COUNTING PRIMITIVE ELEMENTS IN FREE GROUPS

J. Burillo

Dept. MA IV, UPC, Barcelona, Spain

E. Ventura

Dept. MA III, UPC, Barcelona, Spain,
and Math Dept., CCNY, CUNY, New York.

Primitive elements in free groups are those that can be part of a free basis of the group. In this paper we will be interested in their distribution inside the free group with relation to the concentric balls of elements. In particular, we will prove that the set of primitive elements is increasingly sparse in subsequent balls. We will define two notions of *density*, which will be used to show these results on the distribution of primitive elements inside the free group. With this terminology, we will prove that the set of primitive elements has natural density zero.

To study the set of primitive elements of the free group of rank p, F_p, and inspired by the definition of the Whitehead graph of an element of the free group, we define *graphical sets*. Graphical sets are sets of elements of F_p for which there is a graph which specifies which letters can be adjacent to each other. The vertices of such a graph are the generators of F_p and their inverses, and an edge goes from x to y if the subword xy is allowed.

Whitehead's cut vertex lemma ([W] and for a more modern version, [St]) deals with the connectivity of the Whitehead graph of a primitive element. It states that a necessary (although not sufficient) condition for an element to be primitive is that its Whitehead graph has a cut vertex, i.e. has connectivity one. In our case, what is important to us is that the Whitehead graph of a primitive element is not a complete graph. We will prove that the set of elements with noncomplete graphs has density zero, computing exactly the exponential density of the graphical sets of noncomplete graphs in terms of their adjacency matrices. As a corollary, we will obtained the desired result, showing that the set of primitive elements has natural density zero.

1 Densities

We give the definitions of *natural* and *exponential density*.

Definition 1 *Let G be a finitely generated group with finite generating set X, and let $B_X(n)$ be the ball of radius n, i.e. the set of elements with length at most n in the generators of X. For any subset $S \subseteq G$ we define the* natural density *of S with respect to X, denoted $\delta_X(S)$, as*

$$\delta_X(S) = \limsup_{n\to\infty} \frac{|S \cap B_X(n)|}{|B_X(n)|}.$$

And we define the exponential density *of S with respect to X, denoted $d_X(S)$, as*

$$d_X(S) = \limsup_{n\to\infty} \left(\frac{|S \cap B_X(n)|}{|B_X(n)|}\right)^{1/n}.$$

2 Graphical sets in free groups

Let p be a positive integer and F_p be a free group of rank p. Let X be a basis for F_p. The $2p$ elements in $X^{\pm 1}$ will be denoted by $a_1, \dots, a_{2p}$.

Let Z and Z' be two graphs, possibly having loops (i.e. edges with same initial and terminal vertices), having no multiple edges (i.e. at most one edge is allowed from a given vertex to another) and having $X^{\pm 1}$ as the set of vertices. We will use the notation (a, b) to refer to the edge beginning at the vertex a and ending at the vertex b, for $a, b \in X^{\pm 1}$. We use EZ (a subset of $(X^{\pm 1})^2$) to denote the set of edges of Z. An edge (a, b) is said to be *redundant* when $ab = 1$; otherwise, it is called *irredundant*. We say that Z is *irredundant* when EZ does not contain redundant edges. We say that Z is *simpler* than Z', denoted $Z \leq Z'$, when $EZ \subseteq EZ'$.

The *inverse* graph of Z is the new graph $\overline{Z}$ with the following set of edges:

$$E\overline{Z} = \{(a, b) \in (X^{\pm 1})^2 \,|\, (a, b^{-1}) \in EZ\}.$$

Observe that, when inverting, redundant edges become loops, and loops become redundant edges. So, Z is irredundant if and only if $\overline{Z}$ has no loops. Furthermore, note that $\overline{\overline{Z}} = Z$.

Definition 2 *A* graphical *set is a subset of F_p of the form*

$$S(Z) = \{w \in F_p \,|\, \text{ if } w \text{ contains } a \cdot b \text{ then } (a, b) \in EZ\} \subseteq F_p,$$

where Z is a graph. In the opposite direction, let $w \in F_p$ be a word. The local *graph of w, denoted Z_w, is the simplest graph Z such that $w \in S(Z)$.*

The idea behind graphical sets is that the graph gives a rule to specify which generators can be next to each other in the word. An edge (a, b) implies that the subword ab is allowed in the word. Note that redundant edges correspond to words of the type aa^{-1} which can be simplified. The next result computes the exponential density of graphical sets.

Theorem 3 *Let Z be an irredundant graph. If Z has no nontrivial closed paths, then the exponential density of $S(Z)$ is $1/(2p-1)$. Otherwise,*

$$d_X(S(Z)) = \frac{\rho(\mathrm{ad}(Z))}{2p-1}$$

where $\rho(\mathrm{ad}(Z))$ is the spectral radius of the adjacency matrix of Z.

And the main theorem for this section shows that the union of all graphical sets for noncomplete graphs has small exponential density and hence natural density zero. In other words, almost all the words of the free group have a complete Whitehead graph.

Theorem 4 *Let X be a free basis of the free group, F_p, of rank $p \geq 2$, and let S be the following set*

$$S = \{w \in F_p \mid Z_w < \overline{K}_{2p}\} \subseteq F_p.$$

Then,

$$d_X(S) = \frac{\gamma_p}{2p-1} < 1,$$

where $\gamma_p \in (2p-2, 2p-1)$ is the largest root of the polynomial

$$x^3 - (2p-2)x^2 - (2p-1)x + (2p-2).$$

In particular, $\delta_X(S) = 0$.

3 Primitive elements in the free group

Whitehead's cut vertex lemma ([W]) is crucial to study primitive elements.

Theorem 5 *[Whitehead] Let w be a cyclically reduced word of the free group F_p. If w is primitive then W_w has a cut vertex.*

This result implies in particular, that the set of primitive elements of the free group is included in the set of elements with noncomplete Whitehead graphs. Using then the result in the previous section, we deduce our main result.

Theorem 6 *Let F_p be the free group of rank $p \geq 2$, and let $S \subseteq F_p$ be the set of primitive words. For every free basis X of F_p, we have*

$$\frac{2p-3}{2p-1} \leq d_X(S) \leq \frac{\gamma_p}{2p-1} < 1.$$

Consequently, $\delta_X(S) = 0$.

References

[St] J.R. Stallings, *Whitehead graphs on handlebodies*. Geometric Group Theory Down Under (Canberra, 1996), de Gruyter, Berlin, 1999, 319–330

[W] J.H.C. Whitehead, *On certain sets of elements in a free group*, Proc. London Math. Soc., **41** (1936), 48–56.