Fixed subgroups are compressed in free groups

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Abstract

In this paper we prove that the fixed subgroup of an arbitrary family of endomorphisms ψ_i, $i \in I$, of a finitely generated free group F, is F-super-compressed. In particular, $r(\cap_{i\in I}\mathrm{Fix}\ \psi_i) \leq r(M)$ for every subgroup $M \leq F$ containing $\cap_{i\in I}\mathrm{Fix}\ \psi_i$. This provides new evidence towards the inertia conjecture for fixed subgroups of free groups. As a corollary, we show that, in the free group of rank n, every strictly ascending chain of fixed subgroups has length at most $2n$. This answers a question of G. Levitt.

1 Introduction

Let F be a free group. We write $r(F)$ to denote the rank of F, which is the minimal cardinality of a (free) generating set for F. Since every subgroup of a free group is free, the same notation applies to subgroups H of F. In the present paper, although some results are general, we will mostly deal with finitely generated free groups, i.e. free groups with finite rank.

Let endomorphisms of F act on the right, $x \mapsto x\psi$. Given an endomorphism ψ of F, its *fixed subgroup*, denoted Fix ψ, is the subgroup of elements in F fixed by ψ, $\mathrm{Fix}\ \psi = \{x \in F \mid x\psi = x\}$. If $\Psi = \{\psi_i \mid i \in I\}$ is a family of endomorphisms of F, we denote by $\mathrm{Fix}\ \Psi$ the subgroup of elements simultaneously fixed by all ψ_i, $\mathrm{Fix}\ \Psi = \cap_{i\in I}\mathrm{Fix}\ \psi_i$.

Following the terminology introduced in [10], we say that a subgroup $H \leq F$ is *1-endo-fixed* if $H = \mathrm{Fix}\ \psi$ for some endomorphism $\psi \in \mathrm{End}(F)$. And we say that H is *endo-fixed* if $H = \mathrm{Fix}\ \Psi$ for some family of endomorphisms $\Psi \subseteq \mathrm{End}(F)$. The concepts of *1-auto-fixed*, *1-mono-fixed*, *auto-fixed* and *mono-fixed* subgroups are analogously defined.

Bestvina-Handel [2] proved that, if $\psi\colon F \to F$ is an automorphism, then $r(\mathrm{Fix}\ \psi) \leq r(F)$, improving the previously celebrated result of Gersten [6] about finite generation of such fixed subgroups. Immediately after Bestvina and Handel announced their result, Imrich-Turner published [7], where they extended it to arbitrary endomorphisms. Some years latter, Dicks-Ventura [4] introduced the concept of inertia and showed that the fixed subgroup of an injective endomorphism of F is F-inert. A subgroup $H \leq F$ is called *F-inert* when $r(H \cap K) \leq r(K)$ for every $K \leq F$. This result was another extension of the Bestvina-Handel Theorem. Since the family of F-inert subgroups is closed under intersections, an easy corollary was that the rank of an arbitrary mono-fixed subgroup of F is also bounded above by the rank of F. However, in [4] problem 2, it was asked if fixed subgroups of arbitrary endomorphisms of F are necessarily F-inert, and in [17] it was conjectured that, in fact, they are. This is still open and we will refer to it as the "inertia conjecture".

Using inertia, Bergman [1] showed that the rank of an arbitrary endo-fixed subgroup of F is always bounded above by the rank of F, even without the injectivity hypothesis on the involved endomorphisms. This was the first evidence in favour of the inertia conjecture.

In [4] the concept of compression of a subgroup of F was introduced, being a necessary condition for its inertia. We re-estate it here along with some technical variations.

Let $F^{\mathrm{ab}} = F/F'$ be the abelianization of F and let $\pi\colon F \to F^{\mathrm{ab}}$ be the corresponding projection. For any given subgroup $H \leq F$, the *abelian rank of H with respect to F*, denoted $r^{\mathrm{ab}}(H; F)$, is defined as the rank of the free abelian group $H\pi = HF'/F'$, that is, the rank (as an abelian group) of the image of H under the global abelianization π. Note that, in general, this is not the same as $r(H^{\mathrm{ab}})$.

Definition 1.1 Let F be a finitely generated free group and $H \leq F$.

We say that H is *F-compressed* if $r(H) \leq r(K)$ for every $H \leq K \leq F$.

Similarly, H is called F-*strictly-compressed* when $r(H) < r(K)$ for every $H < K \leq F$.

Finally, H is called F-*super-compressed* if for any subgroup $H < K \leq F$, one has both $r(H) \leq r(K)$ and $r(H)+r^{\mathrm{ab}}(H;F) < r(K)+r^{\mathrm{ab}}(K;F)$. Equivalently, H is F-super-compressed if H is F-compressed and, for every $K \leq F$ strictly containing H but having the same rank, $r^{\mathrm{ab}}(H;F) < r^{\mathrm{ab}}(K;F)$ is satisfied.

Clearly, every F-strictly compressed and every F-super-compressed subgroups of F are F-compressed. Also, every F-inert subgroup of F is F-compressed, but it is not known if the converse is true. This was stated in [4] as Problem 1, and in [17] as the "compressed-inert conjecture".

The main result in the present paper is Theorem 3.4, where a simple argument (making use of theorems of Bergman, Takahasi, Martino-Ventura, Bestvina-Handel and Dyer-Scott) is given to prove that arbitrary endo-fixed subgroups of F are F-super-compressed. In particular, they are F-compressed, providing new evidence in support of the inertia conjecture.

For this purpose, we will also make use of the concept of algebraic extension. Following [8], a pair of subgroups $H \leq K$ of F is called an *algebraic extension* if H is not contained in any proper free factor of K. Note that, in general, if $\psi\colon F \to F$ is an endomorphism and $H \leq K$ is an algebraic extension, then $H\psi \leq K\psi$ need not be algebraic (while in fact it is, if ψ is an automorphism). A result of Takahasi [14] states that any finitely generated subgroup H of F has a finite number of algebraic extensions, i.e. there are only finitely many subgroups $K \leq F$ such that H is contained in K but not in any of its proper free factors. Simpler arguments for this result were recently given independently by Ventura [16], Margolis-Sapir-Weil [9] and Kapovich-Myasnikov [8]. This result will be crucial in the proof of our main result.

The structure of the present paper is the following. In section 2 we prove some general properties of the three concepts of compression for subgroups of free groups. In section 3 the main result (Theorem 3.4), namely the F-super-compression of endo-fixed subgroups of F, is proven. Finally, in section 4 we apply the results to better understanding ascending chains of endo-fixed subgroups of free groups, and to answer a question of G. Levitt.

2 Properties of compression

In this section we prove that the three concepts of compression behave well with respect to free products.

Lemma 2.1 *Let U, V be finitely generated subgroups of a free abelian group. Then,*

$$r(U+V) = r(U) + r(V) - r(U \cap V).$$

Proof. Noting that $r(U) = \dim_{\mathbb{Q}}(U \otimes_{\mathbb{Z}} \mathbb{Q})$, the formula follows from the corresponding result in linear algebra. □

A subgroup $R \leq F$ is called a *retract of* F when the identity $Id\colon R \to R$ extends to a homomorphism $\rho\colon F \to R$, called a *retraction*. For example, free factors of F are retracts of F.

Lemma 2.2 *Let F be a free group and let $R \leq F$ be a retract of F. For every $H \leq R$, $r^{\mathrm{ab}}(H; R) = r^{\mathrm{ab}}(H; F)$.*

Proof. The equality between the abelian ranks with respect to R and F is clear if we show $F' \cap R = R'$. One of the inclusions is obvious. To show the other, let $\rho\colon F \to R$ be a retraction and note that if the commutator of $x, y \in F$ lies in R then $[x, y] = [x, y]\rho = [x\rho, y\rho] \in R'$, since $x\rho, y\rho \in R$. Thus, $F' \cap R \leq R'$. □

Proposition 2.3 *Let F be a free group and $H = A * B$ a finitely generated subgroup.*

i) If H is F-compressed then A is also F-compressed.

ii) If H is F-strictly-compressed then A is also F-strictly-compressed.

iii) If H has the property that any proper extension, $H < K$, either satisfies $r(H) < r(K)$ or $r^{\mathrm{ab}}(H; F) < r^{\mathrm{ab}}(K; F)$, then A has the same property.

iv) If H is F-super-compressed then A is also F-super-compressed.

Proof. Let L be an arbitrary subgroup of F containing A. Consider $K = \langle L, B\rangle$, which is a subgroup of F containing H, and having rank $r(K) \leq r(L) + r(B)$.

Suppose that H is F-compressed. Then,

$$r(A) + r(B) = r(H) \leq r(K) \leq r(L) + r(B).$$

So, $r(A) \leq r(L)$. This proves (i).

Now, suppose that H is F-strictly-compressed. By (i), A is F-compressed and hence, to see (ii), it only remains to show that $r(A) < r(L)$ whenever $A < L$. Suppose then that $A < L$. If $A < L \leq A * B$ then A is a proper free factor of L and we are done. Otherwise, there exists $x \in L$, $x \notin A * B$. Thus $H < K$ and, by hypothesis,

$$r(A) + r(B) = r(H) < r(K) \leq r(L) + r(B).$$

So, $r(A) < r(L)$. This completes (ii).

Next suppose that for any subgroup K properly containing H, either $r(H) < r(K)$ or $r^{\mathrm{ab}}(H; F) < r^{\mathrm{ab}}(K; F)$. To prove the same for A consider a subgroup L strictly containing A. Now if $L \leq H = A * B$ then A is a proper free factor of L which implies that $r(A) < r(L)$ and we would be done. So suppose that L is not a subgroup of H and hence the subgroup $K = \langle L, B \rangle$ strictly contains H.

If $r(H) < r(K)$ then $r(A) + r(B) = r(H) < r(K) \leq r(L) + r(B)$, which implies that $r(A) < r(L)$.

If, on the other hand, $r^{\mathrm{ab}}(H; F) < r^{\mathrm{ab}}(K; F)$ then, using Lemma 2.1,

$$r(K\pi) = r(L\pi) + r(H\pi) - r(L\pi \cap H\pi),$$

where $\pi \colon F \to F^{\mathrm{ab}}$ denotes the abelianization map. But $A \leq L \cap H$, so

$$\begin{array}{rcl} r^{\mathrm{ab}}(K; F) & \leq & r^{\mathrm{ab}}(L; F) + r^{\mathrm{ab}}(H; F) - r^{\mathrm{ab}}(A; F) \\ & < & r^{\mathrm{ab}}(L; F) + r^{\mathrm{ab}}(K; F) - r^{\mathrm{ab}}(A; F). \end{array}$$

Hence, $r^{\mathrm{ab}}(A; F) < r^{\mathrm{ab}}(L; F)$. This completes (iii).

Finally, (iv) follows directly from (i) and (iii). □

Proposition 2.4 *Let F be a free group, and $A \leq H \leq F$, $B \leq K \leq F$ and $M \leq F$ be subgroups such that $F = H * K * M$.*

i) If A is H-compressed and B is K-compressed, then $A * B$ is F-compressed.

ii) If A is H-strictly-compressed and B is K-strictly-compressed, then $A * B$ is F-strictly-compressed.

iii) If A is H-super-compressed and B is K-super-compressed, then $A*B$ is F-super-compressed.

Proof. Let $A*B \leq L \leq F = H*K*M$. Writing $A \leq L_A = L \cap H \leq H$ and $B \leq L_B = L \cap K \leq K$, the Kurosh Subgroup Theorem ensures us the existence of a subgroup $L' \leq F$ such that $L = L_A * L_B * L'$.

Assume the hypothesis in (i). Then, $r(A) \leq r(L_A)$, $r(B) \leq r(L_B)$ and so,

$$r(A*B) = r(A) + r(B) \leq r(L_A) + r(L_B) + r(L') = r(L).$$

This proves (i).

The same argument works to prove (ii), with the additional observation that if $A*B < L$ then either $A < L_A$, or $B < L_B$ or $L' \neq 1$. Thus, by the hypothesis in (ii), either $r(A) < r(L_A)$, or $r(B) < r(L_B)$ or $L' \neq 1$. In this situation, the inequality in the above computation is strict, that is, $r(A*B) < r(L)$.

Finally, assume the hypothesis in (iii). Using (i) it only remains to prove that if $A*B < L$ and $r(A*B) = r(L)$ then $r^{\mathrm{ab}}(A*B;F) < r^{\mathrm{ab}}(L;F)$. In this situation, $r(A) = r(L_A)$, $r(B) = r(L_B)$ and $L' = 1$. But either $A < L_A$, or $B < L_B$ so, by the hypothesis, either $r^{\mathrm{ab}}(A;H) < r^{\mathrm{ab}}(L_A;H)$ or $r^{\mathrm{ab}}(B;K) < r^{\mathrm{ab}}(L_B;K)$. Thus, using Lemma 2.2,

$$\begin{aligned} r^{\mathrm{ab}}(A*B;F) &= r^{\mathrm{ab}}(A;F) + r^{\mathrm{ab}}(B;F) \\ &= r^{\mathrm{ab}}(A;H) + r^{\mathrm{ab}}(B;K) \\ &< r^{\mathrm{ab}}(L_A;H) + r^{\mathrm{ab}}(L_B;K) \\ &= r^{\mathrm{ab}}(L_A;F) + r^{\mathrm{ab}}(L_B;F) \\ &= r^{\mathrm{ab}}(L_A * L_B;F) \\ &= r^{\mathrm{ab}}(L;F). \ \square \end{aligned}$$

3 Compression of fixed subgroups

In [11], Martino-Ventura gave an explicit description of 1-auto-fixed subgroups of finitely generated free groups. We state it here for later use.

Theorem 3.1 (Martino-Ventura, [11]) *Let F be a (non-trivial) finitely generated free group and let $\psi \in \mathrm{Aut}(F)$ with* $\mathrm{Fix}\,\psi \neq 1$. *Then, there exist integers $r \geq 1$, $s \geq 0$, ψ-invariant non-trivial subgroups $K_1, \ldots, K_r \leq F$, primitive elements $y_1, \ldots, y_s \in F$, a subgroup $L \leq F$, and elements $1 \neq h'_j \in H_j = K_1 * \cdots * K_r * \langle y_1, \ldots, y_j \rangle$, $j = 0, \ldots, s-1$, such that*

$$F = K_1 * \cdots * K_r * \langle y_1, \ldots, y_s \rangle * L$$

and $y_j\psi = h'_{j-1}y_j$ *for* $j = 1, \dots, s$*; moreover,*

$$\mathrm{Fix}\ \psi = \langle w_1, \dots, w_r, y_1^{-1}h_0y_1, \dots, y_s^{-1}h_{s-1}y_s\rangle$$

for some non-proper powers $1 \neq w_i \in K_i$ *and* $1 \neq h_j \in H_j$ *such that* $h_j\psi = h'_jh_jh'^{-1}_j$, $i = 1, \dots, r$, $j = 0, \dots, s-1$. $\square$

Using Theorem 3.1 and the standard covering theory for graphs (see [13], [16] or [8]), we can extend Theorem 2.2 in [16] to arbitrary 1-auto-fixed subgroups of F. Namely, we will show that for any given automorphism $\psi\colon F \to F$, every subgroup of F strictly containing Fix ψ has either bigger rank or bigger abelian rank than those of Fix ψ. However, we shall need a technical lemma in order to accomplish our goal. After this, we will have all the ingredients to prove our main result, stating the F-super-compression of any endo-fixed subgroup of F.

Lemma 3.2 *Let* F *be a finitely generated free group, let* $\psi : F \to F$ *be an automorphism and suppose that* Fix ψ = Fix ψ^m *for all* $m \geq 1$*. Let a basis for* F *and* Fix ψ *be given as in Theorem 3.1. Then,* (Fix $\psi)\pi \leq H_0\pi$*, where* π *is the natural abelianization map* $\pi\colon F \to F^{\mathrm{ab}}$.

Proof. We call an element $w \in F$ *conjugacy-fixed by* ψ if $w\psi = w^g$ for some $g \in F$. For any given $w \in H_s$, we shall refer to the smallest index k such that $w \in H_k$ as the *height* of w. Note that then, $w \notin H_{k-1}$, i.e. the generator y_k occurs in the reduced expression of w.

In order to prove the lemma we shall in fact prove the stronger statement that if $w \in H_s$ is conjugacy-fixed by ψ then $w\pi \in H_0\pi$.

Let us argue by contradiction. So, let us assume the existence of some $j = 1, \dots, s$ for which there are conjugacy-fixed words in H_s with exponent sum of y_j being non-zero. Given such a j, we choose w of minimal height, say k, among those words. Clearly, $1 \leq j \leq k$.

Now observe that we may replace w by any of its cyclically reduced conjugates without changing any of its defining properties. Clearly, being conjugacy-fixed by ψ must remain unchanged, as must the exponent sum of y_j, and the height must remain unchanged due to the minimality assumption. Hence we shall assume that w is cyclically reduced and, in fact, after possibly taking an inverse, we may assume that the last letter of w is y_k. We now prove the following

Claim: If $u \in H_s$ is cyclically reduced of height $k \geq 1$ and the last letter of u is y_k, then $u\psi \in H_s$ is also cyclically reduced of height k with last letter equal to y_k.

Proof of claim. The word u can be written in the form

$$u = g_0 y_k^{\epsilon_1} g_1 y_k^{\epsilon_2} \cdots y_k^{\epsilon_{t-1}} g_{t-1} y_k,$$

where each $\epsilon_i = \pm 1$, each g_i (possibly trivial) is of height less than k (so, it belongs to H_{k-1}), and the product is reduced in the sense that if $\epsilon_i = -\epsilon_{i+1}$, then $g_i \neq 1$. As u is cyclically reduced, this last condition is understood to hold for subscripts modulo t (with $\epsilon_t = 1$). Note that we are using the fact that $H_k = H_{k-1} * \langle y_k \rangle$.

Now, if we apply ψ, (recall that $y_k \psi = h'_{k-1} y_k$ for some $h'_{k-1} \in H_{k-1}$) we get

$$u\psi = (g_0\psi)(h'_{k-1})^{\lambda_1} y_k^{\epsilon_1} (h'_{k-1})^{\mu_1} (g_1\psi)(h'_{k-1})^{\lambda_2} y_k^{\epsilon_2} (h'_{k-1})^{\mu_2} \cdots$$
$$\cdots (h'_{k-1})^{\lambda_{t-1}} y_k^{\epsilon_{t-1}} (h'_{k-1})^{\mu_{t-1}} (g_{t-1}\psi) h'_{k-1} y_k,$$

where $\lambda_i = \frac{\epsilon_i + 1}{2}$ and $\mu_i = \frac{\epsilon_i - 1}{2}$. Note that in the previous expression there is no cancellation between consecutive y_k's since, if $\epsilon_i = -\epsilon_{i+1}$, then the subword of $u\psi$ between them is either $g_i\psi$ or $(h'_{k-1})^{-1}(g_i\psi)h'_{k-1}$, which are both non-trivial. Since this holds modulo t, and H_{k-1} is ψ-invariant, we have shown that the image under ψ of a cyclically reduced word u in H_s, of height k and ending in y_k is another cyclically reduced word in H_s, of height k and ending in y_k. This concludes the proof of the claim.

Applying this claim to our element w, we have that $w\psi^m$ is cyclically reduced for all $m \geq 1$. And, since w is conjugacy-fixed, these are all conjugates. But w has only finitely many cyclically reduced conjugates, hence $w \in \text{Fix } \psi^m$ for some $m \geq 1$. So, $w \in \text{Fix } \psi$, by hypothesis. Now, it is easy to see that

$$\text{Fix } \psi \cap H_k = \text{Fix } \psi_{H_k} = \text{Fix } \psi_{H_{k-1}} * \langle y_k^{-1} h_{k-1} y_k \rangle$$

and, by the other hand, the exponent sum of y_j in $w \in \text{Fix } \psi \cap H_k$ is non-zero. Thus, $j \neq k$ that is, $1 \leq j < k$.

Moreover, the presence of w in $\text{Fix } \psi_{H_{k-1}} * \langle y_k^{-1} h_{k-1} y_k \rangle$ implies that either some word in $\text{Fix } \psi_{H_{k-1}}$ or the element h_{k-1} has non-zero exponent sum in y_j. In either case, we get a word, which is conjugacy-fixed by ψ, whose exponent sum in y_j is non-zero and of height less than k. This contradicts our minimal height assumption and hence concludes the proof. □

Proposition 3.3 *Let F be a finitely generated free group and let $\psi \colon F \to F$ be an automorphism. Then, for every K with $\text{Fix } \psi < K \leq F$, either $r(\text{Fix } \psi) < r(K)$ or $r^{\text{ab}}(\text{Fix } \psi; F) < r^{\text{ab}}(K; F)$.*

Fig. 1.

Proof. The result is clear if Fix $\psi = 1$. So, assume Fix $\psi \neq 1$.

Now, it is easy to see that, for every positive integer s, ψ restricts to a finite order automorphism of Fix ψ^s, whose fixed subgroup is Fix ψ itself. So, by a result of Dyer-Scott [5], Fix ψ is a free factor of Fix ψ^s. Thus by Proposition 2.3 (iii), it is sufficient to prove the current proposition for ψ^s for some positive integer s.

Now, by the main Theorem of [2], each Fix ψ^s has rank bounded by the rank of F. Therefore we can choose an s so that the rank of Fix ψ^s is maximal. Combining this with the fact that Fix ψ^s is a free factor of Fix ψ^{sm}, we have that Fix $\psi^{sm} =$ Fix ψ^s for every $m \geq 1$ (hence, we can apply Lemma 3.2 to ψ^s). Using the previous observation, we can assume that $s = 1$.

Consider the description of Fix ψ given in Theorem 3.1. Now, the argument works exactly as the proof given for Theorem 2.2 in [16]. The core graph Z corresponding to Fix ψ, with base point u and the corresponding labels on edges and circles, is depicted in Fig. 1. Let E' be the set of edges in Z whose deletion disconnects the graph; for $j = 1, \dots, s$ denote by e_j the edge in E' with label y_j. Observe that, since $h_{j-1} \in K_1 * \cdots * K_r * \langle y_1, \dots, y_{j-1} \rangle$, none of the edges in the component of $Z - E'$ containing ιe_j has label y_j.

By Theorem 1.7 in [16], it is enough to prove that any proper extension Fix $\psi < K$ given by a quotient $\overline{Z}$ of Z has either bigger rank or bigger abelian rank than those of Fix ψ. Fix one such K (and the corresponding $\overline{Z} \neq Z$), and let J be the set of those j such that some vertex in the component of $Z - E'$ containing ιe_j gets identified with some vertex of a different component of $Z - E'$.

Suppose $J \neq \emptyset$, and let j_0 be the largest element of J. Clearly, there is a closed path in the quotient graph $\overline{Z}$ based at $\overline{u}$ which determines an element, $x \in K$, crossing exactly one edge labelled y_{j_0}, and only once. Hence $x\pi \notin H_0\pi$. However, by Lemma 3.2, (Fix $\psi)\pi \leq H_0\pi$ which is a direct summand of F^{ab}. Thus, $r^{\mathrm{ab}}(\mathrm{Fix}\ \psi; F) < r^{\mathrm{ab}}(K; F)$.

This leaves the case where $J = \emptyset$, that is, where vertices in different components of $Z - E'$ remain unidentified in $\overline{Z} \neq Z$. In this case, $\overline{Z}$ is the same as Z with every component of $Z - E'$ replaced by some quotient of itself (with at least one of those quotients being proper). For every $i = 1, \ldots, r$ and $j = 0, \ldots, s-1$, the elements w_i and h_j are not proper powers, so the subgroups $\langle w_i \rangle$ and $\langle h_j \rangle$ are F-strictly compressed. Thus, by Proposition 2.4 (ii), $\langle w_1, \ldots, w_r \rangle$ is F-strictly compressed too. We deduce that $r(\mathrm{Fix}\ \psi) < r(K)$. $\square$

Note that Proposition 3.3 is not saying that 1-auto-fixed subgroups of F are F-compressed, since the statement leaves the possibility of the existence of some subgroup $\mathrm{Fix}\ \psi < K \leq F$ with bigger abelian rank but smaller rank than those of Fix ψ (in fact, in the case $J \neq \emptyset$ we have no control on the rank of the graph $\overline{Z}$). In the next theorem we give an argument showing this compression.

Theorem 3.4 *Let F be a finitely generated free group. Any endo-fixed subgroup of F is F-super-compressed.*

Proof. Let $\Psi \subseteq \mathrm{End}(F)$. First, we will prove that Fix Ψ is F-compressed. Then, using Proposition 3.3, we will extend this to show that it is F-super-compressed.

By [6], Fix Ψ is finitely generated. Let r be the minimum among the ranks of all those subgroups of F containing Fix Ψ. Note that $r \leq r(\mathrm{Fix}\ \Psi)$, and the equality holds if and only if Fix Ψ is F-compressed.

Consider $\mathcal{M} = \{M \leq F \mid \mathrm{Fix}\ \Psi \leq M,\ r(M) = r\} \neq \emptyset$. Observe that, by the minimality of r, every $M \in \mathcal{M}$ is an algebraic extension of Fix Ψ. Hence, by [8], [9], [14] or [16], $|\mathcal{M}| < \infty$.

By [10] Corollary 3.4, there exists φ (in the submonoid of $\mathrm{End}(F)$ generated by Ψ) such that Fix Ψ is a free factor of Fix φ. As in the proof of Proposition 3.3, using Dyer-Scott [5], we have that Fix φ is a free factor of Fix φ^s. Hence, Fix Ψ is a free factor of Fix φ^s for every $s \geq 1$.

Now choose an arbitrary $M \in \mathcal{M}$. Since the rank of a subgroup never increases when taking images, it is clear that $M_k = M\varphi^k \in \mathcal{M}$, for every $k \geq 0$. By the finiteness of $\mathcal{M}$, there exists an integer $k \geq 0$ and a positive integer $s \geq 1$ such that $M_{k+s} = M_k\varphi^s = M_k$. Then, φ^s restricts to an automorphism of M_k, say $\varphi^s_{M_k} \in \mathrm{Aut}(M_k)$. Now, using Bestvina-Handel Theorem,

$$r(M_k \cap \mathrm{Fix}\ \varphi^s) = r(\mathrm{Fix}\ \varphi^s_{M_k}) \leq r(M_k) = r.$$

But we noted above that Fix Ψ is a free factor of Fix φ^s. Hence, it is also a free factor of $M_k \cap \mathrm{Fix}\ \varphi^s$. Thus, $r(\mathrm{Fix}\ \Psi) \leq r$ that is, $r(\mathrm{Fix}\ \Psi) = r$. This means that Fix Ψ is F-compressed.

It remains to show $r^{\mathrm{ab}}(\mathrm{Fix}\ \Psi; F) < r^{\mathrm{ab}}(M; F)$, assuming $M \neq \mathrm{Fix}\ \Psi$.

In [15], Turner showed that the *stable image* of φ^s,

$$R = \cap_{i\geq 1} F(\varphi^s)^i = \cap_{i\geq 1} F\varphi^i,$$

is a retract of F where φ^s restricts to an automorphism. Let $\varphi^s_R \in \mathrm{Aut}(R)$ denote the restriction of φ^s to R. Observe that Fix φ^s_R = Fix φ^s since Fix $\varphi^s \leq R$.

Having proved that endo-fixed subgroups of F are F-compressed, and using Proposition 3.3, we see that Fix φ^s = Fix φ^s_R is R-super-compressed. Recall that Fix Ψ is a free factor of Fix φ^s and hence, by Proposition 2.3 (iv), Fix Ψ is also R-super-compressed.

Now, observe that M_k cannot be equal to Fix Ψ. For, if it were, then as φ acts as the identity on Fix Ψ which is a proper subgroup of M, the map $\varphi^k : M \to M_k$ would be a surjective map with non-trivial kernel between two free groups of the same rank. As finitely generated free groups are Hopfian, this cannot be the case and hence Fix $\Psi < M_k$.

But, since $r(\mathrm{Fix}\ \Psi) = r = r(M_k)$ and $M_k \leq R$, the R-super-compression of Fix Ψ implies that

$$r^{\mathrm{ab}}(\mathrm{Fix}\ \Psi; R) < r^{\mathrm{ab}}(M_k; R)$$

and, by Lemma 2.2,

$$r^{\mathrm{ab}}(\mathrm{Fix}\ \Psi; F) < r^{\mathrm{ab}}(M_k; F).$$

Finally, since $\varphi^k : M \to M_k$ induces a surjective homomorphism from MF'/F' to M_kF'/F', we have

$$r^{\mathrm{ab}}(\mathrm{Fix}\ \Psi; F) < r^{\mathrm{ab}}(M_k; F) \leq r^{\mathrm{ab}}(M; F).$$

This completes the proof. □

Corollary 3.5 *Let F be a finitely generated free group. Any endo-fixed subgroup of F is F-compressed.*

4 Ascending chains of fixed subgroups

In this last section, we use the information obtained before to analyze strictly ascending chains of endo-fixed subgroups of a finitely generated free group.

Theorem 4.1 *In a free group F of rank n, every strictly ascending chain of endo-fixed subgroups has length at most $2n$.*

Furthermore, there exist such chains of length $2n-1$, even using only 1-auto-fixed subgroups.

Proof. By the above Theorem 3.4, the function $r(-)+r^{\mathrm{ab}}(-;F)$ strictly increases in any step of such an ascending chain. And, clearly, its minimum and maximum values among endo-fixed subgroups of F, are 0 (for the trivial subgroup) and $2n$ (for F itself), respectively. So, strictly ascending chains of endo-fixed subgroups of F have length at most $2n$.

Now, we need to construct such a chain with length $2n-1$, and using only 1-auto-fixed subgroups. Let $\{x_1, \ldots, x_n\}$ be a basis of F. Given two integers $1 \leq p \leq q \leq n$, let $r, s, t \geq 0$ be such that $p = r$, $q = r+s$ and $n = r+s+t$. Define $\psi_{p,q}$ to be the automorphism of F given by

$$\begin{array}{rcll} \psi_{p,q} \colon F & \mapsto & F & \\ x_i & \mapsto & x_i, & i = 1, \ldots, r \\ x_j & \mapsto & x_1^j x_j, & j = r+1, \ldots, r+s \\ x_k & \mapsto & x_k^{-1}, & k = r+s+1, \ldots, r+s+t. \end{array}$$

It is not difficult to see that

$$\mathrm{Fix}\ \psi_{p,q} = \langle x_1, \ldots, x_r, x_{r+1}^{-1} x_1 x_{r+1}, \ldots, x_{r+s}^{-1} x_1 x_{r+s} \rangle.$$

We have $r^{\mathrm{ab}}(\mathrm{Fix}\ \psi_{p,q}; F) = r = p$ and $r(\mathrm{Fix}\ \psi_{p,q}) = r+s = q$. The following is a strictly ascending chain of 1-auto-fixed subgroups with length $2n-1$:

$$1 < \mathrm{Fix}\ \psi_{1,1} < \cdots < \mathrm{Fix}\ \psi_{1,n} < \mathrm{Fix}\ \psi_{2,n} < \cdots < \mathrm{Fix}\ \psi_{n,n} = F. \quad \square$$

As a corollary, we can answer a question of G. Levitt.

Corollary 4.2 *Let F be a finitely generated free group and let $\Psi \subseteq \mathrm{End}(F)$. Then, there exists a finite subset $\Psi_0 \subseteq \Psi$ such that $\mathrm{Fix}\ \Psi_0 = \mathrm{Fix}\ \Psi$. Moreover, Ψ_0 can be chosen to have at most $2r(F)$ elements.*

Proof. The result is clear if Ψ contains only the identity. Otherwise, pick any $1 \neq \psi_1 \in \Psi$ and let $\Psi_1 = \{\psi_1\}$. If Fix $\Psi_1 =$ Fix Ψ we are done. Otherwise, we can find a $\psi_2 \in \Psi$ such that, putting $\Psi_2 = \Psi_1 \cup \{\psi_2\}$, we have Fix $\Psi_2 <$ Fix Ψ_1. We continue this process and continue to define subsets Ψ_k of Ψ. At each stage, either Fix $\Psi =$ Fix Ψ_k or we can find $\psi_{k+1} \in \Psi$ such that, putting $\Psi_{k+1} = \Psi_k \cup \{\psi_{k+1}\}$, we have Fix $\Psi_{k+1} <$ Fix Ψ_k. In particular, either Fix $\Psi =$ Fix Ψ_k for some $1 \leq k \leq 2r(F)$ or, otherwise, we would have a strictly descending chain of endo-fixed subgroups,

$$\text{Fix } \Psi < \text{Fix } \Psi_{2r(F)} < \cdots < \text{Fix } \Psi_2 < \text{Fix } \Psi_1 < F,$$

with length $2r(F) + 1$, contradicting Theorem 4.1. Hence, Fix $\Psi =$ Fix Ψ_k for some $1 \leq k \leq 2r(F)$. $\square$

Theorem 4.1 leaves the four questions of whether the length of the longest strictly ascending chains of 1-auto-fixed, 1-endo-fixed, auto-fixed or endo-fixed subgroups of the free group of rank n, is either $2n-1$ or $2n$. For $n = 2$ the four questions coincide since the four families of subgroups do coincide (see Theorem 3.9 in [16]). For $n \geq 3$, the families of 1-endo-fixed and 1-auto-fixed subgroups are known to be different (see [12]), while in [10] the families of 1-auto-fixed and auto-fixed subgroups are conjectured to coincide. So, in general, the four questions are not the same.

In the following proposition we show that, in the cases $n = 2$ and $n = 3$, and for 1-auto-fixed subgroups, the exact maximum length is $2n - 1$. In the remark below we point out a reason why it seems difficult to extrapolate the arguments given to the general case.

Let F be a free group of rank $n \geq 1$.

Let $0 \leq p \leq q \leq n$ be two integers. A subgroup $H \leq F$ is said to be *of type* (p, q) if $r^{\text{ab}}(H; F) = p$ and $r(H) = q$. And, given also $0 \leq p' \leq q' \leq n$, an inclusion $H < K$ of subgroups of F is said to be *of type* "$(p, q) < (p', q')$" when H is of type (p, q) and K is of type (p', q'). Note that, by Theorem 3.4, any strict inclusion of endo-fixed subgroups is of type "$(p, q) < (p', q')$", where $p \leq p'$, $q \leq q'$, and at least one of these two inequalities is strict.

By Corollary 3.4 in [10], Corollary 2 in [15], and Proposition 1 in [3], any endo-fixed subgroup of F with rank n is automatically 1-auto-fixed, and contains a primitive element. Thus, no endo-fixed subgroup of F is of type $(0, n)$. Hence, the inclusion types "$(0, n-1) < (0, n)$" and "$(0, n) < (1, n)$" do not appear in any ascending chain of endo-fixed subgroups. Clearly, the type "$(0, 1) < (1, 1)$" is also impossible there, because any endomorphism fixing a power of an element, fixes the element itself. We will see below that all the other possible inclusion types can be realized by 1-auto-fixed subgroups.

Proposition 4.3 *For $n = 2$ and $n = 3$, the length of the longest strictly ascending chains of 1-auto-fixed subgroups of the free group of rank n, is $2n - 1$.*

Proof. Let F be a free group of rank n. We already know the existence of strictly ascending chains of 1-auto-fixed subgroups of F with length $2n - 1$. Assume now that $H_0 < H_1 < \cdots < H_{2n}$ is such a chain with length $2n$ and we will reach a contradiction when $n = 2$ or $n = 3$.

Since in any inclusion the value of $r(-) + r^{\mathrm{ab}}(-; F)$ increases, the length of the chain forces H_0 to be of type $(0, 0)$, H_{2n} of type (n, n), and every inclusion in the chain of type either "$(p, q) < (p+1, q)$" or "$(p, q) < (p, q+1)$" for some p, q.

Suppose $n = 2$. The impossibility of the inclusion types "$(0, 1) < (1, 1)$", "$(0, 1) < (0, 2)$" and "$(0, 2) < (1, 2)$" leads immediately to a contradiction.

Suppose $n = 3$. The impossibility of the inclusion types "$(0, 1) < (1, 1)$", "$(0, 2) < (0, 3)$" and "$(0, 3) < (1, 3)$" implies that H_0, H_1, H_2, H_3, H_5 and H_6 are of type $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 2)$, $(2, 3)$ and $(3, 3)$, respectively. Furthermore, the type of H_4 is either $(2, 2)$ or $(1, 3)$.

By Theorem 3.1, there exists a basis $\{a, b, c\}$ of F and two non-proper powers $w, h \in \langle a, b\rangle'$ such that $H_2 = \langle w, c^{-1}hc\rangle$. Note that, since $w \neq 1$ has trivial abelianization, $K = \langle a, b\rangle$ is the smallest free factor of F containing w. Consider the core graph X_2 corresponding to H_2, which consists of two circles labelled w and h, and an edge e with label c from a vertex ιe in the circle with label h to a vertex τe in the circle with label w (and with τe being the base point). Since $r(H_2) = r(H_3) = 2$, the inclusion $H_2 < H_3$ is algebraic and so, the core graph X_3 corresponding to H_3 is a quotient of X_2. In particular, the image of e, say e', is the unique edge in X_3 with label c. If $X_3 - \{e'\}$ were disconnected then $H_2 = H_3$ (because w and h are not proper powers), which is not the case. Thus, $X_3 - \{e'\}$ is connected and hence, it is a circle with two (possibly trivial) hairs going to $\iota e'$ and $\tau e'$ (the base point being $\tau e'$). Now, let $u \in K$ be the label of a (possibly trivial) path in $X_3 - \{e'\}$ from $\tau e'$ to $\iota e'$. It is clear that $H_3 = \langle w, uc\rangle$.

Let α be the automorphism of F defined by $a \mapsto a$, $b \mapsto b$, $c \mapsto u^{-1}c$. By looking at the corresponding core graph, it is clear that $H_3\alpha = \langle w, c\rangle$, is F-strictly-compressed (again using the fact that $w \in K$ is not a proper power). Hence, H_3 is also F-strictly-compressed, which implies that H_4, and so $H_4\alpha$, is of type $(1, 3)$ (and not $(2, 2)$).

Let $\psi \in \mathrm{Aut}(F)$ be such that $\mathrm{Fix}\,\psi = H_4 \geq H_3 = \langle w, uc \rangle$, and let $\psi' = \alpha^{-1}\psi\alpha$. A simple computation shows that $\mathrm{Fix}\,\psi' = H_4\alpha \geq \langle w, c \rangle$. We noted above that K is the smallest free factor of F containing $w = w\psi'$, so K is ψ'-invariant. Then (and using also the fact that ψ' fixes c) it is easy to see that $H_4\alpha = \mathrm{Fix}\,\psi' = \mathrm{Fix}\,\psi'_K * \langle c \rangle$. But $H_4\alpha$ is of type $(1,3)$, hence $\mathrm{Fix}\,\psi'_K \leq K$ is of type $(0,2)$. This is a contradiction with the fact that $\mathrm{Fix}\,\psi'_K$ is a 1-auto-fixed subgroup of the free group K, of rank 2. $\Box$

Remark 4.4 Note that the argument given in the previous observation for the case $n = 3$ is global in the sense that, assuming the existence of a chain of length $2n$, we get a contradiction by analyzing more than one consecutive inclusion. It is impossible to do this locally, that is, looking only at some particular inclusion in the chain, because, as we see below, all possible types of inclusions except "$(0, n-1) < (0, n)$", "$(0, n) < (1, n)$" and "$(0, 1) < (1, 1)$" can be realized by 1-auto-fixed subgroups.

Let $\{x_1, \ldots, x_n\}$ be a basis of F.

Let $1 \leq p \leq q \leq n$ be two integers and choose $r, s, t \geq 0$ and $\psi_{p,q}$ as in the proof of Theorem 4.1.

Suppose that $p + 1 \leq q$. Then, $s \geq 1$ and changing the $\psi_{p,q}$-image of x_{r+1} from $x_1^{r+1}x_{r+1}$ to x_{r+1} we obtain another automorphism ψ of F such that

$$\mathrm{Fix}\,\psi = \langle x_1, \ldots, x_r, x_{r+1}, x_{r+2}^{-1}x_1x_{r+2}, \ldots, x_{r+s}^{-1}x_1x_{r+s} \rangle.$$

Hence, $\mathrm{Fix}\,\psi_{p,q} < \mathrm{Fix}\,\psi$ is an strict inclusion of 1-auto-fixed subgroups of type "$(p, q) < (p+1, q)$".

Suppose that $q + 1 \leq n$. Then, $t \geq 1$ and changing the $\psi_{p,q}$-image of x_{r+s+1} from x_{r+s+1}^{-1} to $x_1^{r+s+1}x_{r+s+1}$ we obtain another automorphism ψ of F such that

$$\mathrm{Fix}\,\psi = \langle x_1, \ldots, x_r, x_{r+1}^{-1}x_1x_{r+1}, \ldots, x_{r+s}^{-1}x_1x_{r+s}, x_{r+s+1}^{-1}x_1x_{r+s+1} \rangle.$$

Hence, $\mathrm{Fix}\,\psi_{p,q} < \mathrm{Fix}\,\psi$ is an strict inclusion of 1-auto-fixed subgroups of type "$(p, q) < (p, q+1)$".

Now, let $0 = p \leq q \leq n-1$ be two integers with $1 \leq q$. Choose $r, s, t \geq 0$ such that $r = 2$, $q + 1 = r + s$ and $n = r + s + t$. Define $\psi_{0,q}$ to be the automorphism of F given by

$$\begin{array}{rcll}
\psi_{0,q}\colon F & \mapsto & F & \\
x_1 & \mapsto & [x_1, x_2]^{-1}x_1[x_1, x_2] & \\
x_2 & \mapsto & [x_1, x_2]^{-1}x_2[x_1, x_2] & \\
x_j & \mapsto & [x_1, x_2]^j x_j, & j = 3, \ldots, r+s \\
x_k & \mapsto & x_k^{-1}, & k = r+s+1, \ldots, r+s+t.
\end{array}$$

It is not difficult to see that

$$\mathrm{Fix}\,\psi_{0,q} = \langle [x_1, x_2], x_3^{-1}[x_1, x_2]x_3, \ldots, x_{r+s}^{-1}[x_1, x_2]x_{r+s}\rangle,$$

which is a 1-auto-fixed subgroup of type $(0, 1+s)$, that is, (p, q).

Suppose $2 \leq q$. Then, $s \geq 1$ and changing the $\psi_{0,q}$-image of x_3 from $[x_1, x_2]^3 x_3$ to x_3 we obtain another automorphism ψ of F such that

$$\mathrm{Fix}\,\psi = \langle [x_1, x_2], x_3, x_4^{-1}[x_1, x_2]x_4, \ldots, x_{r+s}^{-1}[x_1, x_2]x_{r+s}\rangle.$$

Hence, $\mathrm{Fix}\,\psi_{0,q} < \mathrm{Fix}\,\psi$ is an strict inclusion of 1-auto-fixed subgroups of type "$(0, q) < (1, q)$".

Suppose $q \leq n-2$. Then $t \geq 1$ and changing the $\psi_{0,q}$-image of x_{r+s+1} from x_{r+s+1}^{-1} to $[x_1, x_2]^{r+s+1}x_{r+s+1}$ we obtain another automorphism ψ of F such that

$$\mathrm{Fix}\,\psi = \langle [x_1, x_2], x_3^{-1}[x_1, x_2]x_3, \ldots, x_{r+s}^{-1}[x_1, x_2]x_{r+s}, x_{r+s+1}^{-1}[x_1, x_2]x_{r+s+1}\rangle.$$

Hence, $\mathrm{Fix}\,\psi_{0,q} < \mathrm{Fix}\,\psi$ is an strict inclusion of 1-auto-fixed subgroups of type "$(0, q) < (0, q+1)$".

Finally, the only remaining case is $p = q = 0$. And it is obvious that there are inclusions of 1-auto-fixed subgroups of type "$(0, 0) < (0, 1)$". $\square$

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