) = vol(X_K/K)$. Let ϕ be an automorphism of F_n . Obviously $\Sigma' = \phi(\Sigma)$ is a free basis of F_n . Take an \mathbb{R} -tree X as the Cayley graph of F_n with respect to Σ' . Then $vol(X_H/H) = E(\Gamma_{\phi(H)})$ and $vol(X_K/K) = E(\Gamma_{\phi(K)})$, and thus we have $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$.

To show that the additional assertion on “capacities” holds in (2), let μ be a monomorphism from $F(a, b)$ to $F(a, b, c)$ defined by $\mu(a) = c^{-1}ac$ and $\mu(b) = b$. Then $\mu\phi$ is a monomorphism from $F(a, b)$ to $F(a, b, c)$. Since H and K are volume equivalent in $F(a, b)$, $\mu\phi(H)$ and $\mu\phi(K)$ are also volume equivalent in $F(a, b, c)$. Hence $E(\Gamma_{\mu\phi(H)}) = E(\Gamma_{\mu\phi(K)})$, and thus

$$E(\Gamma_{\mu\phi(H)}) - E(\Gamma_{\phi(H)}) = E(\Gamma_{\mu\phi(K)}) - E(\Gamma_{\phi(K)}).$$

It then follows from $E(\Gamma_{\mu\phi(H)}) - E(\Gamma_{\phi(H)}) = cap(\Gamma_{\phi(H)}; \{a^{\pm 1}\})$ and $E(\Gamma_{\mu\phi(K)}) - E(\Gamma_{\phi(K)}) = cap(\Gamma_{\phi(K)}; \{a^{\pm 1}\})$ that

$$cap(\Gamma_{\phi(H)}; \{a^{\pm 1}\}) = cap(\Gamma_{\phi(K)}; \{a^{\pm 1}\}),$$

as required.

Next, we prove the “if” part of (1) and (2). Suppose that F_n ($n \geq 2$) is acting freely, discretely and isometrically on an \mathbb{R} -tree X . Let $Y = X/F_n$ be the quotient graph of X . Since X is a metric tree, the edges e of Y come equipped with the lengths $l(e)$. Thus every edge-path in Y has a length which is the sum of the lengths of the edges of this path. There is an obvious canonical identification between F_n and $\pi_1(Y, v)$, where v is a vertex of Y . Choose an orientation $EY = E^+Y \sqcup E^-Y$ on Y . Then for every maximal tree in Y there is a canonically associated free basis of $\pi_1(Y, v)$.

On the other hand, let X_H be the unique minimal H -invariant subtree. Since $vol(X_H/H)$ is the sum of lengths of the edges of the metric graph X_H/H , we have

$$vol(X_H/H) = \sum_{e \in E^+Y} n(X_H/H; e)l(e),$$

where $n(X_H/H; e)$ is the number of times that e occurs in X_H/H .

We consider two cases separately. First, suppose that $e \in E^+Y$ does not separate Y . Choose a maximal tree not containing e , and consider the associated free basis Δ of $F_n = \pi_1(Y, v)$. Let ϕ be an automorphism of F_n which sends Δ to Σ . Then $n(X_H/H; e) = n(\Gamma_{\phi(H)}; \phi(a_e))$ where $a_e \in \Delta$ is the generator corresponding to e . Hence the assertion follows from Lemma 2.1.

Suppose next that e separates Y , so that $Y = Y_1 \cup e \cup Y_2$. Choose a basis Λ of F_n associated to any maximal tree in Y . This basis is partitioned as $\Lambda = \Lambda_1 \sqcup \Lambda_2$, with $b \in \Lambda_i$ if and only if it corresponds to an edge in Y_i . Let ψ be an automorphism of F_n which sends Λ to Σ . Then $n(X_H/H; e) = cap(\Gamma_{\psi(H)}; \psi(\Lambda_1^{\pm 1}))$. Thus the assertion follows from Lemma 2.2 and the hypothesis of the lemma. \square

As stated in Theorem A, our combinatorial characterization of volume equivalence in F_2 requires one additional condition on “capacities” compared to the case of free groups of higher rank. We justify that this extra condition cannot be dropped by proving the following:

Theorem B. *Let*

$$H = \langle [a, b] \rangle \quad \text{and} \quad K = \langle a, b^2, bab^{-1} \rangle$$

be subgroups of $F(a, b)$. Then we have $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ for every automorphism ϕ of $F(a, b)$, but H and K are not volume equivalent in $F(a, b)$.

Proof. Let ϕ be an arbitrary automorphism of $F_2 = F(a, b)$. Then $\phi([a, b])$ is conjugate to $[a, b]^{\pm 1}$ and hence $E(\Gamma_{\phi(H)}) = 4$. Note that K has index 2 in F_2 . So the subgroup $\phi(K)$ also has index 2 in F_2 . This implies that $\Gamma_{\phi(K)}$ is a 2-fold cover of the wedge of two circles labeled by a, b and hence $E(\Gamma_{\phi(K)}) = 4$.

However, we show that H and K are not volume equivalent in F_2 . One can see this either using Theorem A (indeed, we can check $cap(\Gamma_H; \{a^{\pm 1}\}) = 4 \neq 2 = cap(\Gamma_K; \{a^{\pm 1}\})$) or directly using the definition of volume equivalence. Namely, let Γ be a graph consisting of two disjoint loop-edges x and y with a non-loop edge z joining the base point $*$ of x with a base point of y . Then $\pi_1(\Gamma, *)$ is isomorphic to F_2 via $a = x, b = zyz^{-1}$. The universal cover of Γ is a tree T where we give every edge length 1. Then $T_K/K = T/K$ is the double-cover of Γ and thus has volume 6. The volume of T_H/H is equal to the translation length $\inf_{t \in T} d(t, [a, b]t)$ which we can compute by finding a cyclically reduced loop in Γ representing $[a, b]$. Thus we have $[a, b] = aba^{-1}b^{-1} = xzyz^{-1}x^{-1}zy^{-1}z^{-1}$. Hence $vol(T_H/H) = \inf_{t \in T} d(t, [a, b]t) = 8 \neq 6$. Thus H and K are not volume equivalent in F_2 . \square

3. The effect of automorphisms on the subgroup graphs of F_2

This section is the technical heart of the present paper. Throughout this section, let $F_2 = F(a, b)$, the free group with basis $\{a, b\}$. For two automorphisms ϕ and ψ of F_2 , by writing $\phi \equiv \psi$ we mean the equality of ϕ and ψ over all cyclic words in F_2 , that is, $\phi(w) = \psi(w)$ as cyclic words for every cyclic word w in F_2 . From now on, we let

$$\sigma = (\{a\}, b), \quad \tau = (\{b\}, a), \quad \bar{\sigma} = (\{a\}, b^{-1}), \quad \bar{\tau} = (\{b^{-1}\}, a)$$

be Whitehead automorphisms of F_2 of type (W2). As proved in [5], every automorphism of F_2 can be represented in one of two particular types over all cyclic words of F_2 as follows:

Lemma 3.1. (See Lemma 2.3 of [5].) Every automorphism ϕ of F_2 can be represented as $\phi \equiv \beta\phi'$, where β is a Whitehead automorphism of F_2 of type (W1) and ϕ' is a chain of one of the forms

$$(C1) \quad \phi' \equiv \tau^{m_k} \sigma^{l_k} \dots \tau^{m_1} \sigma^{l_1},$$

$$(C2) \quad \phi' \equiv \bar{\tau}^{m_k} \bar{\sigma}^{l_k} \dots \bar{\tau}^{m_1} \bar{\sigma}^{l_1}$$

with $k \in \mathbb{N}$ and both $l_i, m_i \geq 0$ for every $i = 1, \dots, k$.

Lemma 3.1 provides us with an advantage of dealing with automorphisms of F_2 . Indeed, in order to see how graphs of subgroups of F_2 are transformed by applying automorphisms of F_2 , it suffices to consider how these graphs are affected by only 4 Whitehead automorphisms $\sigma, \tau, \bar{\sigma}$ and $\bar{\tau}$.

Now we introduce the notation that is used throughout the remainder of this paper. Let Γ be a finite cyclically reduced $\{a, b\}$ -labeled directed graph. It is clear that there are 11 possible types of vertices of Γ as shown in Fig. 2. We put

- $A(\Gamma)$ = the number of vertices in Γ of the form II-1, II-2, II-5 and III-3;
- $B(\Gamma)$ = the number of vertices in Γ of the form II-4, II-5, II-6 and III-4;
- $C(\Gamma)$ = the number of vertices in Γ of the form II-5;
- $D(\Gamma)$ = the number of vertices in Γ of the form II-3. (7)

Also we put

$$\bar{A}(\Gamma) = \text{the number of vertices in } \Gamma \text{ of the form II-1, II-3, II-6 and III-1;}$$

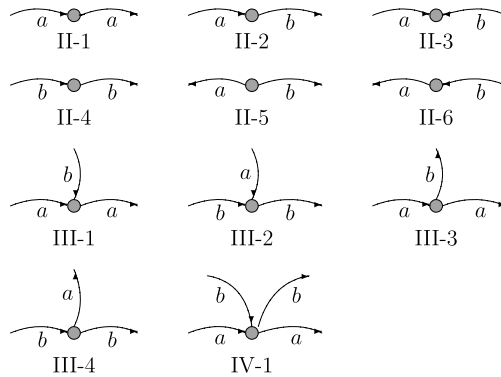


Fig. 2. 11 possible types of vertices of Γ .

$\bar{B}(\Gamma)$ = the number of vertices in Γ of the form II-4, II-5, II-6 and III-4;

$\bar{C}(\Gamma)$ = the number of vertices in Γ of the form II-6;

$\bar{D}(\Gamma)$ = the number of vertices in Γ of the form II-2. (8)

For the remainder of this section, let M be a finitely generated subgroup of F_2 . The following Proposition 3.2–Remark 3.7, that are repeatedly used in Section 4, discuss the effect of Whitehead automorphisms σ , τ , $\bar{\sigma}$ and $\bar{\tau}$ on the subgroup graph Γ_M .

Proposition 3.2. *Under the foregoing notation, we have*

$$E(\Gamma_{\sigma(M)}) - E(\Gamma_M) = A(\Gamma_M) - C(\Gamma_M) - D(\Gamma_M);$$

$$E(\Gamma_{\tau(M)}) - E(\Gamma_M) = B(\Gamma_M) - C(\Gamma_M) - D(\Gamma_M).$$

Proof. Due to a recent result of [8], we have

$$V(\Gamma_{\sigma(M)}) - V(\Gamma_M) = \text{cap}(\Gamma_M; \{a, b\}) - n(\Gamma_M; b);$$

$$V(\Gamma_{\tau(M)}) - V(\Gamma_M) = \text{cap}(\Gamma_M; \{a, b\}) - n(\Gamma_M; a).$$

Since σ and τ are automorphisms of F_2 , $\sigma(M)$ and $\tau(M)$ have the same rank as M does. This implies that the graphs $\Gamma_{\sigma(M)}$ and $\Gamma_{\tau(M)}$ have the same number of faces as Γ_M does. So the vertex difference is the same as the edge difference, and thus we have

$$E(\Gamma_{\sigma(M)}) - E(\Gamma_M) = \text{cap}(\Gamma_M; \{a, b\}) - n(\Gamma_M; b);$$

$$E(\Gamma_{\tau(M)}) - E(\Gamma_M) = \text{cap}(\Gamma_M; \{a, b\}) - n(\Gamma_M; a).$$

Expressing the right-hand sides of these equalities using the notation in (7) and (8), we get

$$E(\Gamma_{\sigma(M)}) - E(\Gamma_M) = A(\Gamma_M) - C(\Gamma_M) - D(\Gamma_M);$$

$$E(\Gamma_{\tau(M)}) - E(\Gamma_M) = B(\Gamma_M) - C(\Gamma_M) - D(\Gamma_M),$$

as required. \square

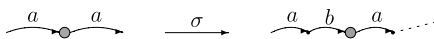


Fig. 3. Claim 1.

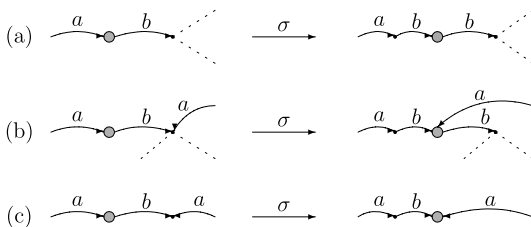


Fig. 4. Claim 2.

Remark 3.3. The same statement as in Proposition 3.2 holds if we replace σ , τ and A, B, C, D by $\bar{\sigma}$, $\bar{\tau}$ and $\bar{A}, \bar{B}, \bar{C}, \bar{D}$, respectively.

Proposition 3.4. Under the foregoing notation, we have

$$A(\Gamma_{\sigma(M)}) = A(\Gamma_M);$$

$$B(\Gamma_{\sigma(M)}) = A(\Gamma_M) + B(\Gamma_M) - C(\Gamma_M) - D(\Gamma_M).$$

Proof. Let μ be the labeling function from the edges of Γ_M to $\{a, b\}^{\pm 1}$. In the following 11 claims, we observe how each vertex in Γ_M is transformed in the process of obtaining $\Gamma_{\sigma(M)}$. In the proof of each claim, we assume that every reduced path p_i consists of at least two edges.

Claim 1. A vertex of the form II-1 in Γ_M is transformed to a vertex of the form II-6 in $\Gamma_{\sigma(M)}$, in which process a vertex of the form II-2 newly occurs in $\Gamma_{\sigma(M)}$.

Proof. Let Q be a vertex of the form II-1 in Γ_M . Also let p_1 and p_2 be reduced paths in Γ_M with terminus Q and labels ending in a and a^{-1} , respectively. Then $\sigma(\mu(p_1))$ and $\sigma(\mu(p_2))$ end in ab and a^{-1} , respectively. This implies that the vertex Q is transformed to a vertex of the form II-6 by σ and in this process a new vertex of the form II-2 occurs as depicted in Fig. 3. \square

Claim 2. A vertex of the form II-2 in Γ_M is transformed to

- either (i) a vertex of the form II-4 or III-2 in $\Gamma_{\sigma(M)}$, in which process a vertex of the form II-2 newly occurs in $\Gamma_{\sigma(M)}$;
- or (ii) a vertex of the form II-3 in $\Gamma_{\sigma(M)}$, in which process a vertex of the form II-2 newly occurs in $\Gamma_{\sigma(M)}$ and a vertex of the form II-3 in Γ_M disappears.

Proof. Let Q be a vertex of the form II-2 in Γ_M . Also let p_1 and p_2 be reduced paths in Γ_M with terminus Q and labels ending in a and b^{-1} , respectively. Then $\sigma(\mu(p_1))$ ends in ab , whereas $\sigma(\mu(p_2))$ ends in a (this happens when $\mu(p_2)$ ends in ab^{-1}) or b^{-1} .

There are three possibilities. First, for every p_2 , $\sigma(\mu(p_2))$ ends in b^{-1} , in which case the vertex Q is transformed to a vertex of the form II-4 as depicted in Fig. 4(a). Next, for some p_2 , $\sigma(\mu(p_2))$ ends in b^{-1} , and for another p_2 , $\sigma(\mu(p_2))$ ends in a , in which case the vertex Q is transformed to a vertex of the form III-2 as depicted in Fig. 4(b). Finally, for every p_2 , $\sigma(\mu(p_2))$ ends in a , in which case the vertex Q is transformed to a vertex of the form II-3 and a vertex of the form II-3 in Γ_M disappears as depicted in Fig. 4(c). In any of three cases above, a new vertex of the form II-2 occurs in $\Gamma_{\sigma(M)}$, because $\sigma(\mu(p_1))$ ends in ab . \square

Claim 3. A vertex of the form II-3 in Γ_M disappears, in which process no other vertex newly occurs or disappears.

Proof. Let Q be a vertex of the form II-2 in Γ_M . Also let p_1 and p_2 be reduced paths in Γ_M with terminus Q and labels ending in a and b , respectively. Then $\sigma(\mu(p_1))$ ends in ab and $\sigma(\mu(p_2))$ ends in b . This implies that Q is transformed to a vertex of degree 1 and then it is removed by the construction of the reduced $\{a, b\}$ -labeled graph $\Gamma_{\sigma(M)}$. \square

Claim 4. A vertex of the form II-4 in Γ_M is transformed to

either (i) a vertex of the form II-4 or III-2 in $\Gamma_{\sigma(M)}$, in which process no other vertex newly occurs or disappears;

or (ii) a vertex of the form II-3 in $\Gamma_{\sigma(M)}$, in which process a vertex of the form II-3 in Γ_M disappears.

Proof. Let Q be a vertex of the form II-4 in Γ_M . Also let p_1 and p_2 be reduced paths in Γ_M with terminus Q and labels ending in b and b^{-1} , respectively. Then $\sigma(\mu(p_1))$ ends in b , whereas $\sigma(\mu(p_2))$ ends in a (this happens when p_2 has label ending in ab^{-1}) or b^{-1} .

There are three possibilities. First, for every p_2 , $\sigma(\mu(p_2))$ ends in b^{-1} , in which case the vertex Q is transformed to a vertex of the form II-4. Next, for some p_2 , $\sigma(\mu(p_2))$ ends in b^{-1} , and for another p_2 , $\sigma(\mu(p_2))$ ends in a , in which case the vertex Q is transformed to a vertex of the form III-2. Finally, for every p_2 , $\sigma(\mu(p_2))$ ends in a , in which case the vertex Q is transformed to a vertex of the form II-3 and a vertex of the form II-3 in Γ_M disappears. \square

Claim 5. A vertex of the form II-5 in Γ_M is transformed to

either (i) a vertex of the form II-5 or III-3 in $\Gamma_{\sigma(M)}$, in which process no other vertex newly occurs or disappears;

or (ii) a vertex of the form II-1 in $\Gamma_{\sigma(M)}$, in which process a vertex of the form II-3 in Γ_M disappears.

Proof. Let Q be a vertex of the form II-5 in Γ_M . Also let p_1 and p_2 be reduced paths in Γ_M with terminus Q and labels ending in a^{-1} and b^{-1} , respectively. Then $\sigma(\mu(p_1))$ ends in a^{-1} , whereas $\sigma(\mu(p_2))$ ends in a (this happens when $\mu(p_2)$ ends in ab^{-1}) or b^{-1} .

There are three possibilities. First, for every p_2 , $\sigma(\mu(p_2))$ ends in b^{-1} , in which case the vertex Q is transformed to a vertex of the form II-5. Next, for some p_2 , $\sigma(\mu(p_2))$ ends in b^{-1} , and for another p_2 , $\sigma(\mu(p_2))$ ends in a , in which case the vertex Q is transformed to a vertex of the form III-3. Finally, for every p_2 , $\sigma(\mu(p_2))$ ends in a , in which case the vertex Q is transformed to a vertex of the form II-1 and a vertex of the form II-3 in Γ_M disappears. \square

Claim 6. A vertex of the form II-6 in Γ_M is transformed to a vertex of the form II-6 in $\Gamma_{\sigma(M)}$, in which process no other vertex newly occurs or disappears.

Proof. Let Q be a vertex of the form II-6 in Γ_M . Also let p_1 and p_2 be reduced paths in Γ_M with terminus Q and labels ending in a^{-1} and b , respectively. Then $\sigma(\mu(p_1))$ ends in a^{-1} and $\sigma(\mu(p_2))$ ends in b . Hence Q is transformed to a vertex of the form II-6. \square

Claim 7. A vertex of the form III-1 in Γ_M is transformed to a vertex of the form II-6 in $\Gamma_{\sigma(M)}$, in which process no other vertex newly occurs or disappears.

Proof. Let Q be a vertex of the form III-1 in Γ_M . Also let p_1 , p_2 and p_3 be reduced paths in Γ_M with terminus Q and labels ending in a , a^{-1} and b , respectively. Then $\sigma(\mu(p_1))$ ends in ab , $\sigma(\mu(p_2))$ ends in a^{-1} , and $\sigma(\mu(p_3))$ ends in b . This implies that Q is transformed to a vertex of the form II-6. Also we can see that there does not occur any new vertex in this process by the construction of the reduced $\{a, b\}$ -labeled graph $\Gamma_{\sigma(M)}$. \square

Claim 8. A vertex of the form III-2 in Γ_M is transformed to

- either (i) a vertex of the form II-4 or III-2 in $\Gamma_{\sigma(M)}$, in which process no other vertex newly occurs or disappears;
or (ii) a vertex of the form II-3 in $\Gamma_{\sigma(M)}$, in which process a vertex of the form II-3 in Γ_M disappears.

Proof. Let Q be a vertex of the form III-2 in Γ_M . Also let p_1, p_2 and p_3 be reduced paths in Γ_M with terminus Q and labels ending in a, b and b^{-1} , respectively. Then $\sigma(\mu(p_1))$ ends in ab , and $\sigma(\mu(p_2))$ ends in b , whereas $\sigma(\mu(p_3))$ ends in a (this happens when p_3 has label ending in ab^{-1}) or b^{-1} .

There are three possibilities. First, for every $p_3, \sigma(\mu(p_3))$ ends in b^{-1} , in which case the vertex Q is transformed to a vertex of the form II-4. Next, for some $p_3, \sigma(\mu(p_3))$ ends in b^{-1} , and for another $p_3, \sigma(\mu(p_3))$ ends in a , in which case the vertex Q is transformed to a vertex of the form III-2. Finally, for every $p_3, \sigma(\mu(p_3))$ ends in a , in which case the vertex Q is transformed to a vertex of the form II-3 and a vertex of the form II-3 in Γ_M disappears. \square

Claim 9. A vertex of the form III-3 in Γ_M is transformed to

- either (i) a vertex of the form III-4 or IV-1 in $\Gamma_{\sigma(M)}$, in which process a vertex of the form II-2 newly occurs in $\Gamma_{\sigma(M)}$;
or (ii) a vertex of the form III-1 in $\Gamma_{\sigma(M)}$, in which process a vertex of the form II-2 newly occurs in $\Gamma_{\sigma(M)}$ and a vertex of the form II-3 in Γ_M disappears.

Proof. Let Q be a vertex of the form III-3 in Γ_M . Also let p_1, p_2 and p_3 be reduced paths in Γ_M with terminus Q and labels ending in a, a^{-1} and b^{-1} , respectively. Then $\sigma(\mu(p_1))$ ends in ab , and $\sigma(\mu(p_2))$ ends in a^{-1} , whereas $\sigma(\mu(p_3))$ ends in a (this happens when p_3 has label ending in ab^{-1}) or b^{-1} .

There are three possibilities. First, for every $p_3, \sigma(\mu(p_3))$ ends in b^{-1} , in which case the vertex Q is transformed to a vertex of the form III-4. Next, for some $p_3, \sigma(\mu(p_3))$ ends in b^{-1} , and for another $p_3, \sigma(\mu(p_3))$ ends in a , in which case the vertex Q is transformed to a vertex of the form IV-1. Finally, for every $p_3, \sigma(\mu(p_3))$ ends in a , in which case the vertex Q is transformed to a vertex of the form III-1 and a vertex of the form II-3 in Γ_M disappears. In any of three cases above, a new vertex of the form II-2 occurs in $\Gamma_{\sigma(M)}$, because $\sigma(\mu(p_1))$ ends in ab . \square

Claim 10. A vertex of the form III-4 in Γ_M is transformed to

- either (i) a vertex of the form III-4 or IV-1 in $\Gamma_{\sigma(M)}$, in which process no other vertex newly occurs or disappears;
or (ii) a vertex of the form III-1 in $\Gamma_{\sigma(M)}$, in which process a vertex of the form II-3 in Γ_M disappears.

Proof. Let Q be a vertex of the form III-4 in Γ_M . Also let p_1, p_2 and p_3 be reduced paths in Γ_M with terminus Q and labels ending in a^{-1}, b and b^{-1} , respectively. Then $\sigma(\mu(p_1))$ ends in a^{-1} , and $\sigma(\mu(p_2))$ ends in b , whereas $\sigma(\mu(p_3))$ ends in a (this happens when p_3 has label ending in ab^{-1}) or b^{-1} .

There are three possibilities. First, for every $p_3, \sigma(\mu(p_3))$ ends in b^{-1} , in which case the vertex Q is transformed to a vertex of the form III-4. Next, for some $p_3, \sigma(\mu(p_3))$ ends in b^{-1} , and for another $p_3, \sigma(\mu(p_3))$ ends in a , in which case the vertex Q is transformed to a vertex of the form IV-1. Finally, for every $p_3, \sigma(\mu(p_3))$ ends in a , in which case the vertex Q is transformed to a vertex of the form III-1 and a vertex of the form II-3 in Γ_M disappears. \square

Claim 11. A vertex of the form IV-1 in Γ_M is transformed to

- either (i) a vertex of the form III-4 or IV-1 in $\Gamma_{\sigma(M)}$, in which process no other vertex newly occurs or disappears;
or (ii) a vertex of the form III-1 in $\Gamma_{\sigma(M)}$, in which process a vertex of the form II-3 in Γ_M disappears.

Table 1

Vertex transformations used in the process $\Gamma_M \rightarrow \Gamma_{\sigma(M)}$.

Vertex transformation	New occurrence	Disappearance
S1: II-1 \rightarrow II-6	II-2	
S2: II-2 \rightarrow II-4	II-2	
S3: II-2 \rightarrow III-2	II-2	
S4: II-2 \rightarrow II-3	II-2	II-3
S5: II-3 disappears		
S6: II-4 \rightarrow II-4		
S7: II-4 \rightarrow III-2		
S8: II-4 \rightarrow II-3		II-3
S9: II-5 \rightarrow II-5		
S10: II-5 \rightarrow III-3		
S11: II-5 \rightarrow II-1		II-3
S12: II-6 \rightarrow II-6		
S13: III-1 \rightarrow II-6		
S14: III-2 \rightarrow II-4		
S15: III-2 \rightarrow III-2		
S16: III-2 \rightarrow II-3		II-3
S17: III-3 \rightarrow III-4	II-2	
S18: III-3 \rightarrow IV-1	II-2	
S19: III-3 \rightarrow III-1	II-2	II-3
S20: III-4 \rightarrow III-4		
S21: III-4 \rightarrow IV-1		
S22: III-4 \rightarrow III-1		II-3
S23: IV-1 \rightarrow III-4		
S24: IV-1 \rightarrow IV-1		
S25: IV-1 \rightarrow III-1		II-3

Proof. Let Q be a vertex of the form III-4 in Γ_M . Also let p_1, p_2, p_3 and p_4 be reduced paths in Γ_M with terminus Q and labels ending in a, a^{-1}, b and b^{-1} , respectively. Then $\sigma(\mu(p_1))$ ends in ab , $\sigma(\mu(p_2))$ ends in a^{-1} , and $\sigma(\mu(p_3))$ ends in b , whereas $\sigma(\mu(p_4))$ ends in a (this happens when p_4 has label ending in ab^{-1}) or b^{-1} .

There are three possibilities. First, for every $p_4, \sigma(\mu(p_4))$ ends in b^{-1} , in which case the vertex Q is transformed to a vertex of the form III-4. Next, for some $p_4, \sigma(\mu(p_4))$ ends in b^{-1} , and for another $p_4, \sigma(\mu(p_4))$ ends in a , in which case the vertex Q is transformed to a vertex of the form IV-1. Finally, for every $p_4, \sigma(\mu(p_4))$ ends in a , in which case the vertex Q is transformed to a vertex of the form III-1 and a vertex of the form II-3 in Γ_M disappears. \square

Claims 1–11 are summarized in Table 1.

Note that $A(\Gamma_M)$ is the total number of vertices of the form II-1, II-2, II-5 or III-3 in Γ_M . Table 1 shows that every vertex of the form II-1, II-2 or III-3 in Γ_M is transformed to a vertex not belonging to $A(\Gamma_{\sigma(M)})$, in the process of which a vertex of the form II-2 newly occurs in $\Gamma_{\sigma(M)}$. Also every vertex of the form II-5 in Γ_M is transformed to a vertex belonging to $A(\Gamma_{\sigma(M)})$, in the process of which no vertex newly occurs in $\Gamma_{\sigma(M)}$. Moreover since every vertex belonging to $\Gamma_{\sigma(M)}$ is obtained from a vertex belonging to $A(\Gamma_M)$ in the process of applying σ , we have

$$A(\Gamma_{\sigma(M)}) = A(\Gamma_M).$$

On the other hand, note that $B(\Gamma_M)$ is the total number of vertices of the form II-4, II-5, II-6 or III-4 in Γ_M . By $[Sj] (1 \leq j \leq 25)$, we mean the number of vertices in Γ_M which are transformed to vertices in $\Gamma_{\sigma(M)}$ by the vertex transformation of type $[Sj]$. From Table 1, we see that

$$D(\Gamma_M) = [S4] + [S8] + [S11] + [S16] + [S19] + [S22],$$

and that

$$\begin{aligned} A(\Gamma_M) + B(\Gamma_M) - C(\Gamma_M) &= [S1] + [S2] + [S3] + [S4] + [S6] + [S7] \\ &\quad + [S8] + [S9] + [S10] + [S11] + [S12] + [S17] \\ &\quad + [S18] + [S19] + [S20] + [S21] + [S22]. \end{aligned}$$

It then follows that

$$\begin{aligned} A(\Gamma_M) + B(\Gamma_M) - C(\Gamma_M) - D(\Gamma_M) &= [S1] + [S2] + [S3] + [S6] + [S7] \\ &\quad + [S9] + [S10] + [S12] - [S16] \\ &\quad + [S17] + [S18] + [S20] + [S21]. \end{aligned} \quad (9)$$

Since every cyclically reduced finite $\{a, b\}$ -labeled graph Γ has the same number of faces as $1/2(\text{the number of vertices in } \Gamma \text{ of the form III-1, III-2, III-3 or III-4}) + (\text{the number of vertices in } \Gamma \text{ of the form IV-1}) + 1$, and since Γ_M has the same number of faces as $\Gamma_{\sigma(M)}$ does, we have

$$[S3] + [S7] + [S10] + [S18] + [S21] = [S13] + [S14] + [S16] + [S23].$$

Using this equality rewrites the right-hand side of (9) as

$$\begin{aligned} A(\Gamma_M) + B(\Gamma_M) - C(\Gamma_M) - D(\Gamma_M) &= [S1] + [S2] + [S6] + [S9] \\ &\quad + [S12] + [S13] + [S14] \\ &\quad + [S17] + [S20] + [S23]. \end{aligned} \quad (10)$$

According to Table 1, the right-hand side of (10) equals $B(\Gamma_{\sigma(M)})$, which completes the proof. \square

Remark 3.5. The same statement as in Proposition 3.4 holds if we replace σ and A, B, C, D by $\bar{\sigma}$ and $\bar{A}, \bar{B}, \bar{C}, \bar{D}$, respectively.

Proposition 3.6. Under the foregoing notation, we have

$$\begin{aligned} A(\Gamma_{\tau(M)}) &= A(\Gamma_M) + B(\Gamma_M) - C(\Gamma_M) - D(\Gamma_M); \\ B(\Gamma_{\tau(M)}) &= B(\Gamma_M). \end{aligned}$$

Proof. As in the proof of Proposition 3.4, we can analyze how each vertex in Γ_M is transformed in the process of obtaining $\Gamma_{\tau(M)}$. We summarize our analysis in Table 2.

We may apply an argument similar to the proof of Proposition 3.4 to get the required result, which completes the proof of the proposition. \square

Remark 3.7. The same statement as in Proposition 3.6 holds if we replace τ and A, B, C, D by $\bar{\tau}$ and $\bar{A}, \bar{B}, \bar{C}, \bar{D}$, respectively.

Table 2
Vertex transformations used in the process $\Gamma_M \rightarrow \Gamma_{\tau(M)}$.

Vertex transformation	New occurrence	Disappearance
T1: II-1 \rightarrow II-1		
T2: II-1 \rightarrow III-1		
T3: II-1 \rightarrow II-3		II-3
T4: II-2 \rightarrow II-2		
T5: II-3 disappears		
T6: II-4 \rightarrow II-2	II-6	
T7: II-5 \rightarrow II-5		
T8: II-5 \rightarrow III-4		
T9: II-5 \rightarrow II-4		II-3
T10: II-6 \rightarrow II-1	II-6	
T11: II-6 \rightarrow III-1	II-6	
T12: II-6 \rightarrow II-3	II-6	II-3
T13: III-1 \rightarrow II-1		
T14: III-1 \rightarrow III-1		
T15: III-1 \rightarrow II-3		II-3
T16: III-2 \rightarrow II-2		
T17: III-3 \rightarrow III-3		
T18: III-3 \rightarrow IV-1		
T19: III-3 \rightarrow III-2		II-3
T20: III-4 \rightarrow III-3	II-6	
T21: III-4 \rightarrow IV-1	II-6	
T22: III-4 \rightarrow III-2	II-6	II-3
T23: IV-1 \rightarrow III-3		
T24: IV-1 \rightarrow IV-1		
T25: IV-1 \rightarrow III-2		II-3

4. An example of volume equivalence without rank equivalence

Throughout this section, let $F_2 = F(a, b)$, the free group with basis $\{a, b\}$. We start with fixing the notation that is used throughout this section. Let Γ be a finite cyclically reduced $\{a, b\}$ -labeled directed graph. In the beginning of Section 3, 11 possible types of vertices of Γ are depicted in Fig. 2. Then we put

$$P_i(\Gamma) = \text{the number of vertices in } \Gamma \text{ of the form II-}i \text{ for each } 1 \leq i \leq 6;$$

$$Q_j(\Gamma) = \text{the number of vertices in } \Gamma \text{ of the form III-}j \text{ for each } 1 \leq j \leq 4;$$

$$R(\Gamma) = \text{the number of vertices in } \Gamma \text{ of the form IV-1.}$$

The following lemma is needed to prove Theorem C from the Introduction.

Lemma 4.1. *For finitely generated subgroups H and K of F_2 , if $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ for every automorphism ϕ of F_2 , then*

$$cap(\Gamma_{\phi(H)}; \{a^{\pm 1}\}) + R(\Gamma_{\phi(H)}) = cap(\Gamma_{\phi(K)}; \{a^{\pm 1}\}) + R(\Gamma_{\phi(K)})$$

for every automorphism ϕ of F_2 .

Proof. Let ϕ be an automorphism of F_2 , and put

$$H_1 = \phi(H) \quad \text{and} \quad K_1 = \phi(K).$$

By the hypothesis of the theorem, we have

$$\begin{aligned} E(\Gamma_{\sigma(H_1)}) - E(\Gamma_{H_1}) &= E(\Gamma_{\sigma(K_1)}) - E(\Gamma_{K_1}); \\ E(\Gamma_{\bar{\sigma}(H_1)}) - E(\Gamma_{H_1}) &= E(\Gamma_{\bar{\sigma}(K_1)}) - E(\Gamma_{K_1}); \\ E(\Gamma_{\tau(H_1)}) - E(\Gamma_{H_1}) &= E(\Gamma_{\tau(K_1)}) - E(\Gamma_{K_1}); \\ E(\Gamma_{\tau^{-1}(H_1)}) - E(\Gamma_{H_1}) &= E(\Gamma_{\tau^{-1}(K_1)}) - E(\Gamma_{K_1}). \end{aligned}$$

In view of Proposition 3.2, the above equalities yield that

$$\begin{aligned} P_1(\Gamma_{H_1}) + P_2(\Gamma_{H_1}) - P_3(\Gamma_{H_1}) + Q_3(\Gamma_{H_1}) &= P_1(\Gamma_{K_1}) + P_2(\Gamma_{K_1}) - P_3(\Gamma_{K_1}) + Q_3(\Gamma_{K_1}); \\ P_1(\Gamma_{H_1}) - P_2(\Gamma_{H_1}) + P_3(\Gamma_{H_1}) + Q_1(\Gamma_{H_1}) &= P_1(\Gamma_{K_1}) - P_2(\Gamma_{K_1}) + P_3(\Gamma_{K_1}) + Q_1(\Gamma_{K_1}); \\ P_4(\Gamma_{H_1}) - P_3(\Gamma_{H_1}) + P_6(\Gamma_{H_1}) + Q_4(\Gamma_{H_1}) &= P_4(\Gamma_{K_1}) - P_3(\Gamma_{K_1}) + P_6(\Gamma_{K_1}) + Q_4(\Gamma_{K_1}); \\ P_4(\Gamma_{H_1}) + P_3(\Gamma_{H_1}) - P_6(\Gamma_{H_1}) + Q_2(\Gamma_{H_1}) &= P_4(\Gamma_{K_1}) + P_3(\Gamma_{K_1}) - P_6(\Gamma_{K_1}) + Q_2(\Gamma_{K_1}). \end{aligned}$$

Adding altogether these equalities, we get

$$2\{P_1(\Gamma_{H_1}) + P_4(\Gamma_{H_1})\} + \sum_{j=1}^4 Q_j(\Gamma_{H_1}) = 2\{P_1(\Gamma_{K_1}) + P_4(\Gamma_{K_1})\} + \sum_{j=1}^4 Q_j(\Gamma_{K_1}),$$

so that

$$P_1(\Gamma_{H_1}) + P_4(\Gamma_{H_1}) + \frac{1}{2} \sum_{j=1}^4 Q_j(\Gamma_{H_1}) = P_1(\Gamma_{K_1}) + P_4(\Gamma_{K_1}) + \frac{1}{2} \sum_{j=1}^4 Q_j(\Gamma_{K_1}). \tag{11}$$

Since $\text{rank } H_1 = 1/2 \sum_{j=1}^4 Q_j(\Gamma_{H_1}) + R(\Gamma_{H_1})$ and $\text{rank } K_1 = 1/2 \sum_{j=1}^4 Q_j(\Gamma_{K_1}) + R(\Gamma_{K_1})$, we obtain from (11) that

$$P_1(\Gamma_{H_1}) + P_4(\Gamma_{H_1}) + \text{rank } H_1 - R(\Gamma_{H_1}) = P_1(\Gamma_{K_1}) + P_4(\Gamma_{K_1}) + \text{rank } K_1 - R(\Gamma_{K_1}). \tag{12}$$

Also, since $E(\Gamma_{H_1}) = E(\Gamma_{K_1})$, we have

$$\sum_{i=1}^6 P_i(\Gamma_{H_1}) + \frac{3}{2} \sum_{j=1}^4 Q_j(\Gamma_{H_1}) + 2R(\Gamma_{H_1}) = \sum_{i=1}^6 P_i(\Gamma_{K_1}) + \frac{3}{2} \sum_{j=1}^4 Q_j(\Gamma_{K_1}) + 2R(\Gamma_{K_1}),$$

so that

$$\begin{aligned} P_1(\Gamma_{H_1}) + P_4(\Gamma_{H_1}) + \text{cap}(\Gamma_{\phi(H)}; \{a^{\pm 1}\}) + \text{rank } H_1 \\ = P_1(\Gamma_{K_1}) + P_4(\Gamma_{K_1}) + \text{cap}(\Gamma_{\phi(K)}; \{a^{\pm 1}\}) + \text{rank } K_1. \end{aligned} \tag{13}$$

Subtract (12) from (13) to obtain

$$\text{cap}(\Gamma_{H_1}; \{a^{\pm 1}\}) + R(\Gamma_{H_1}) = \text{cap}(\Gamma_{K_1}; \{a^{\pm 1}\}) + R(\Gamma_{K_1}),$$

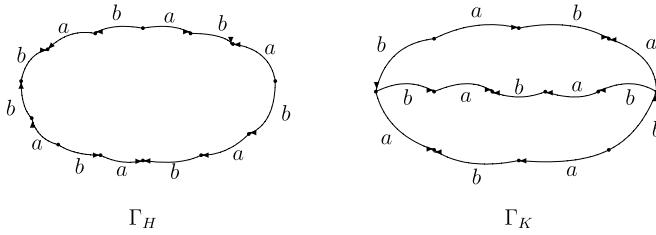


Fig. 5. The subgroup graphs Γ_H and Γ_K .

namely,

$$cap(\Gamma_{\phi(H)}; \{a^{\pm 1}\}) + R(\Gamma_{\phi(H)}) = cap(\Gamma_{\phi(K)}; \{a^{\pm 1}\}) + R(\Gamma_{\phi(K)}),$$

as desired. \square

The following theorem answers, in the negative, the question of whether or not the volume equivalence of two subgroups H and K of F_n necessarily implies that H and K have the same rank (cf. [1]).

Theorem C. *Let*

$$H = \langle [b, a^{-1}]b[a^{-1}]ba^{-1}ba \rangle,$$

$$K = \langle [b, a^{-1}]^2, ba^{-1}[b, a]b^{-2}a \rangle$$

be subgroups of $F(a, b)$. Then H and K are volume equivalent in $F(a, b)$ but $rank(H) \neq rank(K)$.

Proof. The subgroup graphs Γ_H and Γ_K are depicted in Fig. 5.

From Fig. 5, we see that

$$E(\Gamma_H) = E(\Gamma_K), \tag{14}$$

and that

$$R(\Gamma_H) = 0 = R(\Gamma_K).$$

Also, from Fig. 5, it is easy to count

$$\begin{aligned} A(\Gamma_H) &= 6, & B(\Gamma_H) &= 7, & C(\Gamma_H) &= 3, & D(\Gamma_H) &= 3; \\ A(\Gamma_K) &= 5, & B(\Gamma_K) &= 6, & C(\Gamma_K) &= 2, & D(\Gamma_K) &= 3; \\ \bar{A}(\Gamma_H) &= 6, & \bar{B}(\Gamma_H) &= 7, & \bar{C}(\Gamma_H) &= 3, & \bar{D}(\Gamma_H) &= 3; \\ \bar{A}(\Gamma_K) &= 5, & \bar{B}(\Gamma_K) &= 6, & \bar{C}(\Gamma_K) &= 2, & \bar{D}(\Gamma_K) &= 3. \end{aligned} \tag{15}$$

Hence we have

$$A(\Gamma_H) - C(\Gamma_H) - D(\Gamma_H) = A(\Gamma_K) - C(\Gamma_K) - D(\Gamma_K);$$

$$B(\Gamma_H) - C(\Gamma_H) - D(\Gamma_H) = B(\Gamma_K) - C(\Gamma_K) - D(\Gamma_K);$$

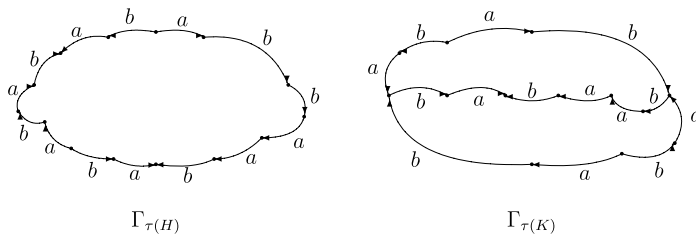


Fig. 6. The subgroup graphs $\Gamma_{\tau(H)}$ and $\Gamma_{\tau(K)}$.

$$\begin{aligned} \bar{A}(\Gamma_H) - \bar{C}(\Gamma_H) - \bar{D}(\Gamma_H) &= \bar{A}(\Gamma_K) - \bar{C}(\Gamma_K) - \bar{D}(\Gamma_K); \\ \bar{B}(\Gamma_H) - \bar{C}(\Gamma_H) - \bar{D}(\Gamma_H) &= \bar{B}(\Gamma_K) - \bar{C}(\Gamma_K) - \bar{D}(\Gamma_K), \end{aligned}$$

which implies that

$$\begin{aligned} E(\Gamma_{\sigma(H)}) - E(\Gamma_H) &= E(\Gamma_{\sigma(K)}) - E(\Gamma_K); \\ E(\Gamma_{\tau(H)}) - E(\Gamma_H) &= E(\Gamma_{\tau(K)}) - E(\Gamma_K); \\ E(\Gamma_{\bar{\sigma}(H)}) - E(\Gamma_H) &= E(\Gamma_{\bar{\sigma}(K)}) - E(\Gamma_K); \\ E(\Gamma_{\bar{\tau}(H)}) - E(\Gamma_H) &= E(\Gamma_{\bar{\tau}(K)}) - E(\Gamma_K). \end{aligned} \tag{16}$$

Let ϕ be a chain of type (C1) or (C2). We define the length of ϕ , denoted by $|\phi|$, to be the number of factors σ and τ in ϕ if ϕ is of type (C1), and the number of factors $\bar{\sigma}$ and $\bar{\tau}$ in ϕ if ϕ is type of (C2). By induction on $|\phi| \geq 1$, we shall prove both $R(\Gamma_{\phi(H)}) = R(\Gamma_{\phi(K)})$ and $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$. Then, in view of Lemma 4.1, the equality $cap(\Gamma_{\phi(H)}; \{a^{\pm 1}\}) = cap(\Gamma_{\phi(K)}; \{a^{\pm 1}\})$ will follow immediately.

Case 1. ϕ is a chain of type (C1).

We may assume that ϕ begins with τ , because the graphs Γ_H and Γ_K remain unchanged by applying σ . The subgroup graphs of $\tau(H)$ and $\tau(K)$ are depicted in Fig. 6.

From Fig. 6, we see that there are no vertices of degree 4 in any of $\Gamma_{\tau(H)}$ and $\Gamma_{\tau(K)}$. Hence, if $|\phi| = 1$, then the equality $R(\Gamma_{\phi(H)}) = R(\Gamma_{\phi(K)})$ is obvious. Also, the equality $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ follows from (14) and (16).

So let $|\phi| > 1$. First, we shall prove that $R(\Gamma_{\phi(H)}) = R(\Gamma_{\phi(K)})$. Obviously $R(\Gamma_{\phi(H)}) = 0$. Let us now consider $R(\Gamma_{\phi(K)})$. From Tables 1 and 2 from Section 3, we see that every vertex of degree 4, if any, in the graph $\Gamma_{\phi(K)}$ can be obtained only from a vertex of type II-5, III-3 or III-4 in the graph $\Gamma_{\tau(K)}$. However, in the graph $\Gamma_{\tau(K)}$, there is no vertex of type III-3 or III-4, and, moreover, every vertex of type II-5 remains unchanged even if we apply further σ and τ . So there cannot occur any vertex of degree 4 in $\Gamma_{\phi(K)}$, which means that $R(\Gamma_{\phi(K)}) = 0$, as desired.

Next, we shall prove that $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$. Observe from Fig. 6 that

$$\begin{aligned} C(\Gamma_{\tau(H)}) &= 2, & D(\Gamma_{\tau(H)}) &= 2; \\ C(\Gamma_{\tau(K)}) &= 2, & D(\Gamma_{\tau(K)}) &= 1, \end{aligned}$$

and, furthermore, that for arbitrary non-negative integers ℓ_i and m_i ($1 \leq i \leq t$),

$$\begin{aligned} C(\Gamma_{\tau^{m_t} \sigma^{\ell_t} \dots \tau^{m_1} \sigma^{\ell_1} \tau(H)}) &= 2, & D(\Gamma_{\tau^{m_t} \sigma^{\ell_t} \dots \tau^{m_1} \sigma^{\ell_1} \tau(H)}) &= 2; \\ C(\Gamma_{\tau^{m_t} \sigma^{\ell_t} \dots \tau^{m_1} \sigma^{\ell_1} \tau(K)}) &= 2, & D(\Gamma_{\tau^{m_t} \sigma^{\ell_t} \dots \tau^{m_1} \sigma^{\ell_1} \tau(K)}) &= 1. \end{aligned} \tag{17}$$

We consider two cases separately.

Case 1.1. ϕ ends in σ .

Again, we divide this case into two subcases.

Case 1.1.a. ϕ ends in σ^2 .

Put

$$\phi = \sigma^2 \phi_1,$$

where ϕ_1 is a chain of type (C1). Also put

$$H_1 = \phi_1(H) \quad \text{and} \quad K_1 = \phi_1(K).$$

It then follows from (17) that

$$\begin{aligned} & C(\Gamma_{\sigma(H_1)}) + D(\Gamma_{\sigma(H_1)}) - C(\Gamma_{H_1}) - D(\Gamma_{H_1}) \\ &= C(\Gamma_{\sigma(K_1)}) + D(\Gamma_{\sigma(K_1)}) - C(\Gamma_{K_1}) - D(\Gamma_{K_1}). \end{aligned} \quad (18)$$

In view of Proposition 3.2, we have

$$\begin{aligned} E(\Gamma_{\sigma(H_1)}) - E(\Gamma_{H_1}) &= A(\Gamma_{H_1}) - C(\Gamma_{H_1}) - D(\Gamma_{H_1}); \\ E(\Gamma_{\sigma(K_1)}) - E(\Gamma_{K_1}) &= A(\Gamma_{K_1}) - C(\Gamma_{K_1}) - D(\Gamma_{K_1}). \end{aligned}$$

Then by the induction hypothesis,

$$A(\Gamma_{H_1}) - C(\Gamma_{H_1}) - D(\Gamma_{H_1}) = A(\Gamma_{K_1}) - C(\Gamma_{K_1}) - D(\Gamma_{K_1}). \quad (19)$$

Again, by Proposition 3.2, we have

$$\begin{aligned} E(\Gamma_{\sigma^2(H_1)}) - E(\Gamma_{\sigma(H_1)}) &= A(\Gamma_{\sigma(H_1)}) - C(\Gamma_{\sigma(H_1)}) - D(\Gamma_{\sigma(H_1)}); \\ E(\Gamma_{\sigma^2(K_1)}) - E(\Gamma_{\sigma(K_1)}) &= A(\Gamma_{\sigma(K_1)}) - C(\Gamma_{\sigma(K_1)}) - D(\Gamma_{\sigma(K_1)}). \end{aligned} \quad (20)$$

Here, in view of Proposition 3.4,

$$A(\Gamma_{\sigma(H_1)}) = A(\Gamma_{H_1}) \quad \text{and} \quad A(\Gamma_{\sigma(K_1)}) = A(\Gamma_{K_1}).$$

These equalities together with (18), (19) and (20) yield that

$$E(\Gamma_{\sigma^2(H_1)}) - E(\Gamma_{\sigma(H_1)}) = E(\Gamma_{\sigma^2(K_1)}) - E(\Gamma_{\sigma(K_1)}).$$

Combine this with the induction hypothesis to finally get

$$E(\Gamma_{\sigma^2(H_1)}) = E(\Gamma_{\sigma^2(K_1)}),$$

that is,

$$E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)}).$$

Case 1.1.b. ϕ ends in $\sigma\tau$.

Put

$$\phi = \sigma\tau\phi_2,$$

where ϕ_1 is a chain of type (C1). Also put

$$H_2 = \phi_2(H) \quad \text{and} \quad K_2 = \phi_2(K).$$

It then follows from (15) and (17) that

$$\begin{aligned} C(\Gamma_{\tau(H_2)}) + D(\Gamma_{\tau(H_2)}) - C(\Gamma_{H_2}) - D(\Gamma_{H_2}) \\ = C(\Gamma_{\tau(K_2)}) + D(\Gamma_{\tau(K_2)}) - C(\Gamma_{K_2}) - D(\Gamma_{K_2}). \end{aligned} \quad (21)$$

Apply Proposition 3.2 to get

$$\begin{aligned} E(\Gamma_{\tau(H_2)}) - E(\Gamma_{H_2}) &= B(\Gamma_{H_2}) - C(\Gamma_{H_2}) - D(\Gamma_{H_2}); \\ E(\Gamma_{\tau(K_2)}) - E(\Gamma_{K_2}) &= B(\Gamma_{K_2}) - C(\Gamma_{K_2}) - D(\Gamma_{K_2}). \end{aligned}$$

Then by the induction hypothesis,

$$B(\Gamma_{H_2}) - C(\Gamma_{H_2}) - D(\Gamma_{H_2}) = B(\Gamma_{K_2}) - C(\Gamma_{K_2}) - D(\Gamma_{K_2}). \quad (22)$$

Also, by Proposition 3.2 and the induction hypothesis,

$$A(\Gamma_{H_2}) - C(\Gamma_{H_2}) - D(\Gamma_{H_2}) = A(\Gamma_{K_2}) - C(\Gamma_{K_2}) - D(\Gamma_{K_2}). \quad (23)$$

Apply again Proposition 3.2 to obtain

$$\begin{aligned} E(\Gamma_{\sigma\tau(H_2)}) - E(\Gamma_{\tau(H_2)}) &= A(\Gamma_{\tau(H_2)}) - C(\Gamma_{\tau(H_2)}) - D(\Gamma_{\tau(H_2)}); \\ E(\Gamma_{\sigma\tau(K_2)}) - E(\Gamma_{\tau(K_2)}) &= A(\Gamma_{\tau(K_2)}) - C(\Gamma_{\tau(K_2)}) - D(\Gamma_{\tau(K_2)}). \end{aligned} \quad (24)$$

Here, in view of Proposition 3.6,

$$\begin{aligned} A(\Gamma_{\tau(H_2)}) &= A(\Gamma_{H_2}) + B(\Gamma_{H_2}) - C(\Gamma_{H_2}) - D(\Gamma_{H_2}); \\ A(\Gamma_{\tau(K_2)}) &= A(\Gamma_{K_2}) + B(\Gamma_{K_2}) - C(\Gamma_{K_2}) - D(\Gamma_{K_2}). \end{aligned}$$

These equalities together with (21)–(24) yield that

$$E(\Gamma_{\sigma\tau(H_2)}) - E(\Gamma_{\tau(H_2)}) = E(\Gamma_{\sigma\tau(K_2)}) - E(\Gamma_{\tau(K_2)}).$$

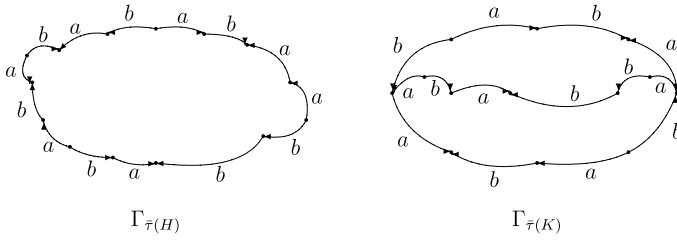


Fig. 7. The subgroup graphs $\Gamma_{\bar{\tau}(H)}$ and $\Gamma_{\bar{\tau}(K)}$.

Combine this with the induction hypothesis to finally get

$$E(\Gamma_{\sigma\tau(H_2)}) = E(\Gamma_{\sigma\tau(K_2)}),$$

that is,

$$E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)}).$$

Case 1.2. ϕ ends in τ .

This case is treated similarly to Case 1.1 to obtain the equality $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$.

Case 2. ϕ is a chain of type (C2).

We may assume that ϕ begins with $\bar{\tau}$, because the graphs Γ_H and Γ_K remain unchanged by applying $\bar{\sigma}$. The subgroup graphs of $\bar{\tau}(H)$ and $\bar{\tau}(K)$ are depicted in Fig. 7.

From Fig. 7, we see that there are no vertices of degree 4 in any of $\Gamma_{\bar{\tau}(H)}$ and $\Gamma_{\bar{\tau}(K)}$. Hence, if $|\phi| = 1$, then the equality $R(\Gamma_{\phi(H)}) = R(\Gamma_{\phi(K)})$ is clear. Also, the equality $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$ immediately follows from (14) and (16).

So let $|\phi| > 1$. First, we shall prove that $R(\Gamma_{\phi(H)}) = R(\Gamma_{\phi(K)})$. It is obvious that $R(\Gamma_{\phi(H)}) = 0$. So, let us consider $R(\Gamma_{\phi(K)})$. There are two vertices of type III-1 in the graph $\Gamma_{\bar{\tau}(H)}$. Denote any one of these vertices by v_K . Let p_1, p_2 and p_3 be arbitrary reduced paths in $\Gamma_{\bar{\tau}(H)}$ with terminus v_K and labels ending in a, a^{-1} and b , respectively. Then we see that $\mu(p_1), \mu(p_2)$ and $\mu(p_3)$ always end in $b^{-1}a, ba^{-1}$ and $a^{-1}b$, respectively. This yields that the vertex v_K always remains as a vertex of degree 3 even if we apply further $\bar{\sigma}$ and $\bar{\tau}$. So there cannot occur any vertex of degree 4 in $\Gamma_{\phi(K)}$, which implies $R(\Gamma_{\phi(K)}) = 0$, as desired.

Next, we shall prove that $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$. Observe from Fig. 7 that

$$\begin{aligned} \bar{C}(\Gamma_{\bar{\tau}(H)}) &= 2, & \bar{D}(\Gamma_{\bar{\tau}(H)}) &= 2; \\ \bar{C}(\Gamma_{\bar{\tau}(K)}) &= 1, & \bar{D}(\Gamma_{\bar{\tau}(K)}) &= 2, \end{aligned}$$

and that for arbitrary non-negative integers ℓ_i and m_i ($1 \leq i \leq t$),

$$\begin{aligned} \bar{C}(\Gamma_{\bar{\tau}^{m_t} \bar{\sigma}^{\ell_t} \dots \bar{\tau}^{m_1} \bar{\sigma}^{\ell_1} \bar{\tau}(H)}) &= 2, & \bar{D}(\Gamma_{\bar{\tau}^{m_t} \bar{\sigma}^{\ell_t} \dots \bar{\tau}^{m_1} \bar{\sigma}^{\ell_1} \bar{\tau}(H)}) &= 2; \\ \bar{C}(\Gamma_{\bar{\tau}^{m_t} \bar{\sigma}^{\ell_t} \dots \bar{\tau}^{m_1} \bar{\sigma}^{\ell_1} \bar{\tau}(K)}) &= 1, & \bar{D}(\Gamma_{\bar{\tau}^{m_t} \bar{\sigma}^{\ell_t} \dots \bar{\tau}^{m_1} \bar{\sigma}^{\ell_1} \bar{\tau}(K)}) &= 2. \end{aligned}$$

This situation is similar to (17). So, from now on, we can follow the proof of Case 1 to derive the equality $E(\Gamma_{\phi(H)}) = E(\Gamma_{\phi(K)})$.

Now the proof of Theorem C is completed. \square

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