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FREE-BY-CYCLIC GROUPS HAVE SOLVABLE CONJUGACY PROBLEM

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Abstract

We show that the conjugacy problem is solvable in [finitely generated free]-by-cyclic groups, by using a result of O. Maslakova that one can algorithmically find generating sets for the fixed subgroups of free group automorphisms, and one of P. Brinkmann that one can determine whether two cyclic words in a free group are mapped to each other by some power of a given automorphism. We also solve the power conjugacy problem and give an algorithm to recognize if two given elements of a finitely generated free group are twisted conjugated to each other with respect to a given automorphism.

1. *Introduction*

A *free-by-cyclic* group is a group G having a free normal subgroup F with cyclic quotient $C = G/F$. If F can be chosen to be finitely generated then G is a *[f.g. free]-by-cyclic* group (note that the parenthesis are relevant here since surface groups are both free-by-cyclic and finitely generated, but most of them are not [f.g. free]-by-cyclic).

We shall be concerned with [f.g. free]-by-cyclic groups. Let $\{x_1, \dots, x_n\}$ be a basis for F and let t be a pre-image in G of a generator of C. Right conjugation by t in G induces an automorphism of F, which we denote ϕ. The following is then a presentation for G

$$\langle x_1, \dots, x_n, t \ : \ t^{-1}x_i t = x_i\phi, \ t^m = h\rangle,$$

where m is the cardinal of C and h is some element in F, understanding that the relation $t^m = h$ is not present when $m = \infty$. Note that we shall write ϕ on the right so that the image of $w \in F$ under ϕ will be denoted $w\phi$.

To refer such group and such presentation we shall use the notation M_ϕ. It is well known that, up to isomorphism, M_ϕ does not depend on ϕ but only on the conjugacy class in $Out(F)$ of its induced outer automorphism. Our main result is the following

THEOREM 1.1. *The conjugacy problem in [f.g. free]-by-cyclic groups is solvable.*

For some special cases, this result is already known. The automorphism ϕ is said to have no periodic conjugacy classes if one cannot find an integer $k \neq 0$ and elements $g, h \in F$ such that $1 \neq g\phi^k = h^{-1}gh$. If C is infinite, this is equivalent to saying that M_ϕ has no $\mathbb{Z} \oplus \mathbb{Z}$ subgroups and hence, by [**8**], [**3**] and [**4**], that M_ϕ is hyperbolic. Also, if C is finite, the group M_ϕ is virtually free and hence hyperbolic. In all these cases, then, it is well known that M_ϕ has solvable conjugacy problem.

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But, clearly, not all [f.g. free]-by-cyclic groups are hyperbolic. It has even been announced, in [**7**], that they fail to be automatic, in general.

Some other partial results are already known in this direction. For example, in the preprint [**2**] the authors consider the case where some power of ϕ is an inner automorphism, and give an algorithm to decide if two given elements in M_ϕ of the form tu, tv for $u, v \in F$ are conjugated by some element in F.

Our proof of Theorem 1.1 will work, in general, for arbitrary [f.g. free]-by-cyclic groups, including the previously known particular cases. The algorithm provided also computes a conjugating element, when it exists. Our proof relies on the following recent theorems.

Theorem 1.2 Maslakova, [**16**]. *There exists an algorithm to compute a finite generating set for the fixed point subgroup of an arbitrary automorphism of a free group of finite rank.*

Theorem 1.3 Brinkmann, [**9**]. *Given a finitely generated free group F, two elements u, v of F and an automorphism ϕ of F it is decidable whether there exists an integer k such that $u\phi^k$ is conjugate to v.*

Our solution to the conjugacy problem for M_ϕ proceeds by showing that, in the light of Theorem 1.3, it can be reduced to the *twisted conjugacy problem* for F. Then, we solve this classical problem providing an algorithm to recognize Reidemeister classes with respect to automorphisms of finitely generated free groups.

Let G be an arbitrary group and ϕ an automorphism of G. Two elements $u, v \in G$ are said to be *ϕ-twisted conjugated*, denoted $u \sim_\phi v$, if there exists $g \in G$ such that $(g\phi)^{-1}ug = v$. The equivalence relation $\sim_\phi$ was first introduced by Reidemeister in [**18**], and has an important role in Nielsen fixed point theory. A couple of interesting references are [**13**], where it is proven that the number of ϕ-twisted conjugacy classes is always infinite when G is hyperbolic, and [**11**], where Problem 3(i) in the Open Problem section asks for an algorithm recognizing ϕ-twisted conjugacy classes.

For a given $\phi \in Aut(G)$, it is said that the *ϕ-twisted conjugacy problem* is solvable in G if, for any elements $u, v \in G$, we can algorithmically decide if $u \sim_\phi v$ (for example, the id-twisted conjugacy problem is the standard conjugacy problem in G). And it is said that the *twisted conjugacy problem* is solvable in G if the ϕ-twisted conjugacy problem is solvable for any $\phi \in Aut(G)$. This twisted conjugacy problem is also part of a more general problem posted by G. Makanin in Question 10.26(a) of [**20**].

We first prove the following.

Proposition 1.4. *Let F be a finitely generated free group. If the twisted conjugacy problem is solvable in F then the standard conjugacy problem is solvable in M_ϕ, for every $\phi \in Aut(F)$.*

Then, we give a solution for the twisted conjugacy problem in a finitely generated free group F.

Theorem 1.5. *Let F be a finitely generated free group. The twisted conjugacy problem is solvable in F.*

Now, Theorem 1.1 follows immediately from Proposition 1.4 and Theorem 1.5.
Finally, in the last section we develop a few technical lemmas that will allow us to extend Theorem 1.1 (essentially with the same proof) to the following result.

THEOREM 1.6. *The power conjugacy problem in [f.g. free]-by-cyclic groups is solvable.*

In an arbitrary group, two elements u and v are said to be *power conjugated* when there exist integers p, q such that u^p and v^q are non-trivial and conjugated to each other in the group. The *power conjugacy problem* in a group consists on deciding whether two given elements are power conjugated (and finding such exponents and conjugating element if they exist). An early reference for this concept can be found in [**1**].

Before going into the details of the algorithm, we make some remarks.
M. Lustig has a recent series of two preprints, [**15**], indicating a solution to the conjugacy problem in $Aut(F)$ and $Out(F)$. As a consequence, he obtains also an algorithm for computing the fixed subgroup of any automorphism of F, thus providing an alternative proof for Theorem 1.2. It also seems that these preprints implicitly contain a solution for the twisted conjugacy problem in F.
It was pointed out to us by Ilya Kapovich that many one-relator groups are [f.g. free]-by-cyclic (one can prove this by imposing a few assumptions on the relator). It seems that there is evidence to think that these assumptions are quite weak, meaning that most relators satisfy them and, hence, most of 1-relator groups fall in the family of groups considered in this paper. However, this has not been expressed yet in a precise form at this time.
It is worth mentioning that in Chapter 3 of [**17**] there is an explicit construction of a [f.g. free]-by-[f.g. free] group with unsolvable conjugacy problem. So, Theorem 1.1 is no longer true if we replace “cyclic” by “finitely generated free”. However, three of the four authors of the present paper have the preprint [**6**] where the techniques developed here are extended to a bigger class of groups and, in particular, a characterization of the solvability of the conjugacy problem for [f.g. free]-by-[f.g. free] groups is obtained.
Finally, we state the following lemma for later use.

LEMMA 1.7. *Let ϕ be an automorphism of a free group F. Then, any ϕ-twisted conjugacy class in F is a union of ϕ-orbits.*

Proof. It is sufficient to prove that if two elements from F lie in the same ϕ-orbit, then they lie in the same ϕ-twisted conjugacy class. By induction, this reduces to prove that, for every $u \in F$, $u \sim_\phi u\phi$. And this fact is obvious since $u = (u\phi)^{-1}(u\phi)u$. □

2. *The conjugacy problem*

First note that, using the relations $wt = t(w\phi)$ and $wt^{-1} = t^{-1}(w\phi^{-1})$ for $w \in F$, every element in M_ϕ can be algorithmically re-written as a word of the form $t^r u$, where r is an integer and $u \in F$. In the case where C is finite and we also have the relation $t^m = h$, we may further assume that $0 \leq r \leq m-1$. In either case, we get a unique representation for elements of M_ϕ, which is algorithmically computable from a given arbitrary word on the generators.

If we conjugate $t^r u$ by an arbitrary element $t^k g$, we obtain

$$(t^k g)^{-1}(t^r u)(t^k g) = t^r (g\phi^r)^{-1} t^{-k} u t^k g = t^r (g\phi^r)^{-1}(u\phi^k) g.$$

Hence, two elements in M_ϕ, say $t^r u$ and $t^s v$ (with $0 \leq r, s \leq m-1$ in the case where $|C| = m < \infty$), are conjugate in M_ϕ if and only if $r = s$ and $v \sim_{\phi^r} (u\phi^k)$ for some integer k. This is the key fact in the following discussion.

Proof of Proposition 1.4. Suppose two elements in M_ϕ are given, say $t^r u$ and $t^s v$. We have to decide if they are conjugate to each other in M_ϕ, and find a conjugating element if it exists.

We first deal with the case where $r = 0$. Note that u is only conjugate in M_ϕ to other elements v of the base group F. Moreover, u is conjugate to v in M_ϕ if and only if, some power of the automorphism ϕ, maps u to a conjugate of v. This is decidable by Theorem 1.3, so we can decide if $u, v \in F$ are conjugate in M_ϕ.

For the case $r \neq 0$ note that, by Lemma 1.7, $u\phi^k \sim_{\phi^r} u\phi^{k \pm r}$. Hence, $t^r u$ and $t^s v$ are one conjugate to the other in M_ϕ if, and only if, $r = s$ and $v \sim_{\phi^r} (u\phi^k)$ for some integer $0 \leq k \leq |r| - 1$. Thus, a solution for the twisted conjugacy problem in F provides a solution for the standard conjugacy problem in M_ϕ. □

Proof of Theorem 1.5. Let ϕ be an automorphism of F, and suppose $u, v \in F$ are given. We need to algorithmically decide whether $u \sim_\phi v$.

Choose a free basis for F and, adding a new letter z, we get a free basis for $F' = F * \langle z \rangle$. Let $\phi' \in Aut(F')$ be the extension of ϕ defined by $z\phi' = uzu^{-1}$. Let γ_y denote the inner automorphism of F' given by right conjugation by $y \in F'$, $x\gamma_y = y^{-1}xy$.

We claim that $u \sim_\phi v$ if, and only if, $Fix\,(\phi'\gamma_v)$ contains an element of the form $g^{-1}zg$ for some $g \in F$ (and, in this case, g itself is a valid ϕ-twisted conjugating element). In fact, suppose that $v = (g\phi)^{-1}ug$ for some $g \in F$. Then,

$$\begin{aligned} (g^{-1}zg)\phi'\gamma_v &= v^{-1}(g\phi)^{-1}uzu^{-1}(g\phi)v \\ &= g^{-1}u^{-1}(g\phi)(g\phi)^{-1}uzu^{-1}(g\phi)(g\phi)^{-1}ug \\ &= g^{-1}zg. \end{aligned}$$

Conversely, if $g^{-1}zg$ is fixed by $\phi'\gamma_v$ for some $g \in F$, then

$$g^{-1}zg = (g^{-1}zg)\phi'\gamma_v = v^{-1}(g\phi)^{-1}uzu^{-1}(g\phi)v$$

and so, $gv^{-1}(g\phi)^{-1}u$ commutes with z. This implies $gv^{-1}(g\phi)^{-1}u = 1$, since this word contains no occurrences of z. Hence, $v = (g\phi)^{-1}ug$ and $u \sim_\phi v$ (with g being a ϕ-twisted conjugating element).

Since, by Theorem 1.2, we can algorithmically find a generating set for $Fix\,(\phi'\gamma_v)$, we can also decide if this subgroup contains an element of the form $g^{-1}zg$ for some $g \in F$. One can, for example, look at the corresponding (finite) core-graph for $Fix\,(\phi'\gamma_v)$ (algorithmically computable from a set of generators) and see if there is some loop labelled z at some vertex connected to the base-point by a path whose label does not use the letter z. If this is the case, the label of such a path provides the g, i.e. the required ϕ-twisted conjugating element.

(It is not difficult to show that $Fix\,(\phi'\gamma_v)$ contains an element of the form $g^{-1}zg$ if, and only if, it contains some word involving the letter z; and, in this case, the longest initial F-segment in such a word provides the ϕ-twisted conjugating element. With this observation, one can slightly simplify the algorithm given, by just checking to see whether any of the generators of $Fix\,(\phi'\gamma_v)$ involve z.) □

3. *The power conjugacy problem*

With the help of a few technical lemmas, the argument given to solve the conjugacy problem in M_ϕ also works, in much the same way, to solve the power conjugacy problem.

Theorem 1.2 can be extended to consider periodic subgroups. Recall that, given an automorphism ϕ of F, the *periodic subgroup* of ϕ is the subgroup

$$Per\,\phi = \{w \in F : w\phi^k = w \text{ for some } k > 0\} = \cup_{k=1}^{\infty} Fix\,\phi^k.$$

PROPOSITION 3.1. *There exists an algorithm to compute a finite generating set for the periodic subgroup of any given automorphism, ϕ, of a finitely generated free group F. More precisely, there exists a computable integer p_0 (even independent of ϕ) such that $Per\,\phi = Fix\,\phi^{p_0}$.*

Proof. It is well known that, for a finitely generated free group F of rank $n \geq 0$, the group $Aut(F)$ has bounded torsion (Stallings first proved this in [**19**]). What we need here is a computable integer p_0 (only depending on n) such that the order of any finite order element in $Aut(F)$ divides p_0. A possible direct proof follows (see [**14**] and [**12**] for better bounds that can possibly reduce the complexity of our algorithm).

Clearly, if $n = 0$ or $n = 1$, we can take $p_0 = 2$. So, let us assume $n \geq 2$.

In this case, we can invoke Theorem 2.1 of [**10**], which implies that every finite order element of $Out(F)$ can be realised as a graph automorphism of a finite graph Z with rank n. Deleting the degree 1 and degree 2 vertices in Z, we can assume that Z contains no such vertices. It is easy to see then that Z has at most $3n-3$ edges, which total to a maximum of $6n-6$ oriented edges. Hence, every finite order element of $Out(F)$ has order dividing $p_0 = (6n-6)!$. The same is true for $Aut(F)$, since the natural map $Aut(F) \to Out(F)$ has torsion free kernel. By Corollaries 3.6 and 3.7 of [**19**], there exists $s \geq 1$ such that $Per\,\phi = Fix\,\phi^s$ (we assume further that s is minimal possible). In particular, $Per\,\phi$ has rank $r \leq n$ (see [**5**]), and ϕ restricts to an automorphism $\phi' \in Aut(Per\,\phi)$ of order s. So, s either divides 2 (if $r = 0, 1$) or $(6r-6)!$ (otherwise). In any case, s divides $p_0 = (6n-6)!$. So, $Per\,\phi = Fix\,\phi^{p_0}$.

Finally, using Theorem 1.2, we are done. □

Let ϕ be an automorphism of a finitely generated free group F.

For any $p \geq 1$, and any $w \in F$, we define $w_{\phi,\,p} = (w\phi^{p-1})(w\phi^{p-2})\cdots(w\phi)w$. This notation will be useful because, for every integer r and every $u \in F$, we have $(t^r u)^p = t^{rp} u_{\phi^r,\,p}$ in M_ϕ. Note that $w_{\mathrm{id},\,p} = w^p$. Note also that, for every automorphism ψ commuting with ϕ, we have $w_{\phi,\,p}\psi = (w\psi)_{\phi,\,p}$. Also, $u \sim_\phi v$ implies $u\psi \sim_\phi v\psi$.

LEMMA 3.2. *Let $u, v \in F$. If $u \sim_\phi v$ then $u_{\phi,\,p} \sim_{\phi^p} v_{\phi,\,p}$ for every $p \geq 1$.*

Proof. Assume the existence of an element $g \in F$ satisfying $(g\phi)^{-1}ug = v$. Then, applying ϕ^i on both sides, we obtain $(g\phi^{i+1})^{-1}(u\phi^i)(g\phi^i) = v\phi^i$. Now, multiplying all these equations,

$$(g\phi^p)^{-1}u_{\phi,\,p}g = \Pi^0_{i=p-1}(g\phi^{i+1})^{-1}(u\phi^i)(g\phi^i) = \Pi^0_{i=p-1}v\phi^i = v_{\phi,\,p}.$$

This proves that $u_{\phi,\,p} \sim_{\phi^p} v_{\phi,\,p}$ (with the same twisted conjugating element g). □

Adapting the proof of Theorem 1.5, we can obtain the following technical result.

LEMMA 3.3. *Given $u, v \in F$, one can algorithmically decide if $u_{\phi,\,p} \sim_{\phi^p} v_{\phi,\,p}$ for some $p \geq 1$.*

Proof. As before, add a new generator z to F and consider the extension $\phi' \in Aut(F * \langle z\rangle)$ of ϕ given by $z\phi' = uzu^{-1}$. Exactly the same arguments as above show now that, for $p \geq 1$ and $g \in F$, $v_{\phi,\,p} = (g\phi^p)^{-1}u_{\phi,\,p}g$ if and only if $g^{-1}zg \in Fix\,(\phi'\gamma_v)^p$ (to do this computation, note that $(\phi'\gamma_v)^p = \phi'^p\gamma_{v_{\phi,\,p}}$ and $z\phi'^p = u_{\phi,\,p}zu^{-1}_{\phi,\,p}$). So, we are done by invoking Proposition 3.1 (and the computability of p_0 there ensures that we can compute the value of p here). □

Now, we can adapt the proof of Proposition 1.4 to solve the power conjugacy problem in M_ϕ.

Proof of Theorem 1.6. Suppose we are given two elements, $t^r u$ and $t^s v$, from M_ϕ, $r,\, s \in \mathbb{Z}$, $u, v \in F$. We need to decide whether they are power conjugated in M_ϕ. Note that, if these two elements have infinite order, this is the same as deciding whether there exist non-zero exponents p and q such that $(t^r u)^p$ and $(t^s v)^q$ are conjugated to each other in M_ϕ.

As before, we deal first with the case $r = s = 0$. Here, given $1 \neq u, v \in F$ we have to decide whether for some integers $p, q \neq 0$, u^p is mapped to a conjugate of v^q by some power of ϕ. In a free group we can algorithmically find *roots* of non-trivial elements. That is, there is a unique element $\hat{u}$ of F such that u is a positive power of $\hat{u}$, and $\hat{u}$ is not itself a proper power. Similarly, there exists a root $\hat{v}$, for v. Thus there exist computable integers k_1, k_2 such that $u^p = \hat{u}^{k_1}$ and $v^q = \hat{v}^{k_2}$.

Since roots are unique in free groups, u^p is mapped to a conjugate of v^q by some power of ϕ if, and only if, $\hat{u}$ is mapped to a conjugate of $\hat{v}^\epsilon$, for $\epsilon = \pm 1$ and $k_1 = \epsilon k_2$. Thus, we are done by invoking Theorem 1.3 applied to the roots of u and v.

Now, if the cyclic part of M_ϕ is finite (i.e. $|C| = m < \infty$), then any element of M_ϕ raised to the power m lies in F. Moreover, $t^r u$ has infinite order if and only if, $(t^r u)^m \neq 1$. Thus, if $t^r u$ and $t^s v$ are to be power conjugated, then either $(t^r u)^m = 1 = (t^s v)^m$ or $1 \neq (t^r u)^m, (t^s v)^m \in F$. In the former case we can solve the power conjugacy problem by finitely many checks of the standard conjugacy problem for M_ϕ. In the latter case, we are done by the argument in the preceding paragraph, applied to the pair $(t^r u)^m, (t^s v)^m$.

So, we can restrict our attention to the case $m = \infty$. In particular, M_ϕ is torsion-free.

By applying the future algorithm twice (once for the pair of elements $t^r u$, $t^s v$, and again for $t^r u$, $(t^s v)^{-1}$) we may restrict our attention to positive exponents, p, q. Note that, if for some integers $p, q \geq 1$, $(t^r u)^p = t^{rp}u_{\phi^r,\,p}$ and $(t^s v)^q = t^{sq}v_{\phi^s,\,q}$ are conjugated to each other in M_ϕ, then $rp = sq$. In particular, $r = 0$ if and only if $s = 0$. And if both r and s are not zero then $t^r u$ and $t^s v$ are power conjugated in

M_ϕ if and only if $(t^r u)^s$ and $(t^s v)^r$ also are. Thus, our problem reduces to the case $r = s$.

Since we have dealt with the case $r = s = 0$ above, it remains to consider the situation where $r = s \neq 0$ (and hence, $p = q$). That is, we are given elements of the form $t^r u$ and $t^r v$ with $r \neq 0$, and we have to decide if there exists an integer $p \geq 1$ such that $(t^r u)^p$ and $(t^r v)^p$ are conjugate to each other in M_ϕ, i.e. such that $v_{\phi^r, p} \sim_{\phi^{rp}} (u_{\phi^r, p}\phi^k)$ for some integer k.

Here, we claim that $(u_{\phi^r, p}\phi^k) \sim_{\phi^{rp}} (u_{\phi^r, p}\phi^{k\pm r})$ for every $p \geq 1$. In fact, by Lemma 1.7, we have $u \sim_{\phi^r} (u\phi^{\pm r})$ so, using Lemma 3.2, $u_{\phi^r, p} \sim_{\phi^{rp}} (u\phi^{\pm r})_{\phi^r, p} = u_{\phi^r, p}\phi^{\pm r}$. Then, $(u_{\phi^r, p}\phi^k) \sim_{\phi^{rp}} (u_{\phi^r, p}\phi^{k\pm r})$, for every $k, p \in \mathbb{Z}$, $p \geq 1$. Thus it only remains to decide whether there exists an integer $p \geq 1$ such that $v_{\phi^r, p} \sim_{\phi^{rp}} (u_{\phi^r, p}\phi^k)$ for some integer $0 \leq k \leq |r| - 1$. But $(u_{\phi^r, p}\phi^k) = (u\phi^k)_{\phi^r, p}$ so, using Lemma 3.3 at most r times, we are done. □

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