Examples of retracts in free groups that are not the fixed subgroup of any automorphism

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Abstract

Let n be an arbitrary cardinal, and let F_n be a free group of rank n. The *fixed subgroup* of an endomorphism ψ of F_n is the subgroup of elements in F_n fixed by ψ. In this paper, the relationship between the family of fixed subgroups of endomorphisms of F_n and the family of fixed subgroups of automorphisms of F_n will be studied. We prove that these two families of subgroups do not coincide for $n \geq 3$, by showing an infinite sequence of explicit examples of retracts of F_n –and so, fixed subgroups of endomorphisms of F_n– which are not fixed subgroups of any automorphism of F_n.

Key words: Free group, automorphism, 1-auto-fixed subgroup, retract.

1 Introduction

Let F be a free group. The *rank* of F, denoted $r(F)$, is the cardinality of any free generating set (also called a *basis*), which is known to depend only on F. Conversely, the isomorphism type of a free group F is determined by its rank.

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This way, for any given cardinal $n \geq 0$, the free group of rank n is usually denoted F_n. The classical theorem of Nielsen-Schreier says that any subgroup of a free group is free and so it has bases and its own rank. However, there are free groups F_n with subgroups whose rank is strictly bigger than n; for example, $F_{\aleph_0}$ can be viewed as a subgroup of F_2.

In almost all the paper we shall deal with finitely generated free groups F_n, where n is a non-negative integer. Theorem 19 and Corollary 20 are the only results in the paper that apply to free groups of arbitrary rank.

Let $n \geq 0$ be an arbitrary cardinal.

Throughout, we let endomorphisms of F_n act on the right, so $x \mapsto x\psi$. Given an endomorphism ψ of F_n, a subgroup $H \leq F_n$ is said to be *ψ-invariant* if $H\psi \leq H$ setwise.

As usual, $\mathrm{Aut}(F_n)$ denotes the *automorphism group* of F_n. We shall use exponential notation for inner automorphisms, $(\ )^u \colon F_n \to F_n$, $x \mapsto x^u = u^{-1}xu$.

Definition 1 The *fixed subgroup* of an endomorphism ψ of F_n, denoted $\mathrm{Fix}\,\psi$, is the subgroup of elements in F_n fixed by ψ:

$$\mathrm{Fix}\,\psi = \{x \in F_n : x\psi = x\}.$$

Following the terminology introduced in [7], a subgroup H of F_n is called *1-endo-fixed* when there exists an endomorphism ψ of F_n such that $H = \mathrm{Fix}\,\psi$. If, additionally, ψ can be chosen to be an automorphism (monomorphism), we further say that H is a *1-auto-fixed* (*1-mono-fixed*) subgroup of F_n. An *endo-fixed* subgroup of F_n is an arbitrary intersection of 1-endo-fixed subgroups and, analogously, a *mono-fixed* subgroup of F_n is an intersection of 1-mono-fixed subgroups, and an *auto-fixed* subgroup of F_n is an intersection of 1-auto-fixed subgroups.

A series of results by Bestvina-Handel [2], Imrich-Turner [5], Dicks-Ventura [4] and Bergman [1] proved that, in the finitely generated case, all these types of subgroups of F_n have rank at most n (if n is infinite, the result is clear by reasons of cardinality). However, the exact relationship between these six families of subgroups is not completely known.

In Theorem 11 below, it is shown that the families of 1-mono-fixed and 1-auto-fixed subgroups of F_n coincide (and, hence, the mono-fixed and auto-fixed families also do). In [7] it was conjectured that the families of auto-fixed and 1-auto-fixed subgroups of F_n coincide too. There, the authors gave only a partial result in this direction, proving that, if n is finite, every endo-fixed (auto-fixed) subgroup of F_n is a free factor of certain 1-endo-fixed (1-auto-fixed) subgroup. In general, the conjecture is only known to be true for $n \leq 2$, and in the case where the endo-fixed subgroup has maximal rank.

The aim of this paper is to study the relationship between the families of 1-auto-fixed and 1-endo-fixed subgroups of F_n. For $n = 0$ and $n = 1$ these families clearly coincide, and the same is true, although not so obvious, in the case $n = 2$ (see Corollary 2 of [10]). In the present paper we prove Theorem 7 describing the exact relation between these two families, in the finitely generated case. Also, Theorem 19, which is the main result, works for arbitrary $n \geq 3$, and provides explicit examples of subgroups $H \leq F_n$ with given rank between 2 and $n-1$ (understand n if n is infinite), which are retracts and so fixed subgroups of idempotent endomorphisms of F_n, but are not the fixed subgroup of any automorphism of F_n. This proves that, for $n \geq 3$, the family of 1-auto-fixed subgroups of F_n is strictly contained in that of 1-endo-fixed subgroups. The construction of such examples is first done in F_3 (see Proposition 18) and is strongly based on Theorem 1.3 of [8] (Theorem 9 below), which provides a sufficiently explicit description of 1-auto-fixed subgroups of F_n, for n finite. Finally, Theorem 19 extends the construction to arbitrary ranks $n \geq 3$ (finite or infinite).

2 Preliminaries

For all this section, let n be a non-negative integer, F_n a (finitely generated) free group of rank n, and let $X = \{a_1, \ldots, a_n\}$ be a basis for F_n.

We first quote some well known facts about free groups, which can be found in [6].

It is well known that every element $w \in F_n$ can be expressed in a unique way as a *reduced* word in $X^{\pm 1}$, say $w = x_1 \cdots x_r$, where $r \geq 0$, $x_i \in X^{\pm 1}$ and $x_i \neq x_{i+1}^{-1}$ (for this reason, elements in F_n are also called *words*). An element of F_n with reduced expression of the form $x_j \cdots x_k$, for $1 \leq j \leq k \leq r$ is called a *subword* of w.

It is said that a word $w \in F_n$ is *cyclically reduced* when its reduced expression, $w = x_1 \cdots x_r$, satisfies $x_r \neq x_1^{-1}$. Clearly, every element $w \in F_n$ has a cyclically reduced conjugate, though this is not in general unique.

A subgroup $H \leq F_n$ is called a *free factor* of F_n if it admits a basis which can be extended to a basis of F_n. For any free factor $H \leq F_n$, we have $r(H) \leq n$ with equality if and only if $H = F_n$. It is well known that if H is a free factor of F_n and $K \leq F_n$, then $H \cap K$ is a free factor of K. Moreover, if K is a free factor of H and H is a free factor of F_n then K is also a free factor of F_n (in particular, intersections of free factors are free factors). Note also that automorphisms send free factors to free factors.

Definition 2 It is usual to call $w \in F_n$ an F_n*-primitive* word (simply *primitive* if there is no risk of confusion) when there exist words $w_2, \ldots, w_n \in F_n$ such that $\{w, w_2, \ldots, w_n\}$ is a basis of F_n (that is, when $\langle w \rangle$ is a free factor of F_n). Alternatively, a primitive element is the image of a free generator under an automorphism of F_n. In particular, w is primitive if and only if w^u is primitive for every $u \in F_n$.

Similarly, a finite number of words $S = \{w_1, \ldots, w_m\}$ are called (F_n-)*associated primitives* if they simultaneously form part of a basis for F_n. More generally, they are called (F_n-)*associated primitives up to conjugation* whenever $\{w_1^{h_1}, \ldots, w_m^{h_m}\}$ are associated primitives for some $h_1, \ldots, h_m \in F_n$ (or, equivalently, if $\{w_1, w_2^{h_2}, \ldots, w_m^{h_m}\}$ are associated primitives for some choice of $h_2, \ldots, h_m \in F_n$). Note that if $S = \{w_1, \ldots, w_m\}$ are F_n-associated primitives up to conjugation, then $m \leq n$.

The natural epimorphism from F_n to $F_n^{\mathrm{ab}} = F_n/[F_n, F_n] \simeq \mathbb{Z}^n$ will be denoted $(\ )^{\mathrm{ab}}_{F_n} \colon F_n \to F_n^{\mathrm{ab}}$, or simply $(\ )^{\mathrm{ab}}$ when there is no risk of confusion. The image of a subgroup $H \leq F_n$ under this abelianisation epimorphism will be referred to as the (F_n-)*abelianisation* of H, which is not in general isomorphic to $H/[H,H]$. Since the kernel of $(\ )^{\mathrm{ab}}$ is characteristic in F_n, $(\ )^{\mathrm{ab}}$ induces a map from $\mathrm{Aut}(F_n)$ to $\mathrm{Aut}(F_n^{\mathrm{ab}}) \simeq GL_n(\mathbb{Z})$, which is known to be surjective.

Remark 3 Let $H \leq F_n$, let $i \colon H \to F_n$ be the inclusion, and consider the two epimorphisms $(\ )^{\mathrm{ab}}_{F_n} \colon F_n \to F_n^{\mathrm{ab}}$ and $(\ )^{\mathrm{ab}}_{H} \colon H \to H/[H,H]$. Note that, since $[H,H] \leq [F_n, F_n]$, we have the epimorphism $\pi \colon H/[H,H] \to H^{\mathrm{ab}}_{F_n} \leq F_n^{\mathrm{ab}}$, which is not in general injective (note that the rank of $H/[H,H]$ as free abelian group coincides with the rank of H, which can be bigger than n). Clearly, $(h)^{\mathrm{ab}}_{H}\pi = (hi)^{\mathrm{ab}}_{F_n}$ for every $h \in H$.

If H is a free factor of F_n, then $H \cap [F_n, F_n] = [H,H]$ and so, π is an isomorphism. Hence, in this case, $(\ )^{\mathrm{ab}}_{H}$ can be viewed as the restriction of $(\ )^{\mathrm{ab}}_{F_n}$ to H. Furthermore, $H^{\mathrm{ab}}_{F_n} = H^{\mathrm{ab}}_{H}$ is a direct summand of F_n^{ab}.

Finally, let us recall Whitehead graphs and the Whitehead cut vertex Lemma, a technique introduced by H.C. Whitehead in [11] and strongly used in the arguments below.

Definition 4 Let $w \in F_n$ and let $w = x_1 \cdots x_r$ be its reduced expression in the basis $X = \{a_1, \ldots, a_n\}$. The *Whitehead graph* of w, denoted W_w, is the graph with vertex set $VW_w = X^{\pm 1}$ and whose edge set EW_w contains exactly r edges, from x_1 to x_2^{-1}, from x_2 to x_3^{-1}, ... , from x_{r-1} to x_r^{-1}, and from x_r to x_1^{-1}.

Similarly, if $S \subseteq F_n$, the *Whitehead graph* of S, denoted W_S, is the graph with the same vertex set, $VW_S = X^{\pm 1}$, and edge set $EW_S = \cup_{w \in S} EW_w$.

Clearly, if w is cyclically reduced then W_w has no loops (i.e. edges starting and ending at the same vertex). Moreover, there may be several edges in W_w starting and ending at the same pair of vertices (as many as occurrences of the corresponding pair of consecutive letters in the reduced expression of w).

Definition 5 Let Z be a graph and $v \in VZ$ a vertex. We say that v is a *cut vertex* of Z if the graph obtained by deleting v together with all its adjacent edges is disconnected.

In particular, every vertex incident to a non-loop edge in a non-connected graph is a cut vertex. Observe that every non-connected graph contains a cut vertex, with the only exception of those with only two vertices.

The most important result concerning Whitehead graphs is the Whitehead cut vertex Lemma. The reformulation that we establish here for latter use, is essentially contained in the original paper [11] and can also be obtained as a direct corollary of Theorem 2.4 in [9], which is an extension of the classical Whitehead cut vertex Lemma.

Theorem 6 (Whitehead cut vertex Lemma, [11], [9]) *Let F_n be a finitely generated free group, and let $S \subset F_n$ be a set of cyclically reduced F_n-associated primitives up to conjugation. Then, W_S has a cut vertex.*

In particular, the Whitehead graph of any cyclically reduced primitive word in F_n has a cut vertex.

3 The 1-endo-fixed and 1-auto-fixed families

As mentioned in the introduction, the families of 1-endo-fixed and 1-auto-fixed subgroups of F_n do coincide for $n = 0, 1, 2$. Let us analyze now the situation for $n \geq 3$.

It is easy to see that every 1-endo-fixed subgroup H of F_n is *pure* (i.e., $x^r \in H$ implies $x \in H$). So, for a non-trivial cyclic subgroup $\langle w \rangle \leq F_n$, the following three assertions are equivalent:

(i) $\langle w \rangle$ is 1-endo-fixed,
(ii) w is not a proper power,
(iii) $\langle w \rangle$ is 1-auto-fixed.

In fact, in this case, $\langle w \rangle$ is the fixed subgroup of conjugating by w. Thus, for cyclic subgroups, the families of 1-endo-fixed and 1-auto-fixed subgroups of

F_n do coincide.

A subgroup $H \leq F_n$ is called a *retract* of F_n if the identity $Id: H \to H$ extends to a homomorphism $r: F_n \to H$. Since, in this case, the composition of r with the inclusion gives an idempotent endomorphism of F_n, retracts can alternatively be defined as images (or, equivalently, fixed subgroups) of idempotent endomorphisms of F_n. Observe that, from the definition, one can deduce that retracts of F_n have rank at most n, and that, in the finitely generated case, the only one with rank n is the whole group F_n.

Clearly, free factors of F_n are examples of retracts of F_n. However, for $n \geq 3$, the family of retracts is much larger (see problem 15 on page 140 of [6], or Proposition 1 of [10]).

For any endomorphism ψ of F_n, the *stable image* of ψ is defined to be the subgroup $F_n\psi^\infty = \cap_{i\geq 0} F_n\psi^i$. In [10] Theorem 1, E. Turner proved that stable images of endomorphisms of finitely generated free groups are always retracts. From this result, he obtained a corollary saying that, if n is finite, every endomorphism of F_n having fixed subgroup of rank n has to be an automorphism. In particular, in the maximal rank case, the families of 1-endo-fixed and 1-auto-fixed subgroups of finitely generated free groups do coincide.

We can obtain another easy corollary of Turner's result, describing in general the relationship between the 1-endo-fixed and the 1-auto-fixed families of subgroups of F_n, also in the finitely generated case.

Theorem 7 *Let F_n be a finitely generated free group of rank n. The family of 1-endo-fixed subgroups of F_n is precisely the family of 1-auto-fixed subgroups of retracts of F_n.*

Proof. Let $H \leq F_n$ be a retract of F_n and $r: F_n \to H$ be a homomorphism such that $\mathrm{Im}\, r = \mathrm{Fix}\, r = H$. For every $\phi \in Aut(H)$, it is clear that we have $\mathrm{Fix}\,(r\phi i) = \mathrm{Fix}\, \phi$, where $i: H \to F_n$ is the inclusion. So, 1-auto-fixed (in fact, 1-endo-fixed) subgroups of retracts of F_n are 1-endo-fixed subgroups of F_n.

Conversely, let ψ be an endomorphism of F_n. By [5] Theorem 1, the stable image of ψ is ψ-invariant and ψ restricts to an automorphism there. Since $\mathrm{Fix}\, \psi \leq F_n\psi^\infty$, we deduce from Turner's theorem that 1-endo-fixed subgroups of F_n are 1-auto-fixed subgroups of retracts of F_n. □

Remark 8 Consider a free group F_n with $n \geq \aleph_0$, and let $\psi: F_n \to F_n$ be an epimorphism with non-trivial kernel. Clearly, the stable image of ψ is the whole F_n, while ψ is not an automorphism of F_n. So, the arguments in Theorem 7 do not work in the case of infinite rank. We do not know if Theorem 7 is valid for free groups of infinite rank. In fact, for $n \geq \aleph_0$, the relationship between the families of 1-endo-fixed subgroups and 1-auto-fixed subgroups of F_n is quite obscure. Essentially, we only know that these two families do not coincide, as shown later in Theorem 19.

By sending the complementary generators to their own inverses, we see that 1-auto-fixed subgroups of free factors of F_n are themselves 1-auto-fixed subgroups of F_n. And we noted above that, for $n \geq 3$, the family of retracts of F_n is much larger than the family of free factors. So, in view of Theorem 7, it is reasonable to expect that the family of 1-endo-fixed subgroups of F_n is also larger than that of 1-auto-fixed subgroups, at least in the finitely generated case. However, it is not easy to find a subgroup $H \leq F_n$, $n \geq 3$, which is the fixed subgroup of an endomorphism of F_n, but is not the fixed subgroup of any automorphism.

The rest of the paper is dedicated to constructing such subgroups. As observed above, we have to look for them among subgroups $H \leq F_n$ with $2 \leq r(H) \leq n-1$ (understand $2 \leq r(H) \leq n$ if n is infinite), and so $n \geq 3$. Natural candidates are the fixed subgroups of conveniently chosen idempotent endomorphisms of F_n. Of course, the main difficulty will be to prove that such subgroups are not the fixed subgroup of any automorphism of F_n (or, equivalently, that any automorphism of F_n fixing such a subgroup has to fix something else). For this purpose, we shall use Theorem 1.3 in [8], which provides a sufficiently explicit description of what 1-auto-fixed subgroups of finitely generated free groups look like. We state it here for later reference.

Theorem 9 (Martino-Ventura, [8]) *Let ϕ be an automorphism of a finitely generated free group F_n. Then, either* Fix ϕ *is cyclic or there exists a non-trivial free factorisation $F_n = H * K$ such that H is ϕ-invariant and one of the following holds:*

(i) Fix $\phi \leq H$,

(ii) *K is also ϕ-invariant and* Fix $\phi = (H \cap \text{Fix } \phi) * (K \cap \text{Fix } \phi)$, *where* $r(K \cap \text{Fix } \phi) = 1$,

(iii) *there exist non-trivial elements $y \in F_n$, $h, h' \in H$, such that $K = \langle y \rangle$, $y\phi = h'y$, h is not a proper power,* Fix $\phi = (H \cap \text{Fix } \phi) * \langle y^{-1}hy \rangle$ *and* $h\phi = h'hh'^{-1}$.

The following (stated here for later use) is an immediate corollary, which was first proved by Collins-Turner in [3]. One can think of the statement as saying that any automorphism of F_2 which has a fixed subgroup of rank 2 can be realised as a Dehn twist on a punctured torus.

Corollary 10 (Collins-Turner, [3]) *Let ϕ be an automorphism of F_2 such that* $r(\text{Fix } \phi) = 2$. *Then, either* Fix $\phi = F_2$ *(and $\phi = Id$) or there is a basis* $\{a, b\}$ *of F_2 such that* Fix $\phi = \langle a, b^{-1}ab \rangle$.

To finish the present section, we remark the following easy consequence of [10] and [5].

Theorem 11 *Let F_n be a finitely generated free group of finite rank n. The families of 1-mono-fixed and 1-auto-fixed subgroups of F_n do coincide. Hence, those of mono-fixed and auto-fixed subgroups also do.*

Proof. Let $\psi\colon F_n \to F_n$ be a monomorphism. By [5] Theorem 1, the stable image of ψ is ψ-invariant and ψ restricts to an automorphism there. But, by [10] and the injectivity of ψ, the stable image of ψ is a free factor of F_n. Hence, Fix ψ is a 1-auto-fixed subgroup of a free factor of F_n and so, a 1-auto-fixed subgroup of F_n. □

4 The Examples

In order to construct examples of 1-endo-fixed subgroups of F_n which are not 1-auto-fixed, we first prove some technical results about F_2 and F_3. Theorem 16 gives conditions on a subgroup $L \leq F_3$ which are enough to ensure that L is not 1-auto-fixed. Then, Proposition 18 provides an infinite sequence of examples of 1-endo-fixed not 1-auto-fixed subgroups of F_3. And finally, these examples are generalized to arbitrary (finite or infinite) ranks in Theorem 19.

Lemma 12 *Let $F_2 = \langle a, b \rangle$ and let $g \in F_2$ be a word such that $g^{\mathrm{ab}} = b^{\mathrm{ab}}$. Then, there exists $u \in F_2$ such that g^u is cyclically reduced and either $g^u = b$ or g^u has $ba^\delta, a^\delta b^{-1}, b^{-1}a^\epsilon$ and $a^\epsilon b$ as subwords, for some $\epsilon, \delta = \pm 1$. In the latter case, W_{g^u} contains a cycle and therefore has no cut vertex.*

Proof. We shall assume that g is not conjugate to b and find $\epsilon, \delta = \pm 1$ and $u \in F_2$ such that g^u is cyclically reduced and has $ba^\delta, a^\delta b^{-1}, b^{-1}a^\epsilon$ and $a^\epsilon b$ as subwords.

Choose $u \in F_2$ such that the reduced expression of g^u begins with $b^{\pm 1}$ and ends with $a^{\pm 1}$ (and so, g^u is cyclically reduced). That is,

$$g^u = b^{n_1} a^{m_1} \cdots b^{n_k} a^{m_k}$$

where $m_i, n_i \neq 0$. Since g has the same abelianisation as b, we have $k \geq 2$ and we can assume $n_1 > 0$ and $n_2 < 0$. Hence, g^u has subwords $ba^\delta, a^\delta b^{-1}$ where $\delta = \pm 1$ depending on whether m_1 is positive or negative.

Now, consider the least index i such that $n_i < 0$ and $n_{i+1} > 0$ (where it is understood that $n_{k+1} = n_1$). If $i < k$ then g has $b^{-1} a^{m_i} b$ as a subword, and this would end the proof of the lemma.

Otherwise, $i = k$ which means that $n_i < 0$ for all $i \neq 1$, and thus $n_1 \geq 2$. Hence, g has a conjugate of the form

$$g^{ub} = b^{n_1-1}a^{m_1} \cdots b^{n_k}a^{m_k}b,$$

where all the exponents are non-zero. This is a cyclically reduced conjugate of g with the required properties. □

Lemma 13 *Let $F_3 = \langle a, b, c\rangle$ and let $g \in \langle a, b\rangle$ be a cyclically reduced word having $ba^\delta, a^\delta b^{-1}, b^{-1}a^\epsilon$ and $a^\epsilon b$ as subwords, for some $\epsilon, \delta = \pm 1$. Then, any F_3-primitive element lying in the subgroup $\langle g, c\rangle$ is conjugate to $(cg^m)^{\pm 1}$ for some integer m.*

Proof. Let $w \in \langle g, c\rangle$ be a F_3-primitive element. By the hypotheses we have on $g \in \langle a, b\rangle$, W_g contains a cycle and therefore has no cut vertex. Then, by the Whitehead cut vertex Lemma 6, g is not $\langle a, b\rangle$-primitive and so, it is not F_3-primitive either. Hence, the reduced expression of w and of any of its conjugates by elements in $\langle g, c\rangle$, as a word in g and c, must involve c.

If some conjugate of w is a power of c, then the exponent must be ± 1 and we are done, by taking $m = 0$. Hence, we may assume that the reduced expression of any conjugate of w by elements in $\langle g, c\rangle$ involves both g and c. Changing w to w^{-1} if necessary, we may choose $u \in \langle g, c\rangle$ such that

$$w^u = c^{n_1}g^{m_1} \cdots c^{n_k}g^{m_k} \tag{1}$$

with $k \geq 1$, $m_i, n_i \neq 0$ and $n_1 \geq 1$. Also, changing g to g^{-1} and changing the signs of $m_1, \ldots, m_k$ if necessary, we may assume that $m_1 \geq 1$.

Since g, and so w^u, are cyclically reduced words in $\{a, b, c\}$, g is a subword of w^u and the graph W_{w^u} contains at least four edges joining b with a^{-1}, a with b, b^{-1} with a^{-1}, and a with b^{-1}, respectively. Write $g = x_1 \cdots x_r$, $x_i \in \{a^{\pm 1}, b^{\pm 1}\}$ for the cyclically reduced expression of g. Note that W_{w^u} also has an edge joining c with x_1^{-1}, and either x_r with c^{-1} if $m_k > 0$, or x_1^{-1} with c^{-1} and $c^{\pm 1}$ with x_r if $m_k < 0$.

Furthermore, if $|n_i| \geq 2$ for some i, then W_{w^u} would have another edge joining c with c^{-1} and, since $x_r \neq x_1^{-1}$, it would have no cut vertex, contradicting the F_3-primitivity of w^u. Thus, $n_i = \pm 1$ for every $i = 1, \ldots, k$.

Now, we claim that consecutive n_i's have alternating signs. In fact, suppose that $n_i = n_{i+1}$ for some $i = 1, \ldots, k-1$. Inverting w and changing u if necessary, we can assume $i = 1$ and $n_1 = n_2 = 1$. Now, consider the automorphism φ of F_3 which sends c to cg^{-m_1} and fixes a and b. The image of w^u is

$$(w^u)\varphi = c^{n_1}g^{m'_1} \cdots c^{n_k}g^{m'_k}, \tag{2}$$

where $m'_i = m_i - \frac{n_i+1}{2}m_1 + \frac{1-n_{i+1}}{2}m_1$ (with $n_{k+1} = 1$), i.e. m'_i equals m_i subtracting m_1 if $n_i = 1$, and also adding m_1 if $n_{i+1} = -1$. Note that m'_i can be equal to zero but no c can cancel in (2), and that the reduced expression for $(w^u)\varphi$ has the same form as (1), with possibly smaller k and the exponents of c not necessarily equal to ± 1. But $m'_1 = m_1 - m_1 = 0$ so, $(w^u)\varphi$ begins with c^2. Now, the argument in the previous paragraph contradicts the F_3-primitivity of $(w^u)\varphi$. And this contradiction shows that consecutive n_i's in (1) must have alternating signs.

If $k = 1$ we are done. So, assume $k \geq 2$ and let us find a contradiction.

We know that the Whitehead graph W_{w^u} has at least four edges joining b with a^{-1}, a with b, b^{-1} with a^{-1}, and a with b^{-1}, respectively. On the other hand, by looking at the reduced expression (1) (where we know that $k \geq 2$, $m_1 \geq 1$, $n_1 = 1$ and the $n_i = \pm 1$ have alternating signs), we see that W_{w^u} has at least four more edges joining each of c and c^{-1} with the two distinct vertices $x_1^{-1}, x_n \in \{a^{\pm 1}, b^{\pm 1}\}$. This implies that W_{w^u} has no cut vertices, contradicting again the F_3-primitivity of w^u. This contradiction completes the proof. □

Proposition 14 *Let $F_3 = \langle a, b, c\rangle$ and let $g \in \langle a, b\rangle$ be such that $L = \langle g, c\rangle$ abelianises to a rank two direct summand of F_3^{ab}. Then, either*

(a) L is a free factor of F_3, or
(b) L contains no pair of F_3-associated primitives up to conjugation.

Proof. First, note that, since L^{ab} is a rank two direct summand of F_3^{ab}, g^{ab} generates a non-trivial direct summand of $\langle a^{\mathrm{ab}}, b^{\mathrm{ab}}\rangle$. Thus, there is an automorphism of $\langle a^{\mathrm{ab}}, b^{\mathrm{ab}}\rangle$ sending g^{ab} to b^{ab}. Since the abelianisation map from $\mathrm{Aut}(F_2)$ to $\mathrm{Aut}(F_2^{\mathrm{ab}})$ is surjective, there is an automorphism of $\langle a, b\rangle$ which sends g to an element with the same abelianisation as b. Extend this to an automorphism $\varphi_1\colon F_3 \to F_3$ by just fixing c. Changing g to $g\varphi_1$ and L to $L\varphi_1$, we may assume that $g^{\mathrm{ab}} = b^{\mathrm{ab}}$.

We now invoke Lemma 12 to find an element $u \in \langle a, b\rangle$ such that g^u satisfies the conclusion of the lemma. Consider the automorphism $\varphi_2\colon F_3 \to F_3$ given by $a \mapsto a^u$, $b \mapsto b^u$, $c \mapsto c$. Changing L to $L\varphi_2$ and g to $g\varphi_2 = g^u$, we may assume that $g \in \langle a, b\rangle$ is cyclically reduced and is either equal to b or has $ba^\delta, a^\delta b^{-1}, b^{-1}a^\epsilon$ and $a^\epsilon b$ as subwords, for some $\epsilon, \delta = \pm 1$.

In the former case, $L = \langle g, c\rangle = \langle b, c\rangle$ is a free factor of F_3 and we are done. Assume the latter case, suppose that there exist a pair of F_3-associated primitives up to conjugation $w_1, w_2 \in L$, and let us find a contradiction.

By Lemma 13, there are integers m_1, m_2 such that w_1 and w_2 are conjugate to $(cg^{m_1})^{\pm 1}$ and $(cg^{m_2})^{\pm 1}$, respectively. Conjugating and inverting w_1 and/or w_2 if necessary, we may assume that $w_1 = cg^{m_1}$ and $w_2 = cg^{m_2}$. By applying the automorphism $\varphi_3 \colon F_3 \to F_3$ given by $a \mapsto a$, $b \mapsto b$, $c \mapsto cg^{-m_1}$, we deduce that $w_1\varphi_3 = c$ and $w_2\varphi_3 = cg^m$ form another pair of F_3-associated primitives up to conjugation, where $m = m_2 - m_1$. In particular, $m \neq 0$.

Since g is cyclically reduced and has $ba^\delta, a^\delta b^{-1}, b^{-1}a^\epsilon$ and $a^\epsilon b$ as subwords, the Whitehead graph W_{cg^m} contains at least four edges joining b with a^{-1}, a with b, b^{-1} with a^{-1}, and a with b^{-1}, respectively, as well as two more edges joining two different vertices in $\{a^{\pm 1}, b^{\pm 1}\}$ with c and c^{-1}, respectively. On the other hand, the Whitehead graph of c has a single edge joining c with c^{-1}. Hence, $W_{\{c, cg^m\}}$ has no cut vertices, contradicting Theorem 6. □

Proposition 15 *Let $F_3 = \langle a, b, c\rangle$ and let $g, h \in \langle a, b\rangle$ be such that the subgroup $L = \langle g, c^{-1}hc\rangle$ abelianises to a rank two direct summand of F_3^{ab}. Then, either*

(a) L is a free factor of F_3, or
(b) L contains no pair of F_3-associated primitives up to conjugation.

Proof. First, note that, since L^{ab} is a rank two direct summand of F_3^{ab}, we have $\langle g^{\mathrm{ab}}, h^{\mathrm{ab}}\rangle = \langle a^{\mathrm{ab}}, b^{\mathrm{ab}}\rangle$. Consider the automorphism of $\langle a^{\mathrm{ab}}, b^{\mathrm{ab}}\rangle$ defined by $g^{\mathrm{ab}} \mapsto a^{\mathrm{ab}}$ and $h^{\mathrm{ab}} \mapsto b^{\mathrm{ab}}$. Since the abelianisation map from $\mathrm{Aut}(F_2)$ to $\mathrm{Aut}(F_2^{\mathrm{ab}})$ is surjective, there is an automorphism of $\langle a, b\rangle$ which sends g and h to elements with the same abelianisation as a and b, respectively. Extend this to an automorphism $\varphi_1 \colon F_3 \to F_3$ by just fixing c. Changing g to $g\varphi_1$, h to $h\varphi_1$ and L to $L\varphi_1$, we may assume that $g^{\mathrm{ab}} = a^{\mathrm{ab}}$ and $h^{\mathrm{ab}} = b^{\mathrm{ab}}$.

We now invoke Lemma 12, twice. There exists $u \in \langle a, b\rangle$ such that g^u is cyclically reduced and is either equal to a or has $ab^\delta, b^\delta a^{-1}, a^{-1}b^\epsilon$ and $b^\epsilon a$ as subwords, for some $\epsilon, \delta = \pm 1$. Similarly, there exists $v \in \langle a, b\rangle$ such that h^v is cyclically reduced and is either equal to b or has $ba^{\delta'}, a^{\delta'}b^{-1}, b^{-1}a^{\epsilon'}$ and $a^{\epsilon'}b$ as subwords, for some $\epsilon', \delta' = \pm 1$. Consider the automorphism $\varphi_2 \colon F_3 \to F_3$ given by $a \mapsto a^u$, $b \mapsto b^u$, $c \mapsto u^{-1}vc$, and note that $g\varphi_2 = g^u$, $h\varphi_2 = h^u$, but

$$(c^{-1}hc)\varphi_2 = c^{-1}v^{-1}uh^u u^{-1}vc = c^{-1}h^v c.$$

So, changing g to g^u, h to h^v and L to $L\varphi_2 = \langle g^u, c^{-1}h^v c\rangle$, we may simultaneously assume that $g, h \in \langle a, b\rangle$ are cyclically reduced, that g is either equal to a or has $ab^\delta, b^\delta a^{-1}, a^{-1}b^\epsilon$ and $b^\epsilon a$ as subwords, for some $\epsilon, \delta = \pm 1$, and that h is either equal to b or has $ba^{\delta'}, a^{\delta'}b^{-1}, b^{-1}a^{\epsilon'}$ and $a^{\epsilon'}b$ as subwords, for some $\epsilon', \delta' = \pm 1$.

If both $g = a$ and $h = b$, then $L = \langle a, c^{-1}bc\rangle$ is a free factor of F_3 and we are done. Otherwise, suppose that $g \neq a$ or $h \neq b$.

We claim that if $g \neq a$ then every F_3-primitive lying in L is conjugate to $h^{\pm 1}$. So assume that $g \neq a$, pick a F_3-primitive element $w \in L$, suppose that

w is not conjugate to $h^{\pm 1}$, and find a contradiction. For every $u \in L$, the reduced expression of w^u as a word in $g, c^{-1}hc$ involves g. Also, since g has $ab^\delta, b^\delta a^{-1}, a^{-1}b^\epsilon$ and $b^\epsilon a$ as subwords, it is not $\langle a, b\rangle$-primitive, so it is not F_3-primitive either, and hence, the above expression for w^u involves $c^{-1}hc$ too. Choose $u \in L$ such that

$$w^u = g^{n_1}(c^{-1}h^{m_1}c) \cdots g^{n_k}(c^{-1}h^{m_k}c), \tag{3}$$

where $m_i, n_i \neq 0$ and $k \geq 1$. Since g, h are cyclically reduced, in their reduced expressions as words in a, b, say $g^{\mathrm{sign}(n_1)} = x_1 \cdots x_r$ and $h^{\mathrm{sign}(m_1)} = y_1 \cdots y_s$ with $x_i, y_i \in \{a^{\pm 1}, b^{\pm 1}\}$, we have $x_1^{-1} \neq x_r$ and $y_1^{-1} \neq y_s$. Then, replacing these expressions in (3), we obtain the reduced expression of w^u as a word in $\{a, b, c\}$. Now, look at the Whitehead graph W_{w^u}. Since $g \neq a$, it contains four edges joining a with b^{-1}, b with a, a^{-1} with b^{-1}, and b with a^{-1}, respectively. Additionally, it also contains two edges joining c with the two different vertices $x_1^{-1}, x_r \in \{a^{\pm 1}, b^{\pm 1}\}$, and two more edges joining c^{-1} with the two different vertices $y_1^{-1}, y_s \in \{a^{\pm 1}, b^{\pm 1}\}$. So, W_{w^u} has no cut vertices, contradicting Theorem 6.

A symmetric argument shows that if $h \neq b$ then any F_3-primitive in L is conjugate to $g^{\pm 1}$. Thus, either all F_3-primitives in L are conjugate to $h^{\pm 1}$, or they all are conjugate to $g^{\pm 1}$. From this we conclude that L contains no pair of F_3-associated primitives up to conjugation. (Note that if simultaneously $g \neq a$ and $h \neq b$ then L contains no F_3-primitive element, since L^{ab} has rank two and so, $g^{\pm 1}$ is not conjugate to $h^{\pm 1}$). □

We can now give a list of conditions on a subgroup $L \leq F_3$ which are enough to ensure that L is not 1-auto-fixed.

Theorem 16 *Let $L \leq F_3$ be a subgroup such that $r(L) = 2$, L^{ab} is a rank two direct summand of F_3^{ab}, L is not a free factor of F_3, and it contains a pair of F_3-associated primitives up to conjugation. Then, L is not 1-auto-fixed.*

Proof. Let us argue by contradiction and suppose the existence of an automorphism $\phi \in Aut(F_3)$ such that $L = \mathrm{Fix}\,\phi$.

Apply Theorem 9. Since L is not cyclic, there exists a non-trivial free factorisation $F_3 = H * K$ such that H is ϕ-invariant and (i), (ii) or (iii) as in the theorem holds.

Case (i) is impossible, since in this case $r(H) = 2$, and Corollary 10 would then imply that either $L = H$ or L^{ab} is cyclic (note that, by remark 3, $L^{\mathrm{ab}}_{F_3} = L^{\mathrm{ab}}_H$); and both possibilities contradict our hypotheses. Thus, there exists a basis $\{a, b, c\}$ for F_3 such that L equals $\langle g, c\rangle$ or $\langle g, c^{-1}hc\rangle$ for some $g, h \in H = \langle a, b\rangle$. By Propositions 14 and 15, we then deduce that either L is a free factor of

F_3 or it does not contain a pair of F_3-associated primitives up to conjugation. Again, both possibilities contradict the hypotheses on L. □

Corollary 17 *Let $F_3 = \langle a, b, c\rangle$ be a free group of rank 3, let $w_1, w_2 \in \langle a, b\rangle$ be two words in the normal closure $\langle\langle a\rangle\rangle$ of a, and let $L = \langle b, cw_1cw_2c^{-1}\rangle$. Then,*

(i) L is a retract of F_3,
(ii) either L is a free factor of F_3 or it is not a 1-auto-fixed subgroup.

Proof. Consider the endomorphism $\psi\colon F_3 \to F_3$ defined by $a \mapsto 1$, $b \mapsto b$, $c \mapsto cw_1cw_2c^{-1}$. Since $w_1cw_2c^{-1}$ belongs to $\langle\langle a\rangle\rangle \leq \ker\psi$, we see that $\psi^2 = \psi$. Hence, Fix ψ = Im $\psi = \langle b, cw_1cw_2c^{-1}\rangle$ is a retract of F_3. This proves (i).

Clearly, L has rank two and abelianises to a rank two direct summand of F_3^{ab}. Furthermore, since $w_1, w_2 \in \langle a, b\rangle$, we deduce that $\{b, w_1cw_2, a\}$ is a basis for F_3 and then $b, c(w_1cw_2)c^{-1} \in L$ form a pair of F_3-associated primitives up to conjugation. Now, either L is a free factor of F_3 or, otherwise, Theorem 16 implies that L is not a 1-auto-fixed subgroup of F_3. This completes the proof of (ii). □

Simple choices of w_1 and w_2 (for example $w_1, w_2 \in \{1, a, a^{-1}\}$) make L a free factor of F_3. However, for w_1, w_2 complicated enough, L will not be a free factor. Then, by Corollary 17, such a L will be a retract of F_3 which is not a 1-auto-fixed subgroup. The following statement provides an infinite family of such subgroups of F_3.

Proposition 18 *Let $F_3 = \langle a, b, c\rangle$ be a free group of rank 3 and consider the subgroup $L_{r,s,t} = \langle b, ca^rcb^sa^tb^{-s}c^{-1}\rangle$, where r, s, t are integers. Then,*

(i) $L_{r,s,t}$ is a retract of F_3 (and so, a 1-endo-fixed subgroup), but
(ii) $L_{r,s,t}$ is not a 1-auto-fixed subgroup of F_3 if and only if $rst \neq 0$.

Proof. Taking $w_1 = a^r$ and $w_2 = b^sa^tb^{-s}$ in Corollary 17, we know that $L_{r,s,t}$ is a retract of F_3, and that it is not a 1-auto-fixed subgroup unless it is a free factor of F_3. So, it only remains to prove that $L_{r,s,t}$ is a free factor of F_3 if and only if $rst = 0$.

Suppose $rst = 0$. It is easy to see that $\{b, c^2b^sa^tb^{-s}c^{-1}, cb^sab^{-s}c^{-1}\}$ is a basis for F_3. So, if $r = 0$, the subgroup $L_{r,s,t} = \langle b, c^2b^sa^tb^{-s}c^{-1}\rangle$ is a free factor of F_3. On the other hand, $\{b, ca^rca^tc^{-1}, cac^{-1}\}$ is another basis for F_3. Hence, if $s = 0$, the subgroup $L_{r,s,t} = \langle b, ca^rca^tc^{-1}\rangle$ is also a free factor of F_3. Finally, if $t = 0$, then $L_{r,s,t} = \langle b, ca^r\rangle$ is again a free factor.

Conversely, suppose that $L_{r,s,t}$ is a free factor of F_3, assume that $r, s, t \neq 0$ and find a contradiction. Consider a new letter d, embed F_3 into $F_4 = F_3 * \langle d\rangle$, and note that there exists $w \in \langle a, b, c\rangle$ such that $\{b, ca^rcb^sa^tb^{-s}c^{-1}, w, d\}$ is a basis of F_4. So, the word $dbdca^rcb^sa^tb^{-s}c^{-1}$ is F_4-primitive. However, the Whitehead graph $W_{dbdca^rcb^sa^tb^{-s}c^{-1}}$ has 8 vertices none of which is a cut vertex (in order to draw this graph, note that it does not depend on the sign of $t \neq 0$ and that, by changing a to a^{-1} if necessary, we can assume $r \geq 1$; then, distinguish the cases $s \geq 1$ and $s \leq -1$). This contradiction completes the proof. □

One of the simplest examples in this family of 1-endo-fixed not 1-auto-fixed subgroups of F_3 is $L_{1,1,-1} = \langle b, cacba^{-1}b^{-1}c^{-1}\rangle = \langle b, c[a, cb]\rangle$.

In the following theorem, we extend the previous construction to arbitrary ranks $n \geq 3$ (finite or infinite), thus providing examples of retracts of F_n which are not 1-auto-fixed, and with prescribed rank m between 2 and $n-1$ (as noted in section 3, there are no such examples neither with rank 1, nor with rank $n < \aleph_0$). In the proof we make use of the subgroup $L_{1,1,-1}$ of F_3, but exactly the same construction and arguments work replacing this by an arbitrary subgroup of the form $\langle b, cw_1cw_2c^{-1}\rangle$ which is not a free factor of $F_3 = \langle a, b, c\rangle$, where $w_1, w_2 \in \langle a, b\rangle \cap \langle\langle a\rangle\rangle$. Thus, the proof of the following result provides lots of examples of retracts of any given admissible rank, that are not 1-auto-fixed subgroups of F_n, $n \geq 3$.

Theorem 19 *Let $n \geq 3$ and $m \geq 2$ be cardinals such that $m \leq n-1$ (understand $m \leq n$ if n is infinite). Then, the free group of rank n contains retracts of rank m which are not 1-auto-fixed.*

Proof. Let F_n be a free group of rank $n \geq 3$, let X be a basis for F_n, and let $a, b, c \in X$ be three different basis elements. Let U, V be disjoint subsets of X such that $U \cup V = X \setminus \{a, b, c\}$ and $|U| = m-2$ (understand $|U| = m$ if m is infinite). Note that, if n (and so m) is finite, we are simply labelling the n elements of X as $X = \{a, b, c, u_1, \ldots, u_{m-2}, v_1, \ldots, v_{n-m-1}\}$, $u_i \in U, v_j \in V$. Consider now the endomorphism ψ of F_n defined by $a \mapsto 1$, $b \mapsto b$, $c \mapsto c[a, cb]$, $u \mapsto u$, $v \mapsto 1$, for $u \in U$ and $v \in V$. Clearly, ψ is idempotent and hence, $\mathrm{Fix}\,\psi = \mathrm{Im}\,\psi = \langle\{b, c[a, cb]\} \cup U\rangle$ is a retract of F_n with rank m.

Assume that $\mathrm{Fix}\,\psi = \mathrm{Fix}\,\phi$ for some $\phi \in \mathrm{Aut}(F_n)$, and let us find a contradiction.

Let M be the intersection of all those free factors of F_n containing $\langle b, c[a, cb]\rangle$. Note that $M \leq \langle a, b, c\rangle$ and so, it is a free factor of $\langle a, b, c\rangle$. Thus, either $r(M) = 2$ or $M = \langle a, b, c\rangle$.

But M, and so $M\phi$, are free factors of F_n. So, $M \leq M\phi$ and, by the coincidence of their ranks, $M = M\phi$. This means that M is ϕ-invariant. Let $\phi_M \in Aut(M)$ be the restriction of ϕ to M. Clearly, $\mathrm{Fix}\,\phi_M = M \cap \mathrm{Fix}\,\phi = \langle b, c[a, cb]\rangle$ so, Proposition 18 excludes the possibility $M = \langle a, b, c\rangle$.

Thus, $r(M) = 2$. Now, using Corollary 10 and the fact that the abelianisation $(\langle b, c[a, cb]\rangle)^{\text{ab}}_M = (\langle b, c[a, cb]\rangle)^{\text{ab}}_{F_n} = \langle b^{\text{ab}}, c^{\text{ab}}\rangle$ is not cyclic (see remark 3), we deduce that $M = \langle b, c[a, cb]\rangle$. This is a contradiction since we know that $\langle b, c[a, cb]\rangle$ is not a free factor of $\langle a, b, c\rangle$. $\square$

Corollary 20 *Let n be a cardinal. In the free group of rank n the families of 1-endo-fixed and 1-auto-fixed subgroups coincide if and only if $n \leq 2$.*

Proof. Obviously, these two families coincide for $n = 0, 1$. By Corollary 2 of [10], they also coincide if $n = 2$. And, conversely, Theorem 19 ensures that they do not coincide if $n \geq 3$. $\square$

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References

[1] G.M. Bergman, Supports of derivarions, free factorizations and ranks of fixed subgroups in free groups, *Trans. Amer. Math. Soc.*, **351** (1999), 1531-1550.

[2] M. Bestvina, M. Handel, Train tracks and automorphisms of free groups, *Ann. of Math.*, **135** (1992), 1-51.

[3] D.J. Collins, E.C. Turner, All automorphisms of free groups with maximal rank fixed subgroups, *Math. Proc. Cambridge Philos. Soc.*, **119** (1996), 615-630.

[4] W. Dicks, E. Ventura, The group fixed by a family of injective endomorphisms of a free group, *Contemp. Math.*, **195** (1996), 1-81.

[5] W. Imrich, E.C. Turner, Endomorphisms of free groups and their fixed points, *Math. Proc. Cambridge Philos. Soc.*, **105** (1989), 421-422.

[6] W. Magnus, A. Karrass, D. Solitar, *Combinatorial Group Theory*, Interscience Publishers, New York, (1966).

[7] A. Martino, E. Ventura, On automorphism-fixed subgroups of a free group, *J. Algebra*, **230** (2000), 596-607.

[8] A. Martino, E. Ventura, A description of auto-fixed subgroups in a free group, *preprint*, 41 pages, http://xxx.soton.ac.uk/abs/math.GR/0306380.

[9] John R. Stallings, Whitehead graphs on handlebodies, Geometric group theory down under (Canberra, 1996), de Gruyter, Berlin (1999), 317–330.

[10] E.C. Turner, Test words for automorphisms of free groups, *Bull. London Math. Soc.*, **28** (1996), 255-263.

[11] J.H.C. Whitehead, On certain sets of elements in a free group, *Proc. London Math. Soc.*, **41** (1936), 48-56.