THE GROUP FIXED BY A FAMILY OF INJECTIVE ENDOMORPHISMS OF A FREE GROUP

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Let F be a finitely generated, free group, let $\mathrm{Aut}(F)$ denote the group of automorphisms of F, and let $\mathrm{End}(F)$ denote the set of endomorphisms of F.

The *rank of* F, denoted $\mathrm{r}(F)$, is the cardinal of a free generating set of F. The *reduced rank of* F, denoted $\tilde{\mathrm{r}}(F)$, is $\max\{\mathrm{r}(F) - 1, 0\}$, that is, one less than the rank, except for the trivial group where the reduced rank coincides with the rank, which is zero. By an *inert* subgroup of F, we mean a subgroup H such that $\mathrm{r}(H \cap K) \leq \mathrm{r}(K)$ for every subgroup K of F. This is quite a restrictive condition, since the rank of intersections of subgroups of F in general can behave like the order of the *product* of the ranks. In particular, if H is inert in F then $\mathrm{r}(H) \leq \mathrm{r}(F)$.

If $S \subseteq \mathrm{End}(F)$, the *fixed subgroup of* S, denoted $\mathrm{Fix}(S)$, is the subgroup of F consisting of all those elements of F which are fixed by all elements of S, so $\mathrm{Fix}(S) = \{x \in F \mid (x)\beta = x \text{ for all } \beta \in S\}$. For any $\beta \in \mathrm{End}(F)$, we abbreviate $\mathrm{Fix}(\{\beta\})$ to $\mathrm{Fix}(\beta)$.

Building on work of Dyer-Scott, Gersten, Jaco-Shalen, Cohen, and other authors, Bestvina-Handel proved the conjecture of G. P. Scott that, if $\beta \in \mathrm{Aut}(F)$, then $\mathrm{r}(\mathrm{Fix}(\beta)) \leq \mathrm{r}(F)$.

Thomas had used results of Gersten, Stallings, and McCool, to show that if $B \subseteq \mathrm{Aut}(F)$ then $\mathrm{r}(\mathrm{Fix}(B))$ is finite. In this paper we show that if $B \subseteq \mathrm{Aut}(F)$ then $\mathrm{r}(\mathrm{Fix}(B)) \leq \mathrm{r}(F)$. Our proof uses the main result proved by Bestvina (which is actually stronger than the Scott conjecture), in a simplified form which was brought to light by Gaboriau-Levitt-Lustig. We combine this result with graph pullback techniques of Stallings, to show that if $\beta \in \mathrm{Aut}(F)$, then $\mathrm{Fix}(\beta)$ is an inert subgroup of F.

A classical result, recalled as the Inverse Limit Theorem I.4.11 below, bounds the rank of an intersection of subgroups of bounded rank, and this then implies that if $B \subseteq \mathrm{Aut}(F)$ then $\mathrm{Fix}(B)$ is inert in F, so $\mathrm{r}(\mathrm{Fix}(B)) \leq \mathrm{r}(F)$.

The surjectivity property of automorphisms is used nowhere in our proofs, so for any set B of injective endomorphisms of F, $\mathrm{Fix}(B)$ is inert in F, and in particular, $\mathrm{r}(\mathrm{Fix}(B)) \leq \mathrm{r}(F)$.

Concerning general endomorphisms, we can provide the following information about what is known. Imrich-Turner had used the Bestvina-Handel result to show that, if $\beta \in \mathrm{End}(F)$, then $\mathrm{r}(\mathrm{Fix}(\beta)) \leq \mathrm{r}(F)$. Recently, G.M. Bergman has used our result that $\mathrm{r}(\mathrm{Fix}(B)) \leq \mathrm{r}(F)$ for any set B of injective endomorphisms of F, to show that the same inequality holds for any set B of endomorphisms of F, thus proving a conjecture we made in an earlier version of this paper. We conjecture further that, for any set B of endomorphisms of F, $\mathrm{Fix}(B)$ is inert in F; by recent results of Turner it suffices to show that for any idempotent endomorphism β of F, $\mathrm{Fix}(\beta)$ is inert in F. See Problem 2, and the commentary thereto, in the Open Problems section of this article.

The paper is organized as follows.

In Chapter I we develop that part of the theory of groupoids we shall use. In §I.1, we introduce a lot of tedious vocabulary that will allow us to express the Bestvina-Handel topological argument in terms of groupoids, and so make transparent the algebraic nature of their proof. In §I.2, we describe the operations that will be performed on graphs, and the relationship with the free groupoids they generate, and present Stallings' folding lemma which plays an important part throughout. In §I.3, we review the basics concerning free groupoids, including sketches of proofs of theorems by Reidemeister, Goldstein and Nielsen-Schreier. In §I.4, we introduce the notions of inertia for sets acted on by free groups, and for morphisms of free groupoids, and for graph morphism immersions, all related to good behaviour for intersections of subgroups of free groups. Here we sketch proofs of theorems by Imrich-Neumann-Stallings and Magnus-Karrass-Solitar. In §I.5 we give a similar analysis for the fixed subgroupoid of a groupoid endomorphism, sketching a proof of a theorem by Goldstein-Turner-Cohen-Lustig-Bestvina-Handel.

Chapter II introduces the tools for measuring good behaviour of (continuous) self-maps of graphs. In §II.1, we sketch Wielandt's elegant proof of the famous Perron-Frobenius theorem, and describe the consequences which we shall use. In §II.2, we introduce morphisms of finite (pseudo)metric graphs, prove the Bestvina-Feighn-Handel Bounded Cancellation Lemma, and describe the relation with the Perron-Frobenius theorem. In §II.3, we describe five numerical quantities associated with a self-map of a finite graph, the most important being a Perron-Frobenius eigenvalue.

Chapter III is quite a technical interlude checking that certain basic operations can be used to lower, or at least not raise, various of the numerical quantities, especially the Perron-Frobenius eigenvalue.

Chapter IV then studies self-maps of finite graphs which lexicographically minimize, within a similarity class, the five numerical quantities introduced in §2.3. Such self-maps are called minimal representatives, and, in §4.1, it is shown that every similarity class has a minimal representative. In §§IV.2-IV.4, various tremendously useful properties of minimal representatives are obtained, and the implications for the fixed subgroupoid are developed; in essence, there exists an inductive step in which the basis of the fixed subgroupoid increases by at most one edge. In §IV.5, the main results are obtained by analysing this new edge. Thus, in Theorem IV.5.4, we find that the fixed subgroupoid $\mathrm{Fix}(\beta)$, of a π_1-injective groupoid endomorphism β, of a finitely generated, free groupoid G, has the property that all groupoid morphisms equivalent to the inclusion $\mathrm{Fix}(\beta) \to G$ behave well with respect to pullbacks.

We conclude with a selection of open problems.

Most of our presentation, especially §§IV.1-IV.4, consists of a faithful translation of arguments of Bestvina, and Gaboriau-Levitt-Lustig into the language of groupoids, which required a small amount of ingenuity. In spite of our having found some simplifications, on balance the overall proof is now somewhat longer, because we provide many of the details that were originally left to the reader. Let us emphasize then where our original contributions lie. Two of the five numerical quantitities used are new, and this simplifies the original arguments, at the price of obtaining less information which experts may miss, specifically about non-smooth turns which do not occur in indivisible fixed elements. The extension from automorphisms to injective endomorphisms of free groups, although straightforward, is new. The concept of inertia is new, and quite useful, and the main results, Theorems IV.5.4 and IV.5.5, are new.