The Mean Dehn Functions of Abelian Groups

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September 13, 2007

Abstract

While Dehn functions, $D(n)$, of finitely presented groups are very well studied in the literature, mean Dehn functions have received much less attention. M. Gromov introduced the notion of mean Dehn function of a group, $D_{mean}(n)$, suggesting that in many cases it should grow much more slowly than the Dehn function itself. Using only elementary counting methods, this paper presents some computations pointing in this direction. Particularizing them to the case of any finite presentation of an infinite finitely generated abelian group (for which it is well known that $D(n) \sim n^2$ except in the 1-dimensional case), we show that the three variations $D_{osmean}(n)$, $D_{smean}(n)$ and $D_{mean}(n)$ all are bounded above by $Kn(\ln n)^2$, where the constant K depends only on the presentation (and the geodesic combing) chosen. This improves an earlier bound given by Kukina and Roman'kov.

1 Introduction

Throughout this paper, let $A = \{a_1, \ldots, a_r\}$ be an alphabet with r letters, A^* be the free monoid on $A \cup A^{-1}$, F be the free group on A, and $\langle A \,|\, R \rangle$ be a chosen finite presentation of a quotient G of F. We have the natural projections, $A^* \twoheadrightarrow F \twoheadrightarrow G$, and $|w|_A \geqslant |w|_F \geqslant |w|_G$ where $|w|_A$, $|w|_F$ and $|w|_G$ denote the metric lengths of w in A^*, F and G, respectively.

The *Dehn function* of the finite presentation $G = \langle A \,|\, R \rangle$ is the function $D : \mathbb{N} \to \mathbb{N}$ defined by

$$D(n) = \max_{w \in B_G(n)} \{\text{area}(w)\},$$

where area(w) is the minimal number of factors in any expression of w (considered as word in F) as a product of conjugates of relations in $R^{\pm 1}$, and

$$B_G(n) = \{w \in A^* \mid w =_G 1,\ |w|_A \leqslant n\}.$$

The notations $B_G(n)$ and $S_G(n) = B_G(n) \setminus B_G(n-1)$ reflects the idea of *balls* and *spheres*. However, these sets are not real balls or spheres in the metric of G, but sets of closed paths in the corresponding Cayley graph $\Gamma(G)$, with possible backtrackings, and with bounded or given A-length.

It is well known that, on changing to another presentation of the same group, $D(n)$ remains the same up to multiplicative and additive constants, both in the argument and in the range. Many papers in the literature investigate Dehn functions of groups (specially in relation to the word problem). For example, it is well known the Dehn function of word-hyperbolic groups are linear and those of automatic groups are at most quadratic (see [1] for a general exposition). A theorem attributed to Gromov states that every subquadratic Dehn function is in fact linear (see [5] for a detailed proof). Thus there is a gap between linear and quadratic growth of Dehn functions for finitely presented groups. In particular, non-cyclic finitely generated free abelian groups (as automatic but not word-hyperbolic groups) have precisely quadratic Dehn function, i.e., $C_1 n^2 \leqslant D(n) \leqslant C_2 n^2$ for appropriate constants $C_1, C_2 > 0$, and $n \gg 0$.

In the literature, there are interesting variations of the concept of Dehn function, which have still not been deeply investigated. In this paper, we are concerned with mean Dehn functions, first introduced by M. Gromov in [2]. The *mean Dehn function* of the presentation $\langle A \mid R \rangle$ for G, denoted D_{mean}, is the mapping $D_{\text{mean}} \colon \mathbb{N} \to \mathbb{Q}$ defined by

$$D_{\text{mean}}(n) = \frac{\sum\limits_{w \in B_G(n)} \text{area}(w)}{\sharp B_G(n)}$$

(note that the denominator is never 0 since the empty word always belongs to $B_G(n)$). Similarly, the *spherical mean Dehn function*, denoted D_{smean}, is defined as

$$D_{\text{smean}}(n) = \frac{\sum\limits_{w \in S_G(n)} \text{area}(w)}{\sharp S_G(n)},$$

where we understand that $D_{\text{smean}}(n) = 0$ if the sphere $S_G(n)$ is empty. In contrast with the situation for the classical Dehn function, it is still not known in general whether the asymptotic behavior of these averaged versions is also invariant under changing the presentation of G.

The main result in the present paper is the following:

Theorem 1.1 *The mean Dehn function of a finitely generated abelian group G satisfies $D_{mean}(n) = O\big(n(\ln n)^2\big)$ (with the constant depending only on the chosen finite presentation for G). The same assertion is valid for the spherical mean Dehn function of G.*

This improves an earlier result due to E. G. Kukina and V. A. Roman'kov [3], who proved that for finitely generated free abelian groups, $\lim_{n\to\infty} \frac{D_{\text{mean}}(n)}{n^{7/4}} = 0$.

After the present paper was written, a preprint appeared concerning the asymptotic behavior of another Dehn function. Young [7] considers finitely generated nilpotent groups and what he calls their *averaged Dehn function*. His results imply that, for the finitely generated abelian case, this function is $O(n \ln n)$. However, there is a crucial difference between the Dehn functions in the two papers. In [7], the author considers what he calls *lazy words*, which are elements of the free monoid on $A \cup A^{-1} \cup \{e\}$ i.e., formal sequences of the form $a_1 \cdots a_n$ with $a_i \in A \cup A^{-1} \cup \{e\}$. Because the symbol e (which represents the trivial element in G) can be used, a lazy word of length n corresponds to a (non-necessarily reduced) word of length at most n in our terminology. The number of lazy words of length n is $(2r+1)^n$, while the number of our words of length at most n is $(2r)^0 + (2r)^1 + \cdots + (2r)^n = \frac{(2r)^{n+1}-1}{2r-1}$, which grows asymptotically like $(2r)^n \ll (2r+1)^n$. The difference is because every word w of length $m < n$ is counted many times as a lazy word, precisely as many ways there are of expanding w to a sequence of n symbols by adding $n-m$ symbols e between the existing ones. These representations of the same element of G all have the same area and the number of them distorts the average of the corresponding areas. We believe that this may explain the difference between the bound $O(n \ln n)$ in [7], and the bound $O(n(\ln n)^2)$ obtained here. Beyond this discussion, there is the question of which is the most appropriate notion of mean Dehn function from the group-theoretic point of view.

Sapir has recently introduced another variant of the concept of Dehn function, namely his *random Dehn function*. Here, the notion of area of a word $w \in A^*$ (not necessarily equal to 1 in G) is used with respect to a given geodesic combing in $\Gamma(G)$ (see Section 2 below). After choosing such a geodesic combing, say that $f\colon \mathbb{N} \to \mathbb{N}$ is a *random isoperimetric function* for G if

$$\frac{\sharp\{w \in A^* \mid |w|_A \leqslant n,\ \operatorname{area}(w) \geqslant f(n)\}}{\sharp\{w \in A^* \mid |w|_A \leqslant n\}} \to 0,$$

for $n \to \infty$. Then, the *random Dehn function* for G is the smallest random isoperimetric function (which, a priori, depends on the presentation of G and on the chosen combing). M. Sapir claimed (private communication) that for any finite presentation of an abelian group G, and for any geodesic combing in the corresponding Cayley graph $\Gamma(G)$, there exists a constant K such that the random Dehn function of G is dominated by $n \mapsto Kn \ln n$. It would be interesting to investigate the possible relationships between mean and random Dehn functions.

To conclude this introduction, let us clarify some notational uses along the paper: $B_1(n) = \{w \in A^* \mid |w|_A \leqslant n\}$ and $S_1(n) = \{w \in A^* \mid |w|_A = n\}$ will be simply denoted $B(n)$ and $S(n)$, respectively (these are the real balls and spheres in the monoid A^*). To avoid possible confusion with lengths, we shall write the cardinal of a set S as $\sharp S$. We use the term "ln" always meaning neperian logarithm (i.e. $\exp(\ln n) = n$). Simply for technical reasons ($\ln 1 = 0$ and we will need to work with functions $f\colon \mathbb{N} \to \mathbb{R}^+$ taking strictly positive values) the set $\mathbb{N}$ will be taken to be all natural numbers except 1. Also, for every real number x, we shall denote by $\lfloor x \rfloor$ its integral part (i.e. the biggest integer which is less than or equal to x), and similarly by $\lceil x \rceil$ the smallest integer n with $x \leqslant n$. So, $\lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil$. Note that, for a positive integer n, $n \geqslant x$ is equivalent to $n \geqslant \lceil x \rceil$; and $n \leqslant x$ is equivalent to $n \leqslant \lfloor x \rfloor$. Furthermore, for every integer $n > 0$, $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$.

The paper is organized as follows. In Section 2 we introduce the notion of open mean Dehn function and give a general upper bound for it, assuming that the presentation satisfies some technical assumptions. We also give some indications on how to convert this bound into a bound for the spherical mean and the mean Dehn functions. In Section 3 we concentrate on finitely generated abelian groups, making the necessary countings there to ensure that every finite presentation of such a group satisfies the assumptions required in the previous section. Finally, in Section 4 we deduce explicit upper bounds for the mean and the spherical mean Dehn functions of any finite presentation of an abelian group. It is interesting to remark that the techniques developed in Section 2 can probably be applied to other groups as well. As soon as one can find two functions satisfying Assumption 2.1 for his favorite group presentation, an upper bound for the open spherical mean Dehn function of that presentation will easily follow. With further computations, one can also expect to obtain an upper bound for the mean Dehn function of such a presentation.

2 Combings in groups and the open mean Dehn function

Let $\Gamma(G)$ be the Cayley graph associated to the given presentation $\langle A \mid R\rangle$ of G, and let e denote the vertex corresponding to the trivial element in G. There is a natural bijection, $w \longleftrightarrow \gamma_w$, between (possibly non-reduced) words in A^* and paths in $\Gamma(G)$ starting at e, i.e. $\iota\gamma_w = e$ (and possibly with backtrackings); usually, we will not distinguish between w and γ_w at notational level. Clearly, the length of γ_w is $|w|_A$, the length of γ_w after reducing all possible backtrackings is $|w|_F$, and the distance in $\Gamma(G)$ from e to $\tau\gamma_w$ (the terminal point of γ_w) is $|w|_G$. Any path in $\Gamma(G)$ of the minimal possible length from e to $\tau\gamma_w$ (not unique, in general) is called a *geodesic for* $w \in G$, and in fact, corresponds to a word $w' \in A^*$ of the shortest possible A-length such that $w =_G w'$.

Let $w \in A^*$ be such that $w =_G 1$ (i.e. a path in $\Gamma(G)$ closed at e). Classically, the *area* of w, denoted by area(w), is the minimal m for which one has an expression

$$w =_F \prod_{i=1}^{m} f_i^{-1} r_i^{\epsilon_i} f_i,$$

where $f_i \in F$, $r_i \in R$, and $\epsilon_i = \pm 1$. Note that area(w) only depends on the free reduction of w, and that, if $w, w' \in A^*$ map to the identity element in G, then $\mathrm{area}(ww') \leqslant \mathrm{area}(w) + \mathrm{area}(w')$. Furthermore, $\mathrm{area}(w^{-1}) = \mathrm{area}(w)$ and $\mathrm{area}(v^{-1}wv) = \mathrm{area}(w)$ for every $v \in A^*$.

In our arguments, an extension of the concept of area to arbitrary (non-necessarily closed) paths in $\Gamma(G)$ is required. Accordingly, we shall introduce the notion of *open* mean Dehn function averaging over all those words.

A *combing* in $\Gamma(G)$ is a set T consisting of exactly one path from e to every vertex $v \in \Gamma(G)$, denoted $T[e,\, v]$ or simply $T[v]$, and such that $T[e]$ is the trivial path. By left translation, such a set also determines a (unique) path between every given pair of vertices in $\Gamma(G)$, namely $T[u,\, v] = uT[e,\, u^{-1}v]$. A combing T is said to be *geodesic* if $T[v]$ is a geodesic path for every v. In this case, $T[u, v]$ will also be a geodesic path for every pair of vertices u, v. Using a combing T, any path γ in $\Gamma(G)$ can be *closed up* by returning back to its initial vertex through the combing. That is, defining $\widetilde{\gamma} = T[\iota\gamma,\, \tau\gamma]$, we have that $\gamma\widetilde{\gamma}^{-1}$ is a closed path at $\iota\gamma$. Note that if T is geodesic then $|\widetilde{\gamma}|_A \leqslant |\gamma|_A$.

Standard examples of combings are the *tree combings*, i.e. those determined by a maximal tree T in $\Gamma(G)$. In this case, $T[v]$ is the unique reduced path from e to v in T. For example, $\Gamma(\mathbb{Z}^2)$ (with the standard presentation) is the two dimensional integral lattice; and the maximal tree given by the X-axis plus all the vertical lines determines the geodesic combing of $G = \mathbb{Z}^2$ where $T[(r, s)]$ is the path that goes first r steps to the right and then s steps up. Note that, for these tree combings, $T[u,\, v]$ may *not* be the path determined by the tree from u to v.

Using combings, we can define the area of an arbitrary path γ in $\Gamma(G)$ (not-necessarily reduced, neither closed, nor even starting at e). If γ is closed at e, the meaning of area(γ) was given above. If γ is closed at a vertex $u = \iota\gamma = \tau\gamma \neq e$ we define the *area* of γ by first translating γ to e (i.e. reading the same word γ but from the vertex e) or, equivalently, going first to (and then coming back from) u through an arbitrary path (which makes no difference at the level of the area because it is conjugacy invariant):

$$\mathrm{area}(\gamma) = \mathrm{area}(T[e, u]\gamma T[e, u]^{-1})$$

(caution! $T[e, u]^{-1} \neq T[u, e] = uT[e, u^{-1}]$ in general). Finally, suppose γ is an arbitrary path in $\Gamma(G)$ (with $u = \iota\gamma$ and $v = \tau\gamma$ not necessarily equal, neither equal to e). The *area* of γ is defined by first closing it through the combing:

$$\mathrm{area}(\gamma) = \mathrm{area}(\gamma\widetilde{\gamma}^{-1}).$$

Since by definition $T[u,v] = uT[e, u^{-1}v]$, closing up γ and translating the result to e reads the same as translating first γ to e and then closing it up. Note that, for a non-closed path γ, area(γ) and area(γ^{-1}) are not necessarily equal.

To analyze the spherical mean Dehn function of a group G, we must evaluate the sum of areas of all words in A^* mapping to 1 in G, and having a given length. That is, the sum of areas of all paths in $\Gamma(G)$ of a given length, and closed at e. To do this, we will do inductive arguments that force us to consider more general sums, like the sum of areas of all paths in $\Gamma(G)$ starting at e and of a given length (...and being closed or not). The following notation will be useful in order to manipulate these sums.

For a given set of paths P starting at e (i.e. a given $P \subseteq A^*$) we write

$$\mathcal{A}_P = \sum_{\gamma \in P} \operatorname{area}(\gamma).$$

If v is a vertex in $\Gamma(G)$ and n is a positive integer, we denote by $\mathcal{A}_v(n)$ the sum of areas of all paths γ in $\Gamma(G)$ having length n, starting at $\iota\gamma = e$ and ending at $\tau\gamma = v$. Note that, if $|v|_G > n$, then there are no such paths and so $\mathcal{A}_v(n) = 0$. Note also that $\mathcal{A}_e(n)$ is the sum of areas of all closed paths at e with length n, which is precisely the numerator of the spherical mean Dehn function of G evaluated at n. Finally, let $\mathcal{A}(n)$ denote the sum of areas of all paths γ in $\Gamma(G)$ having length n and starting at e. Thus, we have

$$\begin{aligned} \mathcal{A}_v(n) &= \sum_{\substack{|\gamma|_A = n \\ \iota\gamma = e,\, \tau\gamma = v}} \operatorname{area}(\gamma), \\ \mathcal{A}(n) &= \sum_v \mathcal{A}_v(n) = \sum_{\substack{|\gamma|_A = n \\ \iota\gamma = e}} \operatorname{area}(\gamma). \end{aligned}$$

Similarly, we denote by $\mathcal{N}_v(n)$ the number of paths γ in $\Gamma(G)$ having length n, starting at $\iota\gamma = e$ and ending at $\tau\gamma = v$. Of course, $\mathcal{N}_v(n) = 0$ if $|v|_G > n$. Also, $\sum_v \mathcal{N}_v(n) = (2r)^n$. This notation allows us to write

$$D_{\text{smean}}(n) = \frac{\mathcal{A}_e(n)}{\mathcal{N}_e(n)},$$

and suggests defining the *open spherical mean Dehn function* as the averaged area over non-necessarily closed paths:

$$D_{\text{osmean}}(n) = \frac{\mathcal{A}(n)}{(2r)^n} = \frac{\sum_v \mathcal{A}_v(n)}{\sum_v \mathcal{N}_v(n)}.$$

In order to find an upper bound for $D_{\text{osmean}}(n)$, we are guided by the following intuitive idea. Out of the $(2r)^n$ paths of length n, those arriving "far" from e will maybe contribute with a "large" area; but there are "few" of them. And those arriving "close" to e (which are "much more" frequent) will be inductively controlled by shorter paths.

To make this precise, we consider the following technical condition. For all those finite presentations satisfying it, we will be able to give a recurrent estimation of $\mathcal{A}(n)$.

Assumption 2.1 Let $A = \{a_1, \ldots, a_r\}$, F be the free group on A, and $\langle A \,|\, R\rangle$ be a finite presentation of a quotient G of F. We assume the existence of two non-decreasing functions $f, g \colon \mathbb{N} \to \mathbb{R}^+$ such that

$$\sharp\{w \in A^* \mid |w|_A = n,\ |w|_G > f(n)\} = O\left(\frac{(2r)^n}{g(n)}\right),$$

and $\frac{D(2n)}{g(\lfloor \frac{n}{2} \rfloor)} = O(D(4\lceil f(n) \rceil))$ (note that the first assumption is vacuous if $f(n)$ grows faster than linear).

Proposition 2.2 *Let $A = \{a_1, \ldots, a_r\}$, F be the free group on A, and $\langle A \mid R \rangle$ be a finite presentation of a quotient G of F satisfying assumption 2.1. Choose an arbitrary geodesic combing T in $\Gamma(G)$. Then we have*

$$\mathcal{A}(n) \leqslant (2r)^{\lceil n/2 \rceil} \mathcal{A}(\lfloor \frac{n}{2} \rfloor) + (2r)^{\lfloor n/2 \rfloor} \mathcal{A}(\lceil \frac{n}{2} \rceil) + (2r)^n O(D(4\lceil f(n) \rceil)).$$

Proof. Every summand in $\mathcal{A}(n)$ has the form $\text{area}(\gamma) = \text{area}(\gamma \widetilde{\gamma}^{-1})$ and so is bounded above by $D(2n)$ (since $|\widetilde{\gamma}|_A \leqslant |\gamma|_A \leqslant n$). On the other hand, $\mathcal{A}(n)$ is a sum of $(2r)^n$ summands. Let us split $\mathcal{A}(n)$ into two terms in such a way that we can improve one of these two estimates in each. Consider

$$P_1 = \{\gamma \mid |\gamma|_A = n,\, \iota(\gamma) = e,\, |\tau\gamma|_G > f(n)\},$$
$$P_2 = \{\gamma \mid |\gamma|_A = n,\, \iota(\gamma) = e,\, |\tau\gamma|_G \leqslant f(n)\}.$$

Writing

$$\mathcal{A}(n) = \mathcal{A}_{P_1} + \mathcal{A}_{P_2}, \tag{1}$$

we see that the first term has a small number of summands (according to Assumption 2.1), while the second term will be related to $\mathcal{A}(\lfloor \frac{n}{2} \rfloor)$ and $\mathcal{A}(\lceil \frac{n}{2} \rceil)$, allowing us to do a later inductive argument. More precisely,

$$\mathcal{A}_{P_1} \leqslant D(2n) \cdot \sharp P_1 = D(2n) O\left(\frac{(2r)^n}{g(n)}\right). \tag{2}$$

Let us now evaluate the second term in (1). A typical summand there is the area of a path γ of length n, starting at e, and ending at some vertex v such that $|v|_G \leqslant f(n)$. That is, $\text{area}(\gamma\widetilde{\gamma}^{-1})$, where $|\gamma|_A = n$ and $|\widetilde{\gamma}|_A \leqslant f(n)$. Break γ into two parts, $\gamma = \gamma_1\gamma_2$ with $|\gamma_1|_A = \lfloor \frac{n}{2} \rfloor$ and $|\gamma_2|_A = \lceil \frac{n}{2} \rceil$, and denote by u the *middle* point, $\tau\gamma_1 = u = \iota\gamma_2$ (see Figure 1, where $\widetilde{\gamma}_1 = T[e, u]$, $\widetilde{\gamma}_2 = T[u, v]$ and $\widetilde{\gamma} = T[e, v]$). For every such $\gamma \in P_2$, we have

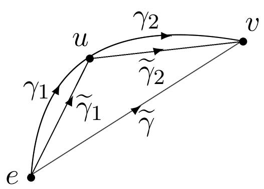

Figure 1: Breaking γ into two parts.

$$\begin{aligned} \text{area}(\gamma) = \text{area}(\gamma_1\gamma_2\widetilde{\gamma}^{-1}) &\leqslant \text{area}(\gamma_1\widetilde{\gamma}_1^{-1}) + \text{area}(\widetilde{\gamma}_1\gamma_2\widetilde{\gamma}_2^{-1}\widetilde{\gamma}_1^{-1}) + \text{area}(\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}^{-1}) \\ &= \text{area}(\gamma_1) + \text{area}(\gamma_2) + \text{area}(\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}^{-1}). \end{aligned}$$

So,

$$\mathcal{A}_{P_2} = \sum_{\gamma \in P_2} \text{area}(\gamma) \leqslant \sum_{\gamma \in P_2} (\text{area}(\gamma_1) + \text{area}(\gamma_2)) + \sum_{\gamma \in P_2} \text{area}(\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}^{-1}). \tag{3}$$

To estimate the first summand in (3), observe that when γ goes over all P_2, γ_1 moves inside the set of words in A^* of length $\lfloor \frac{n}{2} \rfloor$ (and γ_2 inside the set of words of length $\lceil \frac{n}{2} \rceil$). Note also that every word of length $\lfloor \frac{n}{2} \rfloor$ appears as γ_1 at most $(2r)^{\lceil \frac{n}{2} \rceil}$ times (while every word of length $\lceil \frac{n}{2} \rceil$ appears as γ_2 at most $(2r)^{\lfloor \frac{n}{2} \rfloor}$ times). Thus,

$$\sum_{\gamma \in P_2} (\mathrm{area}(\gamma_1) + \mathrm{area}(\gamma_2)) \leqslant (2r)^{\lceil \frac{n}{2} \rceil} \mathcal{A}(\lfloor \frac{n}{2} \rfloor) + (2r)^{\lfloor \frac{n}{2} \rfloor} \mathcal{A}(\lceil \frac{n}{2} \rceil). \tag{4}$$

It remains to estimate the second summand in (3), i.e. the areas of all those geodesic triangles. To do this, we again split P_2 into two disjoint sets, depending on $|u|_G$. Let

$$P_3 = \{\gamma \in P_2 \mid |\widetilde{\gamma}_1|_A = |u|_G > f(\lfloor \frac{n}{2} \rfloor)\}, \quad P_4 = \{\gamma \in P_2 \mid |\widetilde{\gamma}_1|_A = |u|_G \leqslant f(\lfloor \frac{n}{2} \rfloor)\},$$

so that

$$\sum_{\gamma \in P_2} \mathrm{area}(\widetilde{\gamma}_1 \widetilde{\gamma}_2 \widetilde{\gamma}^{-1}) = \sum_{\gamma \in P_3} \mathrm{area}(\widetilde{\gamma}_1 \widetilde{\gamma}_2 \widetilde{\gamma}^{-1}) + \sum_{\gamma \in P_4} \mathrm{area}(\widetilde{\gamma}_1 \widetilde{\gamma}_2 \widetilde{\gamma}^{-1}). \tag{5}$$

With a similar argument as above, we can bound the first summand in (5) by using the fact that it has few summands,

$$\sum_{\gamma \in P_3} \mathrm{area}(\widetilde{\gamma}_1 \widetilde{\gamma}_2 \widetilde{\gamma}^{-1}) \leqslant D(2n) O\left(\frac{(2r)^{\lfloor \frac{n}{2} \rfloor}}{g\left(\lfloor \frac{n}{2} \rfloor\right)} \right) (2r)^{\lceil \frac{n}{2} \rceil}. \tag{6}$$

Finally, the second summand in (5) can be bounded by taking into account that all the involved triangles have perimeter

$$|\widetilde{\gamma}_1|_A + |\widetilde{\gamma}_2|_A + |\widetilde{\gamma}|_A \leqslant 2(|\widetilde{\gamma}_1|_A + |\widetilde{\gamma}|_A) \leqslant 2(f(\lfloor \frac{n}{2} \rfloor) + f(n)) \leqslant 4f(n) \leqslant 4\lceil f(n) \rceil.$$

Hence,

$$\sum_{\gamma \in P_4} \mathrm{area}(\widetilde{\gamma}_1 \widetilde{\gamma}_2 \widetilde{\gamma}^{-1}) \leqslant D(4\lceil f(n) \rceil)(2r)^n. \tag{7}$$

Combining together equations (1) to (7), we conclude the proof:

$$\begin{aligned} \mathcal{A}(n) &\leqslant (2r)^{\lceil \frac{n}{2} \rceil} \mathcal{A}(\lfloor \tfrac{n}{2} \rfloor) + (2r)^{\lfloor \frac{n}{2} \rfloor} \mathcal{A}(\lceil \tfrac{n}{2} \rceil) + \\ &\quad D(2n) O\big(\tfrac{(2r)^n}{g(n)}\big) + D(2n) O\big(\tfrac{(2r)^n}{g(\lfloor \frac{n}{2} \rfloor)}\big) + D(4\lceil f(n) \rceil)(2r)^n \\ &= (2r)^{\lceil \frac{n}{2} \rceil} \mathcal{A}(\lfloor \tfrac{n}{2} \rfloor) + (2r)^{\lfloor \frac{n}{2} \rfloor} \mathcal{A}(\lceil \tfrac{n}{2} \rceil) + (2r)^n O(D(4\lceil f(n) \rceil)), \end{aligned}$$

where we make use of the second assumption. □

To conclude this section, let us unwrap the recurrence given in the previous statement, obtaining an upper bound for the open spherical mean Dehn function of all finite presentations satisfying Assumption 2.1.

Theorem 2.3 *For every finite presentation (and geodesic combing) satisfying Assumption 2.1, and for every non-decreasing function* $h\colon \mathbb{N} \to \mathbb{R}^+$ *satisfying* $h(\lfloor \frac{n}{2} \rfloor) + h(\lceil \frac{n}{2} \rceil) + D(4\lceil f(n) \rceil) \leqslant h(n)$ *for* $n \gg 0$*, we have*

$$D_{osmean}(n) = O\big(h(n)\big).$$

Proof. From Proposition 2.2, there exists a constant M such that, for every $n \geqslant 2$,

$$\mathcal{A}(n) \leqslant (2r)^{\lceil \frac{n}{2} \rceil}\mathcal{A}(\lfloor \frac{n}{2} \rfloor) + (2r)^{\lfloor \frac{n}{2} \rfloor}\mathcal{A}(\lceil \frac{n}{2} \rceil) + M(2r)^n D(4\lceil f(n) \rceil).$$

Now take h as in the statement (for $n \geqslant n_0$), and let $K = \max\{M, \mathcal{A}(2)/h(2), \ldots, \mathcal{A}(n_0)/h(n_0)\}$. Let us prove that, for $n \geqslant 2$,

$$\mathcal{A}(n) \leqslant K(2r)^n h(n).$$

For $n = 2, \ldots, n_0$ the inequality is true, by construction. Fix a value of $n > n_0$, and assume the inequality true for all smaller values. We have

$$\begin{aligned} \mathcal{A}(n) &\leqslant (2r)^{\lceil \frac{n}{2} \rceil}\mathcal{A}(\lfloor \tfrac{n}{2} \rfloor) + (2r)^{\lfloor \frac{n}{2} \rfloor}\mathcal{A}(\lceil \tfrac{n}{2} \rceil) + M(2r)^n D(4\lceil f(n) \rceil) \\ &\leqslant (2r)^{\lceil \frac{n}{2} \rceil} K(2r)^{\lfloor \frac{n}{2} \rfloor} h(\lfloor \tfrac{n}{2} \rfloor) + (2r)^{\lfloor \frac{n}{2} \rfloor} K(2r)^{\lceil \frac{n}{2} \rceil} h(\lceil \tfrac{n}{2} \rceil) + M(2r)^n D(4\lceil f(n) \rceil) \\ &\leqslant K(2r)^n \big(h(\lfloor \tfrac{n}{2} \rfloor) + h(\lceil \tfrac{n}{2} \rceil) + D(4\lceil f(n) \rceil)\big) \\ &\leqslant K(2r)^n h(n). \end{aligned}$$

Hence, $D_{\text{osmean}}(n) = \mathcal{A}(n)/(2r)^n = O(h(n))$ concluding the proof. □

To proceed from Theorem 2.3 to an upper bound for the spherical mean Dehn function, we need to extract some more information from the presentation of G, namely, the proportion of the total $(2r)^n$ paths of length n that are closed. Or, more generally, how sensitive $\mathcal{N}_v(n)$ is in terms of v. This information strongly depends on the group G and on the specific presentation considered. It will be analyzed in Section 4 for the case of abelian groups.

Finally, using the following observation, it is easy to pass from an estimate of the spherical mean Dehn function to an estimate of the mean Dehn function for the same presentation.

Proposition 2.4 *For any finite presentation of a group G, we have*

$$D_{mean}(n) \leqslant \max_{0 \leqslant m \leqslant n} D_{smean}(m).$$

Proof. Directly from the definitions, we have

$$\sum_{w \in B_G(n)} \text{area}(w) = \sum_{m=0}^{n} \sum_{w \in S_G(m)} \text{area}(w) = \sum_{m=0}^{n} D_{\text{smean}}(m) \cdot \sharp S_G(m) \leqslant$$

$$\leqslant \big(\max_{0 \leqslant m \leqslant n} D_{\text{smean}}(m)\big) \sum_{m=0}^{n} \sharp S_G(m) = \big(\max_{0 \leqslant m \leqslant n} D_{\text{smean}}(m)\big) \cdot \sharp B_G(n). \quad \square$$

3 Counting words in abelian groups

Let us now apply the techniques developed in the previous section to any finite presentation of an abelian group, until obtaining explicit upper bounds for $D_{\text{osmean}}(n)$, $D_{\text{smean}}(n)$ and $D_{\text{mean}}(n)$. To do this, we need first to verify that those presentations satisfy assumption 2.1 for appropriate functions f, g. This is the goal of the present section.

We start with a simple and well known fact, which is straightforward to verify by induction.

Lemma 3.1 *Let $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ be two lists of r positive real numbers. Then,*

$$\min\Big\{\frac{x_1}{y_1}, \ldots, \frac{x_r}{y_r}\Big\} \leqslant \frac{x_1 + \cdots + x_r}{y_1 + \cdots + y_r} \leqslant \max\Big\{\frac{x_1}{y_1}, \ldots, \frac{x_r}{y_r}\Big\}. \square$$

Our arguments will make strong use of the following lemma due to Kolmogorov (see Lemma 8.1 in page 378 of [4]). This useful result, proved in 1929, deserves to be better known. We include a self-contained proof extracted from [4]. It uses the following Tchebyshev inequality, which is straightforward to verify.

Lemma 3.2 (Tchebyshev) *Let X be a random variable and $f(x)$ be a nondecreasing real function. Then, for any real number a such that $f(a) > 0$, the following inequality holds:*

$$\Pr(X > a) \leqslant \frac{E(f(X))}{f(a)}.$$

Lemma 3.3 (Kolmogorov) *Consider n pairwise independent random variables $\{X_i\}$, $i = 1, \ldots, n$, with zero means and variances $\sigma_i^2 = E(X_i^2)$, and suppose that $|X_i| \leqslant d < \infty$. Let $S_n = \sum_{i=1}^n X_i$, and let t be a real number such that $0 < td \leqslant s_n$, where $s_n^2 = \mathrm{var}(S_n) = \sum_{i=1}^n \sigma_i^2$. Then, for any $\epsilon > 0$,*

$$\Pr(S_n > \epsilon s_n) \leqslant \exp\bigl(-t\epsilon + \frac{1}{2}t^2(1 + \frac{1}{2}tds_n^{-1})\bigr).$$

Proof. For each X_i, and for every $j \geqslant 2$ we have

$$E(X_i^j) = E(X_i^{j-2}X_i^2) \leqslant d^{j-2}E(X_i^2) = d^{j-2}\sigma_i^2.$$

Furthermore, the following series are absolutely convergent and, since $0 < tds_n^{-1} \leqslant 1$, and $\sum_{j=3}^\infty \frac{2}{j!} = 2(e - 2.5) < 0.5$, we have

$$\begin{aligned}
E\bigl(e^{ts_n^{-1}X_i}\bigr) &= E(\textstyle\sum_{j=0}^\infty \frac{1}{j!}(ts_n^{-1}X_i)^j) \\
&= \textstyle\sum_{j=0}^\infty \frac{1}{j!}(ts_n^{-1})^j E(X_i^j) \\
&\leqslant 1 + 0 + \textstyle\sum_{j=2}^\infty \frac{1}{j!}t^j s_n^{-j} d^{j-2}\sigma_i^2 \\
&= 1 + \tfrac{1}{2}(t\sigma_i s_n^{-1})^2\bigl(\textstyle\sum_{j=2}^\infty \frac{2}{j!}(tds_n^{-1})^{j-2}\bigr) \\
&= 1 + \tfrac{1}{2}(t\sigma_i s_n^{-1})^2\bigl(1 + tds_n^{-1}\textstyle\sum_{j=3}^\infty \frac{2}{j!}(tds_n^{-1})^{j-3}\bigr) \\
&\leqslant 1 + \tfrac{1}{2}(t\sigma_i s_n^{-1})^2(1 + tds_n^{-1}\textstyle\sum_{j=3}^\infty \frac{2}{j!}) \\
&\leqslant 1 + \tfrac{1}{2}(t\sigma_i s_n^{-1})^2(1 + \tfrac{1}{2}tds_n^{-1}) \\
&\leqslant \exp\bigl(\tfrac{1}{2}(t\sigma_i s_n^{-1})^2(1 + \tfrac{1}{2}tds_n^{-1})\bigr).
\end{aligned}$$

Now, using Tchebyshev's inequality (Lemma 3.2) applied to $X = S_n$, $f(x) = e^{ts_n^{-1}x}$ and $a = \epsilon s_n$, we have

$$\begin{aligned}
\Pr\bigl(S_n > \epsilon s_n\bigr) &\leqslant e^{-t\epsilon}E\bigl(e^{ts_n^{-1}S_n}\bigr) \\
&= e^{-t\epsilon}E\bigl(\textstyle\prod_{i=1}^n e^{ts_n^{-1}X_i}\bigr) \\
&= e^{-t\epsilon}\textstyle\prod_{i=1}^n E\bigl(e^{ts_n^{-1}X_i}\bigr) \\
&\leqslant e^{-t\epsilon}\textstyle\prod_{i=1}^n \exp\bigl(\frac{1}{2}(t\sigma_i s_n^{-1})^2(1 + \frac{1}{2}tds_n^{-1})\bigr) \\
&= \exp\bigl(-t\epsilon + \textstyle\sum_{i=1}^n \frac{1}{2}(t\sigma_i s_n^{-1})^2(1 + \frac{1}{2}tds_n^{-1})\bigr) \\
&= \exp\bigl(-t\epsilon + \tfrac{1}{2}t^2(1 + \tfrac{1}{2}tds_n^{-1})\bigr).
\end{aligned}$$

This completes the proof. □

As a corollary, we easily deduce the following result on 1-dimensional random walks.

Proposition 3.4 *Let $A = \{a\}$ and let $F = G \simeq \mathbb{Z}$ be the infinite cyclic group generated by A. Given a real number $c > 0$, the number of words $w \in A^*$ with $|w|_A = n$ and $|w|_\mathbb{Z} > c\sqrt{n \ln n}$ is $O(\frac{2^n}{n^{c-\frac{1}{2}}})$.*

Proof. Let us assume $n \geqslant 2$, and consider a 1-dimensional random walk on $\mathbb{Z}$ of length n, i.e. n independent (and uniform) random variables $\{X_i\}$ with $X_i \in \{-1, 1\}$ and $E(X_i) = 0$, $i = 1, \ldots, n$. We have $\sigma_i^2 = 1$ and $s_n^2 = n$. Now, by applying Kolmogorov's Lemma with $d = 1$, $t = \sqrt{\ln n}$ and $\epsilon = c\sqrt{\ln n}$, we obtain that

$$\begin{aligned} \Pr\left(\sum_{i=1}^n X_i > c\sqrt{n\ln n}\right) &\leqslant \exp\left(-c\ln n + \tfrac{\ln n}{2}\left(1 + \tfrac{\sqrt{\ln n}}{2\sqrt{n}}\right)\right) \\ &= \exp\left((\ln n)\left(-c + \tfrac{1}{2} + \tfrac{1}{4}\sqrt{\tfrac{\ln n}{n}}\right)\right) \\ &= \frac{n^{\frac{1}{4}\sqrt{\frac{\ln n}{n}}}}{n^{c-\frac{1}{2}}} \\ &\leqslant \frac{K}{n^{c-\frac{1}{2}}}, \end{aligned}$$

where the last inequality is due to the fact that $\lim_{n\to\infty} n^{\frac{1}{4}\sqrt{\frac{\ln n}{n}}} = 1$ (we can take, for example, $K = 1.35$).

However, the number of words in A^* of A-length n is 2^n. So, the previous inequality means that the number of words $w \in A^*$ with $|w|_A = n$, $|w|_\mathbb{Z} > c\sqrt{n\ln n}$, and representing positive integers is at most $K\frac{2^n}{n^{c-\frac{1}{2}}}$. By symmetry, the number of words $w \in A^*$ with $|w|_A = n$ and $|w|_\mathbb{Z} > c\sqrt{n\ln n}$ is at most $K\frac{2^{n+1}}{n^{c-\frac{1}{2}}}$. Finally, since K does not depend on n (nor even on c) we have the result. □

The next statement is the analog of Proposition 3.4 for an arbitrary finitely generated abelian group.

Proposition 3.5 *Let $A = \{a_1, \ldots, a_r\}$, F be the free group on A, and $\langle A \mid R\rangle$ be a finite presentation of an abelian quotient G of F. Given a real number $c > 1/2$, the number of words $w \in A^*$ with $|w|_A = n$ and $|w|_G > rc\sqrt{n\ln n}$ is $O(\frac{(2r)^n}{(\sqrt{n\ln n})^{c-\frac{1}{2}}})$.*

Proof. Since G is an r-generated abelian group, the map $F \twoheadrightarrow G$ factors through $\mathbb{Z}^r$, so we have $A^* \twoheadrightarrow F \twoheadrightarrow \mathbb{Z}^r \twoheadrightarrow G$. And as we have observed before, $|w|_{\mathbb{Z}^r} \geqslant |w|_G$. Therefore, it is enough to prove the result for $\mathbb{Z}^r$. So, we are reduced to considering only the case where G is the free abelian group of rank r.

Let $w \in A^*$. For any $i = 1, \ldots, r$, let $w_{a_i} \in \{a_i\}^*$ be the word which can be obtained from w by deleting all letters different from a_i and a_i^{-1}. Clearly, $|w|_A = \sum_{i=1}^r |w_{a_i}|_A$ (note that $|w_{a_i}|_A = |w_{a_i}|_{\{a_i\}}$). Also, $|w|_{\mathbb{Z}^r} = \sum_{i=1}^r |w_{a_i}|_{\mathbb{Z}^r}$.

Now, let $\ell = c\sqrt{n\ln n}$. Note also that $|w|_{\mathbb{Z}^r} > r\ell$ implies $|w_{a_i}|_{\mathbb{Z}^r} > \ell$ for some i. Therefore, we have

$$\frac{\sharp\{w \in S(n) \mid |w|_{\mathbb{Z}^r} > r\ell\}}{\sharp S(n)} \leqslant \frac{\sum_{i=1}^r \sharp\{w \in S(n) \mid |w_{a_i}|_{\mathbb{Z}^r} > \ell\}}{\sharp S(n)}.$$

Furthermore, for every $i = 1, \ldots, r$, we also have

$$\begin{aligned}\frac{\sharp\{w \in S(n) \mid |w_{a_i}|_{\mathbb{Z}^r} > \ell\}}{\sharp S(n)} &\leqslant \frac{\sum_{m=\lceil \ell \rceil}^{n} \sharp\{w \in S(n) \mid |w_{a_i}|_A = m,\, |w_{a_i}|_{\mathbb{Z}^r} > \ell\}}{\sharp S(n)} \\ &\leqslant \frac{\sum_{m=\lceil \ell \rceil}^{n} \sharp\{w \in S(n) \mid |w_{a_i}|_A = m,\, |w_{a_i}|_{\mathbb{Z}^r} > \ell\}}{\sum_{m=\lceil \ell \rceil}^{n} \sharp\{w \in S(n) \mid |w_{a_i}|_A = m\}} \\ &\leqslant \max_{\lceil \ell \rceil \leqslant m \leqslant n} \frac{\sharp\{w \in S(n) \mid |w_{a_i}|_A = m,\, |w_{a_i}|_{\mathbb{Z}^r} > \ell\}}{\sharp\{w \in S(n) \mid |w_{a_i}|_A = m\}},\end{aligned}$$

where the last inequality is justified by Lemma 3.1. However, given a word $v \in \{a_i\}^*$, the number of words $w \in S(n)$ such that $w_{a_i} = v$ do not depend on v, but only on $m = |v|_A = |v|_{\{a_i\}}$. So, for every $\lceil \ell \rceil \leqslant m \leqslant n$, we have

$$\begin{aligned}\frac{\sharp\{w \in S(n) \mid |w_{a_i}|_A = m,\, |w_{a_i}|_{\mathbb{Z}^r} > \ell\}}{\sharp\{w \in S(n) \mid |w_{a_i}|_A = m\}} &= \frac{\sharp\{v \in \{a_i\}^* \mid |v|_{\{a_i\}} = m,\ |v|_{\mathbb{Z}} > \ell\}}{\sharp\{v \in \{a_i\}^* \mid |v|_{\{a_i\}} = m\}} \\ &= \frac{\sharp\{v \in \{a_i\}^* \mid |v|_{\{a_i\}} = m,\ |v|_{\mathbb{Z}} > c\sqrt{n \ln n}\,\}}{2^m} \\ &\leqslant \frac{\sharp\{v \in \{a_i\}^* \mid |v|_{\{a_i\}} = m,\ |v|_{\mathbb{Z}} > c\sqrt{m \ln m}\,\}}{2^m} \\ &\leqslant \frac{K}{m^{c-\frac{1}{2}}},\end{aligned}$$

for an appropriate constant K (according to Proposition 3.4, we can take $K = 2.7$). Thus, collecting all together,

$$\frac{\sharp\{w \in S(n) \mid |w|_{\mathbb{Z}^r} > rc\sqrt{n \ln n}\,\}}{(2r)^n} = \frac{\sharp\{w \in S(n) \mid |w|_{\mathbb{Z}^r} > r\ell\,\}}{\sharp S(n)}$$

$$\leqslant \sum_{i=1}^{r} \Big(\max_{\lceil \ell \rceil \leqslant m \leqslant n} \frac{K}{m^{c-\frac{1}{2}}} \Big) = \frac{rK}{\lceil \ell \rceil^{c-\frac{1}{2}}} \leqslant \frac{rK}{(c\sqrt{n \ln n}\,)^{c-\frac{1}{2}}},$$

where we used $c > 1/2$. This proves that the number of words $w \in A^*$ with $|w|_A = n$ and $|w|_G > rc\sqrt{n \ln n}$ is $O\big(\frac{(2r)^n}{(\sqrt{n \ln n}\,)^{c-\frac{1}{2}}}\big)$. □

4 The mean Dehn function of abelian groups

The next step is to fulfill Assumption 2.1 for finite presentations of abelian groups.

Corollary 4.1 *Let $A = \{a_1, \ldots, a_r\}$, F be the free group on A, and $\langle A \,|\, R \rangle$ be a finite presentation of an abelian quotient G of F. The functions $f(n) = \frac{5}{2} r\sqrt{n \ln n}$ and $g(n) = n \ln n$ satisfy Assumption 2.1.*

Proof. Proposition 3.5 with $c = \frac{5}{2}$ tells us that $\sharp\{w \in A^* \,|\, |w|_A = n,\ |w|_G > f(n)\} = O\big(\frac{(2r)^n}{g(n)}\big)$, so the first requirement in Assumption 2.1 is fulfilled. For the second, note that it is immediate if G is finite (and so, $D(2n)$ bounded) and if G is virtually cyclic (and so, $D(2n)$ growing linearly) because $\frac{D(2n)}{g(\lfloor \frac{n}{2} \rfloor)}$ tend to zero in these cases. Otherwise, it is well known that the Dehn function of G is quadratic, i.e. there exist positive constants L_1 and L_2 such that $L_1 n^2 \leqslant D(n) \leqslant L_2 n^2$. Then, straightforward calculations show that $\frac{D(2n)}{g(\lfloor \frac{n}{2} \rfloor)} = O(D(4\lceil f(n) \rceil))$, for $n \geqslant 4$. □

In this situation, Theorem 2.3 allows us to deduce the following upper bound for the open spherical mean Dehn function of an abelian group.

Theorem 4.2 *Let $A = \{a_1, \ldots, a_r\}$, F be the free group on A, and $\langle A \mid R \rangle$ be a finite presentation of an abelian quotient G of F. Then,*

$$D_{osmean}(n) = O\big(n(\ln n)^2\big).$$

Proof. In our situation, Theorem 2.3 and Corollary 4.1 ensures that $D_{\text{osmean}}(n) = O\big(h(n)\big)$ for every non-decreasing function $h\colon \mathbb{N} \to \mathbb{R}^+$ satisfying $h(\lfloor \frac{n}{2} \rfloor) + h(\lceil \frac{n}{2} \rceil) + D(4\lceil \frac{5}{2} r\sqrt{n \ln n}\rceil) \leqslant h(n)$ for $n \gg 0$. And this is the case of the function $h(n) = Kn(\ln n)^2$, where K is a constant satisfying $D(4\lceil \frac{5}{2} r\sqrt{n \ln n}\rceil) \leqslant Kn \ln n$ (which does exist because $D(n)$ is at most quadratic). Indeed, since the second derivative of $h(x) = Kx(\ln x)^2$ is positive and bounded above by $4K$, it is straightforward to see that $h(\lfloor \frac{n}{2} \rfloor) + h(\lceil \frac{n}{2} \rceil) \leqslant 2h(\frac{n}{2}) + 2K$ (use Taylor expansion of degree 2, for $h(x)$ around $\frac{n}{2}$). Finally, it is easy to see that $2K\frac{n}{2}(\ln \frac{n}{2})^2 + 2K + Kn\ln n \leqslant Kn(\ln n)^2$ for $n \gg 0$. (In fact, one can show that any function growing asymptotically more slowly does not satisfy the required inequality). □

As mentioned at the end of Section 2, in order to estimate the spherical mean Dehn function, we need some more information from the presentation of G, namely how the terms $\mathcal{N}_v(n)$ depend on the vertex v. For abelian groups, this can be deduced from the following more general result.

Theorem 4.3 [6, Chapter VI.5]. *Let $A = \{a_1, \ldots, a_r\}$, F be the free group on A, and $\langle A \mid R \rangle$ be a finite presentation of a virtually nilpotent quotient G of F. Then,*

$$\max_{v \in \Gamma(G)} \{\mathcal{N}_v(n)\} = O\Big(\frac{(2r)^n}{n^{d/2}}\Big),$$

where d is the degree of the (polynomial) growth function of G. Moreover, there exists another constant $L > 0$ such that

$$\mathcal{N}_e(n) \geqslant L\,\frac{(2r)^n}{n^{d/2}},$$

for every even $n \geqslant 2$.

Regardless of the meaning of d (which is very significant within the group G, but is not relevant for the present computations) the previous result allows us to transfer our upper bound to the spherical mean Dehn function.

Theorem 4.4 *Let $A = \{a_1, \ldots, a_r\}$, F be the free group on A, and $\langle A \mid R \rangle$ be a finite presentation of an abelian quotient G of F. Then,*

$$D_{smean}(n) = O\big(n(\ln n)^2\big).$$

Proof. Using the present notation, we have $D_{\text{smean}}(n) = \frac{\mathcal{A}_e(n)}{\mathcal{N}_e(n)}$. We now estimate the numerator again by cutting paths on two halves. Let P be the set of all closed paths in $\Gamma(G)$, based at e and having length n. As in the proof of Proposition 2.2, break every $\gamma \in P$ into two parts, $\gamma = \gamma_1\gamma_2$ with $|\gamma_1|_A = \lfloor \frac{n}{2} \rfloor$ and $|\gamma_2|_A = \lceil \frac{n}{2} \rceil$, and denote by u the *middle* point, $\tau\gamma_1 = u = \iota\gamma_2$. We have

$$\text{area}(\gamma) = \text{area}(\gamma_1\gamma_2) \leqslant \text{area}(\gamma_1\widetilde{\gamma}_1^{-1}) + \text{area}(\widetilde{\gamma}_1\gamma_2) = \text{area}(\gamma_1) + \text{area}(\gamma_2^{-1}).$$

Now, taking into account that $|u|_G \leqslant \lfloor \frac{n}{2} \rfloor$, and applying Theorem 4.3, we have

$$\begin{aligned}
\mathcal{A}_e(n) &= \sum_{\gamma \in P} \operatorname{area}(\gamma) \\
&\leqslant \sum_{\gamma \in P} \operatorname{area}(\gamma_1) + \sum_{\gamma \in P} \operatorname{area}(\gamma_2^{-1}) \\
&= \sum_{0 \leqslant |u|_G \leqslant \lfloor n/2 \rfloor} \mathcal{A}_u(\lfloor \tfrac{n}{2} \rfloor)\mathcal{N}_u(\lceil \tfrac{n}{2} \rceil) + \sum_{0 \leqslant |u|_G \leqslant \lfloor n/2 \rfloor} \mathcal{N}_u(\lfloor \tfrac{n}{2} \rfloor)\mathcal{A}_u(\lceil \tfrac{n}{2} \rceil) \\
&\leqslant \max_{u \in \Gamma(G)} \{\mathcal{N}_u(\lceil n/2 \rceil)\} \cdot \sum_{0 \leqslant |u|_G \leqslant \lfloor n/2 \rfloor} \mathcal{A}_u(\lfloor \tfrac{n}{2} \rfloor) + \max_{u \in \Gamma(G)} \{\mathcal{N}_u(\lfloor n/2 \rfloor)\} \cdot \sum_{0 \leqslant |u|_G \leqslant \lfloor n/2 \rfloor} \mathcal{A}_u(\lceil \tfrac{n}{2} \rceil) \\
&\leqslant \max_{u \in \Gamma(G)} \{\mathcal{N}_u(\lceil n/2 \rceil)\} \cdot \mathcal{A}(\lfloor \tfrac{n}{2} \rfloor) + \max_{u \in \Gamma(G)} \{\mathcal{N}_u(\lfloor n/2 \rfloor)\} \cdot \mathcal{A}(\lceil \tfrac{n}{2} \rceil) \\
&\leqslant M \tfrac{(2r)^{\lceil n/2 \rceil}}{\lceil n/2 \rceil^{d/2}} \mathcal{A}(\lfloor \tfrac{n}{2} \rfloor) + M \tfrac{(2r)^{\lfloor n/2 \rfloor}}{\lfloor n/2 \rfloor^{d/2}} \mathcal{A}(\lceil \tfrac{n}{2} \rceil),
\end{aligned}$$

for an appropriate constant M. Finally, by applying again Theorems 4.3 and 4.2, and collecting together all the constants, we conclude

$$\begin{aligned}
D_{\mathrm{smean}}(n) = \frac{\mathcal{A}_e(n)}{\mathcal{N}_e(n)} &\leqslant \frac{M \frac{(2r)^{\lceil n/2 \rceil}}{\lceil n/2 \rceil^{d/2}} \mathcal{A}(\lfloor \frac{n}{2} \rfloor) + M \frac{(2r)^{\lfloor n/2 \rfloor}}{\lfloor n/2 \rfloor^{d/2}} \mathcal{A}(\lceil \frac{n}{2} \rceil)}{L \frac{(2r)^n}{n^{d/2}}} \\
&\leqslant \tfrac{M}{L} \big(\tfrac{n}{\lfloor \frac{n}{2} \rfloor} \big)^{d/2} \frac{(2r)^{\lceil n/2 \rceil} \mathcal{A}(\lfloor \frac{n}{2} \rfloor) + (2r)^{\lfloor n/2 \rfloor} \mathcal{A}(\lceil \frac{n}{2} \rceil)}{(2r)^n} \\
&\leqslant \tfrac{M}{L} \cdot 3^{d/2} \big(\tfrac{\mathcal{A}(\lfloor \frac{n}{2} \rfloor)}{(2r)^{\lfloor \frac{n}{2} \rfloor}} + \tfrac{\mathcal{A}(\lceil \frac{n}{2} \rceil)}{(2r)^{\lceil \frac{n}{2} \rceil}} \big) \\
&= \tfrac{M}{L} \big(D_{\mathrm{osmean}}(\lfloor \tfrac{n}{2} \rfloor) + D_{\mathrm{osmean}}(\lceil \tfrac{n}{2} \rceil) \big) \\
&= O\big(n(\ln n)^2\big).
\end{aligned}$$

A remark about the parity of the closed paths in $\Gamma(G)$ needs to be made here, since we have used the second part of Theorem 4.3 for an arbitrary n, while it was stated only for the even ones. If all the relations R in our presentation have even length, then all closed paths also have even length, and $D_{\mathrm{smean}}(n) = 0$ for every odd n, by convention. In this case, the above computations form a complete proof of the Theorem, understanding n even everywhere.

Otherwise, let γ_0 be a closed path in $\Gamma(G)$ of the smallest possible odd length, say n_0. Then for every closed path γ of even length n, $\gamma_0\gamma$ is again a closed path, now of odd length $n + n_0$. This proves that $\mathcal{N}_e(n + n_0) \geqslant \mathcal{N}_e(n) \geqslant L \frac{(2r)^n}{n^{d/2}}$. Adjusting the constants appropriately, this shows that the assumption "n even" in the second part of Theorem 4.3 can be removed in this case, assuming $n \geq n_0$. Hence, the proof is complete. □

Finally, a similar result is true for the mean Dehn function.

Theorem 4.5 *Let $A = \{a_1, \ldots, a_r\}$, F be the free group on A, and $\langle A \mid R \rangle$ be a finite presentation of an abelian quotient G of F. Then,*

$$D_{mean}(n) = O\big(n(\ln n)^2\big).$$

Proof. This follows immediately from Theorem 4.4 and Proposition 2.4, since $n(\ln n)^2$ is an increasing function. □

Acknowledgments

The authors thank the Centre de Recerca Matemàtica at Barcelona for the warm hospitality received during the fall semester of 2004, when most of this paper was written. The first author gratefully acknowledges support by a grant from the Complex Integration Projects of SBRAS N 1.9, and by the INTAS grant N 03-51-3663. The second author was jointly funded by the MEC (Spain) and the EFRD (EC) through project BFM2003-06613.

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